

Implicit-Explicit methods for hyperbolic systems with hyperbolic and parabolic relaxation

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In this talk we discuss the problem of constructing effective high order methods for the numerical solution of hyperbolic systems of balance laws, in presence of stiff source. Because of the stiffness, the use of implicit integrators is advisable, so that no restrictions on the time step due to small relaxation time will appear. Two different relaxation systems will be considered, namely hyperbolic and parabolic relaxation. Because of the different nature of the problems, the two cases will be considered separately. A common denominator of both treatments is the choice of space discretization. Most schemes for conservation or balance laws are discretized by finite volume (FV), conservative finite difference (FD), or discontinuous Galerkin (DG). Here we choose conservative finite difference since it is probably the simplest general approach for the construction of high order shock capturing schemes for such problems.

Hyperbolic relaxation The prototype 2×2 hyperbolic system with hyperbolic relaxation takes the form:

$$\begin{cases} u_t + v_x = 0 \\ v_t + p(u)_x = -\frac{1}{\varepsilon}(v - q(u)) \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) \end{cases} \quad (1)$$

with $p'(u) > 0 \forall u \in \mathbb{R}$. Formally, if $\varepsilon \rightarrow 0$, the 2×2 system relaxes to the relation $v = q(u)$ and the single scalar equation for u :

$$u_t + q(u)_x = 0 \quad (2)$$

If the subcharacteristic condition $q'(u)^2 \leq p'(u) \forall u \in \mathbb{R}$ is satisfied, then the solution of system (1) relaxes to the solution of Eq.(2). If the initial data is “well prepared”, i.e. if $v_0(x) = q(u_0(x))$, then the solution will not present any “initial layer”.

Numerical solutions of systems of the form (1) can be effectively obtained by using Implicit-Explicit Runge-Kutta methods in time, coupled with conservative finite-difference in space. The hyperbolic part (which may be non linear and is non local because of the space derivative) may be treated explicitly, since the system is non-stiff (if one is interested in resolving all the waves), while the stiff implicit part can be treated implicitly.

The simplest IMEX scheme is obtained by first implicit-explicit Euler scheme, which for system (1) can be written as

$$\begin{cases} u^{n+1} = u^n - \Delta t Dv^n \\ v^{n+1} = v^n - \Delta t Dp(u^n) - \frac{\Delta t}{\varepsilon}(v^{n+1} - q(u^{n+1})) \end{cases}$$

where D represent a discretization of the space derivative. As $\varepsilon \rightarrow 0$, the numerical solution is projected onto the manifold $v = q(u)$, and the scheme relaxes to the Explicit Euler scheme for the relaxed equation (2).

IMEX-Runge Kutta schemes with s -stages will guarantee higher order accuracy. They are characterized by two coupled Runge-Kutta schemes, the implicit one identified by the $s \times s$ matrix A and the vectors $b, c \in \mathbb{R}^s$, while the explicit scheme is defined by matrix \tilde{A} and vectors \tilde{b} and \tilde{c} . Usually the implicit scheme is diagonally implicit, i.e. matrix A is a lower triangular matrix, while \tilde{A} is lower triangular with zeroes on the diagonal.

In the design of effective IMEX schemes for problems with hyperbolic relaxation several requirements are considered, namely:

1. *Accuracy.* High order in time is achieved by imposing the so-called order conditions obtained by matching Taylor expansion in time or exact and numerical solution. In addition to the usual order conditions of the two RK schemes, one has to satisfy some additional coupling conditions (see [4]). Such conditions guarantee the so called *classical order*, valid for $\varepsilon \approx 1$.
2. *Asymptotic preservation.* We require that the method applied to system (1) becomes a consistent discretization of the relaxed equation (2) as $\varepsilon \rightarrow 0$, possibly maintaining the same order of accuracy in the limit. This property is related to the L -stability of the implicit scheme (see, for example, [6]).
3. *Uniform accuracy.* The accuracy of the method depends on ε , and a degradation of the accuracy is observed for intermediate values of ε . It would be desirable to reduce such degradation. The accuracy dependence is analyzed by comparing the asymptotic expansion in ε of the exact and numerical solution [1]. Based on such comparison, additional conditions are derived, and used to construct new schemes with better uniform accuracy in ε .

