Relative entropy methods in the mathematical theory of complete fluid systems

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1 Navier-Stokes-Fourier system

Relative entropy methods are based on estimating the distance, in a suitable metric, of a solution to a system of partial differential equations to a given function, typically another solution of the same system. We use this approach in the study of *weak solutions* to the full Navier-Stokes-Fourier system describing the motion of a viscous, compressible and heat conducting fluid:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0, \tag{1}$$

$$\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathcal{S} + \varrho \vec{f}, \tag{2}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\vec{u}) + \nabla_x \left(\frac{\vec{q}}{\vartheta} \right) = \sigma, \tag{3}$$

where $\varrho = \varrho(t,x)$ is the fluid density, $\vec{u} = \vec{u}(t,x)$ the velocity field, and $\vartheta = \vartheta(t,x)$ the absolute temperature. Furthermore, $p(\varrho, \vartheta)$ is the pressure, $s = s(\varrho, \vartheta)$ the specific entropy, $\mathcal{S} = \mathcal{S}(\vartheta, \nabla_x \vec{u})$ the viscous stress determined by Newton's law

$$\mathcal{S}(\vartheta, \nabla_x \vec{u}) = \mu(\vartheta)\Big(\nabla_x \vec{u} + \nabla_x^t \vec{u} - \frac{2}{3}\mathrm{div}_x \vec{u} \mathcal{I} \Big) + \eta(\vartheta) \mathrm{div}_x \vec{u} \mathcal{I}, \tag{4}$$

and $\vec{q} = \vec{q}(\vartheta, \nabla_x \vartheta)$ is the heat flux,

$$\vec{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta. \tag{5}$$

Finally, the symbol σ stands for the *entropy production*,

$$\sigma = \frac{1}{\vartheta}\Big(\mathcal{S}(\vartheta, \nabla_x \vec{u}) : \nabla_x \vec{u} - \frac{\vec{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \Big). \tag{6}$$

We suppose that the fluid occupies a bounded domain $\Omega \subset R^3$, the boundary of which is energetically insulated, specifically,

$$\vec{u}|_{\partial \Omega} = 0, \ \vec{q}(\vartheta, \nabla_x \vartheta) \cdot \vec{n}|_{\partial\Omega} = 0. \tag{7}$$

If, moreover, the external force $\vec{f} = \nabla_x F(x)$ is conservative, there are two obvious constants of motion: The *total mass*

$$\int_\Omega \varrho(t, \cdot) \ \mathrm{d}x = M_0$$

and the *total energy*

$$\int_{\Omega} \left(\frac{1}{2} \varrho |\vec{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) (t, \cdot) \, \mathrm{d}x = E_0,$$

where $e = e(\varrho, \vartheta)$ is the specific internal energy interrelated to the pressure and the entropy by means of Gibbs' relation

$$\vartheta D s(\varrho, \vartheta) = D e(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right). \tag{8}$$

2 Thermodynamic stability, ballistic free energy

The so-called *hypothesis of thermodynamic stability* plays a crucial role in the forthcoming analysis:

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0. \tag{9}$$

We introduce *ballistic free energy*

$$H_\Theta(\varrho, \vartheta) = \varrho \Big(e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta) \Big), \ \Theta > 0,$$

together with the *relative entropy* functional

$$\mathcal{E}(\varrho, \vartheta | r, \Theta) = H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho} (\varrho - r) - H_\Theta(r, \Theta). \tag{10}$$

As a direct consequence of (9), we check that

$$\varrho \mapsto H_\Theta(\varrho, \Theta) \text{ is strictly convex for any fixed } \Theta,$$

$$\vartheta \mapsto H_\Theta(\varrho, \vartheta) \text{ is decreasing for } \vartheta < \Theta \text{ and increasing for } \vartheta > \Theta.$$

Consequently,

$$\mathcal{E}(\varrho, \vartheta | r, \Theta) \geq c(K) \Big(|\varrho - r|^2 + |\vartheta - \Theta|^2 \Big) \text{ for } (\varrho, \vartheta) \in K, \tag{11}$$

$$\mathcal{E}(\varrho, \vartheta | r, \Theta) \geq c(K) \Big(1 + \varrho e(\varrho, \vartheta) + \varrho |s(\varrho, \vartheta)| \Big) \text{ for } (\varrho, \vartheta) \in [0, \infty)^2 \setminus K, \tag{12}$$

where $K \subset (0, \infty)^2$ is a compact set containing and open neighbourhood of (r, Θ).

