Non-standard solutions of isentropic Euler with Riemann data

Camillo De Lellis
Institut für Mathematik, Universität Zürich
camillo.delellis@math.uzh.ch

We consider the isentropic compressible Euler equations of gas dynamics in two space dimensions and in the Eulerian formulation. The gas is described by the state vector (ρ, v), where ρ is the density and v the velocity. The balance laws for mass and linear momentum give therefore the following system of 3 scalar equations

$$\begin{cases} \partial_t \rho + \operatorname{div}_x (\rho v) = 0 \\ \\ \partial_t (\rho v) + \operatorname{div}_x (\rho v \otimes v) + \nabla [p(\rho)] = 0\,. \end{cases} \tag{1}$$

The pressure p is required to be a smooth function with $p' > 0$. A largely studied class of examples is given by the pressure law $p(\rho) = \kappa \rho^\gamma$ where $\gamma > 1$. However, the results presented in this talk are, for the moment, not valid for such laws.

We will focus our attention on the Cauchy problem for (1), i.e. on solutions on $\mathbb{R}^2 \times [0, \infty[$ satisfying the initial conditions

$$(\rho, v)(x, 0) = (\rho_0(x), v_0(x))\,. \tag{2}$$

Moreover, we will consider Riemann data having the following very specific form

$$\rho_0(x) = \begin{cases} \rho^+ & \text{if } x_2 > 0 \\ \rho^- & \text{if } x_2 < 0 \end{cases} \tag{3}$$

$$v_0(x) = \begin{cases} v^+ & \text{if } x_2 > 0 \\ v^- & \text{if } x_2 < 0\,. \end{cases} \tag{4}$$

As it is well known solutions to (1) are in general not unique, unless the system is complemented with suitable admissibility criteria. Perhaps the most popular one is the so-called *entropy condition*, which in the case at hand requires the following inequality for the energy density and the energy flux:

$$\partial_t \left(\rho \varepsilon(\rho) + \rho \frac{|v|^2}{2} \right) + \operatorname{div}_x \left[\left(\rho \varepsilon \rho + \rho \frac{|v|^2}{2} + p(\rho) \right) v \right] \le 0\,, \tag{5}$$

where the internal energy density ε is linked to the pressure p by the identity $p(r) = r^2 \varepsilon'(r)$.

Since the pioneering work of Riemann it is known that, if we restrict our attention to the 1-*dimensional Riemann problem*, i.e. to pairs (ρ, v) which are admissible solutions of (1)-(2)-(3)-(4) and depend only on $\frac{x_2}{t}$, then (with some more assumptions of technical natural) there is a unique solution (see for instance [7, Section 4.7]). Surprisingly the situation is radically different if we drop the requirement that (ρ, v) depends only on $\frac{x_2}{t}$.

Theorem 0.1 *There are a smooth pressure law p with $p' > 0$ and constants $\rho^\pm$ and $v^\pm$ for which there exist infinitely many admissible bounded solutions (ρ, v) of* (1), (2), (3) - (4) *with* $\inf \rho > 0$.

The proof builds upon the methods of papers [2]-[3], where László Székelyhidi and the first author had already shown that the admissibility condition (5) does not imply the uniqueness of L^∞ solutions of the Cauchy problem. However, the examples in the paper [3] had very rough initial data and it was not at all clear whether more regular data could be achieved. We indeed were inspired by the recent work of Székelyhidi, who in [6] recasts the vortex-sheet problem of incompressible fluid dynamics in the framework of [2]-[3]. The situation here is, though, considerably more complicated and hence requires some new ideas.

We note a few important things.

- The pressure law p is constructed ad hoc and hence it is still open whether special choices of p might obstruct our construction.
- The data of Theorem 0.1 cannot be generated by Lipschitz compression waves and hence the question whether Theorem 0.1 might hold for regular initial data is still open and currently under investigation.
- In view of the results in [6] and because Theorem 0.1 shares many similarities with them, it seems likely that the Dafermos' entropy rate admissibility criterion does not select the "classical" solution to the Riemann problem, i.e. there might be a "non-standard" solution which is more dissipative than the classical one.
- Finally, though the solutions of Theorem 0.1 are very irregular, it is rather unclear where one wishes to set a boundary. On the one hand the space of BV functions does not seem suitable for an existence theory in more than one space dimension (see the papers [5] and [1]; however, explicit examples of blow-up are, to my knowledge, still missing for the system (1)). On the other hand the recent paper [4] shows the existence of *continuous* solutions to the incompressible Euler equations which dissipate the kinetic energy. This may suggest that the framework of [2]-[3] is likely to produce "strange" piecewise continuous solutions to hyperbolic systems of conservation laws.

My talk will discuss al these issues and give an outlook of several natural questions which these considerations naturally rise.

References

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Joint work with: Elisabetta Chiodaroli (*Universität Zürich*), László Székelyhidi Jr. (*Universität Leipzig*)