Almost sure existence of global weak solutions for supercritical Navier-Stokes equations

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In this talk we show that after suitable data randomization there exists a large set of supercritical periodic initial data for both 2D and 3D Navier-Stokes equations for which global energy bounds are proved. As a consequence we obtain almost sure supercritical global weak solutions. We also show that in 2D these global weak solutions are unique.

To explain the problem more in details let's start by considering the initial value problem for the incompressible Navier-Stokes equations given by

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \mathbb{P}\nabla\cdot(\vec{u}\otimes\vec{u}); \quad x\in\mathbb{T}^d \quad \text{or} \quad \mathbb{R}^d,\, t>0 \\ \nabla\cdot\vec{u} = 0 \\ \vec{u}(x,0) = \vec{f}(x), \end{cases} \tag{1}$$

where f is divergence free and $\mathbb{P}$ is the projection into divergence free vector fields given via

$$\mathbb{P}\vec{h} = \vec{h} - \nabla\frac{1}{\Delta}(\nabla\cdot\vec{h}). \tag{2}$$

It is well-known that global well-posedness of (1) when the space dimension $d=3$ is a long standing open question. This is related to the fact that the equations (1) are so called super-critical when $d>2$. Indeed recall that if the velocity vector field $\vec{u}(x,t)$ solves the Navier-Stokes equations (1) in $\mathbb{T}^d$ then $\vec{u}_\lambda(x,t)$ with

$$\vec{u}_\lambda(x,t) = \lambda\vec{u}(\lambda x, \lambda^2 t),$$

is also a solution to the system (1) for the initial data

$$\vec{u}_{0\,\lambda} = \lambda\vec{u}_0(\lambda x) \ . \tag{3}$$

The spaces which are invariant under such a scaling are called critical spaces for the Navier-Stokes equations. Examples of critical spaces for the Navier-Stokes in $\mathbb{T}^d$ are:

$$\dot{H}^{\frac{d}{2}-1} \hookrightarrow L^d \hookrightarrow \dot{B}^{-1+\frac{d}{p}}_{p|p<\infty,\infty} \hookrightarrow BMO^{-1}. \tag{4}$$

In particular, for Sobolev spaces, $\|\vec{u}_\lambda(x,0)\|_{\dot{H}^{s_c}} = \|\vec{u}(x,0)\|_{\dot{H}^{s_c}}$, when $s_c = \frac{d}{2}-1$. We recall that the exponents s are called *critical* if $s=s_c$, *sub-critical* if $s>s_c$ and *super-critical* if $s<s_c$.

On the other hand, classical solutions to the (1) satisfy the decay of energy which can be expressed as:

$$\|u(x,t)\|^2_{L^2} + \int_0^t \|\nabla u(x,\tau)\|^2_{L^2}\, d\tau = \|u(x,0)\|^2_{L^2}. \tag{5}$$

Note that when $d = 2$ the energy $\|u(x,t)\|_{L^2}$, which is the globally controlled thanks to (5), is exactly the scaling invariant $\dot{H}^{s_c} = L^2$-norm. In this case the equations are said to be *critical.* When $d = 3$, the energy $\|u(x,t)\|_{L^2}$ is at the super-critical level with respect to the scaling invariant $\dot{H}^{\frac{1}{2}}$-norm, and hence the Navier-Stokes equations are said to be *super-critical* and the lack of a known bound for the $\dot{H}^{\frac{1}{2}}$ contributes in keeping the global well-posedness question for the initial value problem (1) still open.

