Stability of the free plasma-vacuum interface

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We consider the free boundary problem for the plasma-vacuum interface in ideal compressible magnetohydrodynamics (MHD).

Plasma-vacuum interface problems appear in the mathematical modeling of plasma confinement by magnetic fields (see, e.g., [2]). In this model the plasma is confined inside a perfectly conducting rigid wall and isolated from it by a vacuum region, due to the effect of strong magnetic fields. In astrophysics, the plasma-vacuum interface problem can be used for modeling the motion of a star or the solar corona.

Let us assume that the interface between plasma and vacuum is given by a hypersurface

$$\Gamma(t) := \{(x', x_3) \in \mathbb{R}^3 \, , \, x_3 = f(t, x')\} \, ,$$

where $t \in [0, T]$ and $x' = (x_1, x_2)$ and let $\Omega^+(t)$ and $\Omega^-(t)$ be space-time domains occupied by the plasma and the vacuum respectively. Then we have $\Omega^\pm(t) = \{x_3 \gtrless f(t, x')\}$.

In the plasma region $\Omega^+(t)$ the flow is governed by the usual compressible MHD equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v - H \otimes H) + \nabla q = 0, \\ \partial_t H - \nabla \times (v \times H) = 0, \\ \partial_t\big(\rho e + \frac{1}{2}|H|^2\big) + \operatorname{div}\big((\rho e + p)v + H \times (v \times H)\big) = 0, \end{cases} \tag{1}$$

where ρ denotes density, $v \in \mathbb{R}^3$ plasma velocity, $H \in \mathbb{R}^3$ magnetic field, $p = p(\rho, S)$ pressure, $q = p + \frac{1}{2}|H|^2$ total pressure, S entropy, $e = E + \frac{1}{2}|v|^2$ total energy, and $E = E(\rho, S)$ internal energy. With a state equation of gas, $\rho = \rho(p, S)$, and the first principle of thermodynamics, (1) is a closed system.

System (1) is supplemented by the divergence constraint

$$\operatorname{div} H = 0 \tag{2}$$

on the initial data.

In the vacuum domain $\Omega^-(t)$, as in [1, 2], we consider the so-called *pre-Maxwell dynamics*

$$\nabla \times \mathcal{H} = 0, \qquad \operatorname{div} \mathcal{H} = 0, \tag{3}$$

describing the vacuum magnetic field $\mathcal{H} \in \mathbb{R}^3$. That is, as usual in nonrelativistic MHD, in the Maxwell equations we neglect the displacement current $(1/c)\,\partial_t E$, where c is the speed of light and $E \in \mathbb{R}^3$ the electric field.

The plasma variable $U = U(t,x) = (q, v, H, S)$ is connected with the vacuum magnetic field $\mathcal{H}$ through the relations [1, 2]

$$\partial_t \varphi = v \cdot N, \quad [q] = 0, \quad H \cdot N = 0, \quad \mathcal{H} \cdot N = 0 \quad \text{on } \Gamma(t), \tag{4}$$

where $N = (-\partial_1 f, -\partial_2 f, 1)$ and $[q] = q|_\Gamma - \frac{1}{2}|\mathcal{H}|^2_{|\Gamma}$ denotes the jump of the total pressure across the interface. Therefore the interface $\Gamma(t)$ moves with the plasma, the total pressure is continuous across $\Gamma(t)$, the magnetic field on both sides is tangent to $\Gamma(t)$. Because of (4), the free interface $\Gamma(t)$ is a characteristic boundary for (1). This fact gives a loss of control of derivatives in the normal direction to the boundary.

We assume that the plasma density does not go to zero continuously at the interface (clearly in the vacuum region $\Omega^-(t)$ the density is identically zero), but has a jump, meaning that it is bounded away from zero in the plasma region and it is identically zero in the vacuum region. This assumption is compatible with the continuity of the total pressure in (4).

It is well-known that for general data the linearization of (1) - (4) may be either violently unstable or weakly (neutrally) stable because of the failure of the uniform Kreiss-Lopatinski condition. For instance, posing $H = \mathcal{H} = 0$ gives the Euler compressible equations in vacuum and the Rayleigh-Taylor instability may occur. The introduction of the magnetic field may have a stabilizing effect and it is of interest to find under which conditions the problem becomes stable.

In our talk we discuss the well-posedness of the problem in suitable anisotropic Sobolev spaces under the stability condition

$$|H \times \mathcal{H}| > 0 \quad \text{on } \Gamma(t), \tag{5}$$

i.e. provided the magnetic fields on the two sides of the free-boundary are not colinear.

References

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