All such points will be addressed during the talk. Two classes of IMEX-RK will be considered. The first one, called type A, has the property that the matrix A is invertible. For such methods it is easy to prove that the IMEX relaxes to the explicit RK applied to the relaxed equation, thus maintaining the order of accuracy in the variable u . The second class is called CK [4]. For them $a_{11} = 0$. Such methods are more difficult to analyze, but are somehow easier to construct than methods of type A, because some simplifying conditions can be applied to their coefficients. Several numerical tests on various problems will illustrate the relative merits of the IMEX schemes presented.

Parabolic relaxation Parabolic relaxation is obtained when one is interested in the long time behavior of the solution of hyperbolic systems with relaxation.

The prototype system takes the form

$$\begin{cases} u_t + v_x = 0 \\ \varepsilon^2 v_t + p(u)_x = -(v - q(u)) \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) \end{cases} \quad (3)$$

As the relaxation parameter vanishes, the variable v obeys the relation $v = q(u) - p(x)_x$, while and the asymptotic behavior of the system is governed by a scalar convection-diffusion equation.

$$u_t + q(u)_x = p(u)_{xx} \quad (4)$$

Notice that the characteristic speeds $\lambda_{\pm} = \pm\sqrt{p'(u)}/\varepsilon$ diverge as $\varepsilon \rightarrow 0$, which makes the numerical treatment of the system more delicate.

Two different kinds of IMEX Runge-Kutta schemes will be considered. The first will be denoted as *partitioned* [2]: the stiffness is associated to the *variable*. The equation for the non stiff variable u will be treated explicitly, while the equation for the stiff variable v will be treated implicitly, according to the following scheme (here for simplicity we consider the case $q = 0$ and $p(u) = u$)

$$\begin{aligned} u_t &= \boxed{-v_x} && \text{[Explicit]} \\ v_t &= \boxed{-(u_x + v)/\varepsilon^2} && \text{[Implicit]} \end{aligned} \quad (\text{Partitioned})$$

The second family will be denoted *additive*: the right hand side is given by the sum of two terms, one of which is treated explicitly, and one implicitly, according to the scheme

$$\begin{aligned} u_t &= \boxed{-v_x} \\ v_t &= \boxed{-u_x/\varepsilon^2} \quad \boxed{-v/\varepsilon^2} \end{aligned} \quad (\text{Additive})$$

[Explicit] [Implicit]

Methods based on the first approach have been more studied in the literature for the diffusion relaxation, because the hyperbolic part becomes stiff when the system relaxes towards the parabolic equation. In fact, the characteristic speeds are $c_{\pm} = \pm 1/\varepsilon$, and classical explicit schemes for the hyperbolic part would suffer by a CFL restriction $\Delta t \leq \varepsilon C \Delta x$, where the maximum CFL number C is of order unity and depends on the particular scheme.

Two main issues will be discussed here. First, we shall show that the use of classical IMEX schemes for hyperbolic systems with stiff relaxation, with L-stable implicit part (see, for example, [3]), applied to system (3) in partitioned form will lead to consistent explicit discretization of the limit convection-diffusion equation (4). Because of this, the limit scheme will suffer of the typical parabolic CFL restriction $\Delta t \propto \Delta x^2$. To overcome such a drawback, in both approaches one can make a wise use of the asymptotic limit, by adding and subtracting the same term, one treated implicitly and one explicitly, so that, in

the limit $\varepsilon \rightarrow 0$, the scheme converges to an implicit method for the diffusion equation (see [2,3]).

The second part concerns the analysis of the *additive approach* [3]. This approach is attractive, because it is the more commonly used one for hyperbolic systems with relaxation, however it has the serious drawback that the hyperbolic part itself is stiff, and therefore it appears almost hopeless to treat a stiff term with an explicit scheme, and get around with prohibitive stability conditions. We show that the simple Explicit-Implicit Euler scheme applied to system (3) (with $q = 0$ and $p = u$) in the additive form converges to explicit Euler schemes applied to the diffusion equation (apart from higher order terms), while other classical IMEX schemes fail. The behavior is explained by an analysis based on an asymptotic expansion in ε of the exact and numerical solution. The analysis introduces additional conditions that need to be satisfied. Using such conditions it is possible to derive new second order IMEX additive schemes that possess the desired AP property.

Several applications to various test cases, including non linear diffusion, and model kinetic equations will be presented.

References

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