In this talk we consider the periodic Navier-Stokes problem in (1) and in particular we address the question of long time existence of weak solutions for supercritical initial data both in $d = 2$, 3, see also [8]. For $d = 2$ we address uniqueness as well. Our goal is to show that by randomizing in an appropriate way the initial data in $H^{-\alpha}(\mathbb{T}^d), d = 2,3$ (for some $\alpha = \alpha(d) > 0$) which is below the critical threshold space $H^{s_c}(\mathbb{T}^d)$, one can construct a global in time weak solution to (1). Such solution is unique when $d = 2$. We note that similar well-posededness results were obtained for the super-critical wave equations by Burq and Tzvetkov in [1,2,3]. The approach of Burq and Tzvetkov was applied in the context of the Navier-Stokes in order to obtain local in time solutions to the corresponding integral equation for randomized initial data in $L^2(\mathbb{T}^3)$, as well as global in time solutions to the corresponding integral equation for randomized initial data that are small in $L^2(\mathbb{T}^3)$ by Zhang and Fang [9] and by Deng and Cui [4] Also in [5], Deng and Cui obtained local in time solution to the corresponding integral equation for randomized initial data in $H^s(\mathbb{T}^d)$, for $d = 2,3$ with $-1 < s < 0$. However our result is the first to offer a construction of a **global in time** weak solution to (1) for randomized initial data (without any smallness assumption) in negative Sobolev spaces $H^{-\alpha}(\mathbb{T}^d), d = 2,3$, for some $\alpha = \alpha(d) > 0$.

Roughly speaking the idea of the proof is the following: we start with a divergence free and mean zero initial data $\vec{f} \in (H^{-\alpha}(\mathbb{T}^d))^d, d = 2,3$ and suitably randomize it to obtain $\vec{f}^\omega$ which in particular preserves the divergence free condition. Then we seek a solution to the initial value problem (1) in the form $\vec{u} = e^{t\Delta}\vec{f}^\omega + \vec{w}$. In this way we single out the linear evolution $e^{t\Delta}\vec{f}^\omega$ and identify the difference equation that $\vec{w}$ should satisfy. At this point it becomes convenient to state an equivalence lemma between the initial value problem for the difference equation and the integral formulation of it. This equivalence is similar to Theorem 11.3 in [7], see also [6]. We will be using the integral equation formulation near time zero and the other one away from zero. The key point of this approach is the fact that although the initial data are in $H^{-\alpha}$ for some $\alpha > 0$, the heat flow of the randomized data gives almost surely improved L^p bounds). These bounds in turn yield improved nonlinear estimates arising in the analysis of the difference equation for $\vec{w}$ almost surely , and consequently a construction of a global weak solution to the difference equation is possible.

[1] Burq, N. and Tzvetkov, N. Random data Cauchy theory for supercritical wave equation I: Local theory, *Invent. math.* **173**, No. 3 (2008), 449–475.

[2] Burq, N. and Tzvetkov, N. Random data Cauchy theory for supercritical wave equation II: A global existence result, *Invent. math.* **173**, No. 3 (2008), 477–496.

[3] Burq, N. and Tzvetkov, N. Probabilistic well-posedness for the cubic wave equation, *Preprint arXiv:1103.2222v1*

[4] C. Deng and S. Cui, Random-data Cauchy problem for the Navier-Stokes equations on $\mathbb{T}^3$, *J. Differential Equations* **251**, No. 4-5 (2011), 902–917.

[5] C. Deng and S. Cui, Random-data Cauchy Problem for the Periodic Navier-Stokes Equations with Initial Data in Negative-order Sobolev Spaces, *Preprint arXiv:1103.6170v.1.*

[6] G. Furioli, P.G. Lemarié-Rieusset and E. Terraneo, Unicité dans $L^3(\mathbb{R}^3)$ et d'autres espaces fonctionnels limites pour Navier-Stokes [Uniqueness in $L^3(\mathbb{R}^3)$ and other functional limit spaces for Navier-Stokes equations], *Rev. Mat. Iberoamericana* **16**, No. 3 (2000), 605–667.

[7] P.G. Lemarié-Rieusset, Recent developments in the Navier-Stokes problem, Chapman & Hall/CRC *Reserach Notes in Mathematics* 431, 2002.

[8] T. Tao, A quantitative formulation of the global regularity problem for the periodic Navier-Stokes equation, *Dyn. Partial Differ. Equ.* **4** (2007), no. 4, 293–302.

[9] T. Zhang and D. Fang, Random data Cauchy theory for the incompressible three dimensional Navier-Stokes equations, *Proc. AMS* **139**, No. 8, (2011), 2827–2837.

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