3 Stability of equilibria

Consider the equilibrium solution $\tilde{\varrho}$, $\overline{\vartheta}$,

$$\nabla_x p(\tilde{\varrho}, \overline{\vartheta}) = \tilde{\varrho} \nabla_x F, \ \tilde{\varrho} = \tilde{\varrho}(x), \ \overline{\vartheta} > 0 \text{ a positive constant},$$

determined by the constraints

$$\int_\Omega \tilde{\varrho}\ \mathrm{d}x = M_0,\ \int_\Omega \Big(\tilde{\varrho}e(\tilde{\varrho},\overline{\vartheta}) - \tilde{\varrho}F\Big)\ \mathrm{d}x = E_0.$$

Solutions of (1 - 3), supplemented with the boundary conditions (7), satisfy the *total dissipation balance*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\vec{u}|^2 + \mathcal{E}(\varrho,\vartheta|\tilde{\varrho},\overline{\vartheta})\right)\ \mathrm{d}x + \overline{\vartheta}\int_\Omega \sigma\ \mathrm{d}x = 0, \tag{13}$$

where $\tilde{\varrho}$, $\overline{\vartheta}$ is the equilibrium solution.

Thus the coercivity properties (11), (12) imply that the functional

$$\int_\Omega \left(\frac{1}{2}\varrho|\vec{u}|^2 + \mathcal{E}(\varrho,\vartheta|\tilde{\varrho},\overline{\vartheta})\right)\ \mathrm{d}x$$

represents a *distance* between the trajectory $t \mapsto \{\varrho(t,\cdot),\vartheta(t,\cdot),\vec{u}(t,\cdot)\}$ to the equilibrium $\{\tilde{\varrho},\overline{\vartheta},0\}$. In particular, relation (13) yields *unconditional* convergence of solutions to equilibria for $t \to \infty$, see [2].

4 Weak solutions and weak-strong uniqueness principle

Weak solutions satisfy equations (1 - 3) in the sense of distributions, where the entropy production rate σ complies with *inequality*

$$\sigma \geq \frac{1}{\vartheta}\Big(\mathcal{S}(\vartheta,\nabla_x\vec{u}) : \nabla_x\vec{u} - \frac{\vec{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\Big), \tag{14}$$

and the whole system is supplemented by the *total energy balance*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho e(\varrho,\vartheta)\right)\ \mathrm{d}x = \int_\Omega \varrho\vec{f}\cdot\vec{u}\ \mathrm{d}x. \tag{15}$$

Such a definition is

• *compatible* in the sense that regular weak solutions satisfy the system in the classical sense, in particular, they satisfy (14) with equality sign;

• weak solutions exist *globally in time* for any finite energy initial data under suitable structural restrictions imposed on the state equation and the viscosity coefficients.

Finally, it can be shown, by the method of relative entropy, that the weak solutions satisfy the *weak-strong uniqueness* principle. The proof is based on using the *relative entropy* functional in the form

$$\int_\Omega \Big(\varrho|\vec{u} - \tilde{\vec{u}}|^2 + \mathcal{E}(\varrho,\vartheta|\tilde{\varrho},\tilde{\vartheta})\Big)\ \mathrm{d}x, \tag{16}$$

where $\{\tilde{\varrho}, \tilde{\vartheta}, \tilde{\vec{u}}\}$ is a (hypothetical) strong solution emanating from the same initial data. It can be shown that the weak and strong solutions coincide as long as the latter exists, see [1].

References

[1] E.Feireisl and A.Novotný, *Arch. Rational Mech. Anal.*, **204** (2012), pp. 683-706

[2] E.Feireisl and D.Pražák, *Asymptotic behavior of dynamical systems in fluid mechanics*, AIMS Springfield, (2010)

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