Hyperbolic Equations on Networks

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The theory of transportation methods has been an active area of research since almost three decades. Early work has been inspired by studying the physics of road networks or large scale production networks. Later ideas, methods and results have been extended to electric, biological or social networks. A major motivation for studying systems on the rather particular geometry of a network stems from the huge economic and sociologic impact of results in this area. The common ground are networks wherein the dynamics is governed by partial differential equations, in particular conservation laws and balance equations where we use the various types of solutions that have been considered in the mathematical literature. In particular in the context of mathematical modeling and analysis a variety of literature exists concerning network flows. These publications range from application in data networks (e.g. [1]) and traffic flow (e.g. [2]) over supply chains (e.g. [3]) to flow of water in canals (e.g. [4,5]). Also in the engineering community, gas flow in pipelines is in general modeled by transient models, see e.g. [6]. More detailed models based on partial differential equations have also been introduced be found e.g. in [7]. Therein, the flow of gas inside the pipes is modeled by a system of balance laws. At junctions of two or more pipes, algebraic conditions couple the solution inside the pipeline segments and yield a network solution. A different physical problem, leading to a similar analytical framework, is that of the flow of water in open channels, considered [8]. Starting with the classical work, several other recent papers deal with the control of smooth solutions [9]. In all these cases an assumption on the C^1-norm of initial and boundary data is necessary. Clearly, the given references are incomplete and we refer to the papers and references therein for more details.

In this talk we want to present recent results on conservation laws on networks starting from questions the modeling of physical processes on these networks and going further to tackle questions of control and stablization of network flows. We will summarize existing and new results and present a common framework of the mathematical discussion of network solutions [10]. Based on the analytical results controllability and optimization issues will be discussed. Controlability issues have also been analysed using Lyapunov functions as well as energy estimates and feedback boundary control laws. Currently, most results are based on single equations and have not been extended to networks. In order to obtain a network formulation, coupling conditions which yield boundary conditions are essential. The formulation of well-defined node conditions that are also reasonable model is still a major issue in the mathematical discussion. Further, we discuss the incoporation of results on feedback laws at boundaries of single controlled partial differential equations within suitable coupling conditions. Concerning optimization problems on networks only a few rigorous results exists so far. We present some approaches and discuss recent results in the direction of the characterization of optimal controls with applications to supply chain networks and traffic flow. Numerical examples will be given.

We want to present a broad view on these problems and also give some directions for open problems and possible future research.

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