Homogenization and boundary layers

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We consider the homogenization of an elliptic system with Dirichlet boundary condition, when the coefficients of both the system and the boundary datum are ε-periodic. We show that, as $\varepsilon \to 0$, the solutions converge in L^2 with a power rate in ε, and identify the homogenized limit system. Due to a boundary layer phenomenon, this homogenized system depends in a non trivial way on the boundary. Our analysis answers a longstanding open problem, raised for instance in the book of Bensoussan, Lions and Papanicolaou. It extends substantially previous results obtained for polygonal domains with sides of rational slopes as well as our previous paper [3] where the case of irrational slopes was considered.

We consider the homogenization of elliptic systems in divergence form

$$-\nabla \cdot \left(A\left(\cdot/\varepsilon\right)\nabla u\right)(x) = 0, \quad x \in \Omega, \tag{1}$$

set in a bounded domain Ω of $\mathbb{R}^d$, $d \geq 2$, with an oscillating Dirichlet data

$$u(x) = \varphi(x, x/\varepsilon), \quad x \in \partial\Omega. \tag{2}$$

As is customary, $\varepsilon > 0$ is a small parameter, and $A = A^{\alpha\beta}(y) \in M_N(\mathbb{R})$ is a family of functions of $y \in \mathbb{R}^d$, indexed by $1 \leq \alpha, \beta \leq d$, with values in the set of $N \times N$ matrices. Also, $u = u(x)$ and $\varphi = \varphi(x, y)$ take their values in $\mathbb{R}^N$. We recall, using Einstein convention for summation, that for each $1 \leq i \leq N$,

$$(\nabla \cdot A\left(\cdot/\varepsilon\right)\nabla u)_i(x) \;:=\; \partial_{x_\alpha}\left[A^{\alpha\beta}_{ij}\left(\cdot/\varepsilon\right)\,\partial_{x_\beta} u_j\right](x).$$

In the sequel, greek letters $\alpha, \beta, ...$ will range between 1 and d and latin letters $i, j, k, ...$ will range between 1 and N.

Systems of type (1) are involved in various domains of material physics, notably in linear elasticity and in thermics In many cases they come with a right hand side f. In the context of thermics, $d = 2$ or 3, $N = 1$, u is the temperature, and $\sigma = A(\cdot/\varepsilon)\nabla u$ is the heat flux given by Fourier law. The parameter ε models heterogeneity, that is short-length variations of the material conducting properties. The boundary term φ in (2) corresponds to a prescribed temperature at the surface of the body. In the context of linear elasticity, $d = 2$ or 3, $N = d$, u is the unkown displacement, f is the external load and A is a fourth order tensor that models Hooke's law.

We make three hypotheses:

i) Ellipticity: For some $\lambda > 0$, for all family of vectors $\xi = \xi_i^\alpha \in \mathbb{R}^{Nd}$

$$\lambda \sum_\alpha \xi^\alpha \cdot \xi^\alpha \;\le\; \sum_{\alpha,\beta,i,j} A_{ij}^{\alpha,\beta}\, \xi_j^\beta\, \xi_i^\alpha \;\le\; \lambda^{-1} \sum_\alpha \xi^\alpha \cdot \xi^\alpha.$$

ii) Periodicity: $\forall y \in \mathbb{R}^d$, $\forall h \in \mathbb{Z}^d$, $\forall x \in \partial\Omega$, $A(y+h) = A(y)$, $\varphi(x,y) = \varphi(x, y+h)$.

iii) Smoothness: The functions A and φ, as well as the domain Ω are smooth. It is actually enough to assume that ϕ and Ω are in some H^s for s big enough, but we will not try to compute the optimal regularity.

We are interested in the limit $\varepsilon \to 0$, *i.e.* the homogenization of system (1)-(2).

For the non-oscillating Dirichlet problem, one shows that u^ε weakly converges in $H^1(\Omega)$ to the solution u^0 of the homogenized system

$$\begin{cases} -\nabla \cdot \left(A^0 \nabla u^0\right)(x) = 0, & x \in \Omega, \\ u^0(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{3}$$

The so-called homogenized matrix A^0 comes from the averaging of the microstructure. It involves the periodic solution $\chi = \chi^\gamma(y) \in M_N(\mathbb{R})$, $1 \le \gamma \le d$, of the *cell problem*:

$$-\partial_{y_\alpha} \left[A^{\alpha\beta}(y)\, \partial_{y_\beta} \chi^\gamma(y)\right] \;=\; \partial_{y_\alpha} A^{\alpha\gamma}(y), \quad \int_{[0,1]^d} \chi^\gamma(y)\, dy = 0. \tag{4}$$

The homogenized matrix is then given by:

$$A^{0,\alpha\beta} \;=\; \int_{[0,1]^d} A^{\alpha\beta} \;+\; \int_{[0,1]^d} A^{\alpha\gamma} \partial_{y_\gamma} \chi^\beta.$$

One may even go further in the analysis, and obtain a two-scale expansion of u^ε. Denoting

$$u^1(x,y) \;:=\; -\chi^\alpha(y) \partial_{x_\alpha} u^0(x), \tag{5}$$

it is proved for instance in the book Bensoussan-Louis and Papanicolaou that

$$u^\varepsilon(x) \;=\; u^0(x) \;+\; \varepsilon u^1(x, x/\varepsilon) \;+\; O(\sqrt{\varepsilon}), \;\text{ in }\; H^1(\Omega). \tag{6}$$

Actually, an open problem in this area is to compute the next term in the expansion in the presence of a boundary. This is actually another motivation for this work.

The main result of this talk is

Theorem 1 (Homogenization in smooth domains)
Let Ω be a smooth bounded domain of $\mathbb{R}^d$, $d \ge 2$. We assume that it is uniformly convex (all the principal curvatures are bounded from below).

Let u^ε be the solution of system (1)-(2)*, under the ellipticity, periodicity and smoothness conditions i)-iii).*

There exists a boundary term φ_* (depending on φ, A and Ω), with $\varphi_* \in L^p(\partial\Omega)$ for all finite p, and a solution u^0 of (3) *with boundary data φ_*, such that:*

$$\|u^\varepsilon - u^0\|_{L^2(\Omega)} \leq C_\alpha \, \varepsilon^\alpha, \quad \textit{for all } 0 < \alpha < \frac{d-1}{3d+5}. \tag{7}$$

References

[1] Marco Avellaneda and Fang-Hua Lin, Compactness methods in the theory of homogenization. *Comm. Pure Appl. Math.*, **40**, (6), (1987)803–847, 1987.

[2] Alain Bensoussan, Jacques-Louis Lions, and George Papanicolaou, Asymptotic analysis for periodic structures, *Studies in Mathematics and its Applications*, bf 5, North-Holland Publishing Co., Amsterdam, (1978).

[3] David Gérard-Varet and Nader Masmoudi, Homogenization in polygonal domains. *J. Eur. Math. Soc. (JEMS)*, **13**, no. 5, (2011), 1477-1503.

[4] David Gérard-Varet and Nader Masmoudi, Homogenization and boundary layers. *Acta. Math (to appear)*, (2012).

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