Two uniqueness results for the two-dimensional continuity equation with velocity having L^1 or measure curl

Gianluca Crippa
Departement Mathematik und Informatik – Universität Basel
gianluca.crippa@unibas.ch
http://www.math.unibas.ch/crippa

In this seminar I will present two results regarding the uniqueness (and further properties) for the two-dimensional *continuity equation*

$$\begin{cases} u_t + \operatorname{div}(bu) = 0 \\ u(0,x) = \bar{u}(x) \end{cases}$$

and the *ordinary differential equation*

$$\begin{cases} \dfrac{\partial \Phi}{\partial t}(t,x) = b\big(t, \Phi(t,x)\big) \\ \\ \Phi(0,x) = x \end{cases}$$

in the case when the vector field

$$b(t,x) : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$$

is bounded, divergence free and satisfies additional conditions on its distributional curl:

$$\operatorname{curl} b = -\partial_{x_2} b^1 + \partial_{x_1} b^2 \,.$$

Such settings appear in a very natural way in various situations, for instance when considering two-dimensional incompressible fluids.

(1) *The case when b is time-independent and its curl is a (locally finite) measure (without any sign condition).* Uniqueness of bounded distributional solutions to the continuity equation follows from a series of papers by Alberti, Bianchini and myself. In such papers, we provide a characterization of two-dimensional bounded divergence free vector fields enjoying uniqueness, in terms of the so-called *weak Sard property.* The weak Sard property is a suitable measure theoretical version of the usual Sard property satisfied by "sufficiently differentiable" maps between Euclidean space. It is possible to prove that the weak Sard property is enjoyed by vector fields with measure curl.

(2) *The case when b is time-dependent and its curl belongs to $L^1(\mathbb{R}^2)$.* This case is covered by a joint result with Bouchut, extending previous works with De Lellis. This time the idea is to work at the level of the ordinary differential equation, deriving effective estimates of stability and compactness under suitable bounds on the velocity. The result with De Lellis was addressing the case of $W^{1,p}$ velocities, with $p > 1$, and the upgrade with Bouchut sets the case in which the derivative of the velocity is a singular integral of a summable function, including in particular the case of L^1 curl. This proves uniqueness, stability and compactness of Lagrangian solutions to the continuity equation.

In the seminar I will present the main steps in the proofs in the two cases described above. I will also sketch some possible improvements and extensions and describe some applications.

References

[1] G. Alberti, S. Bianchini and G. Crippa, A uniqueness result for the continuity equation in two dimensions, preprint (2011), to appear on *Journal of the European Mathematical Society (JEMS).*

[2] G. Alberti, S. Bianchini and G. Crippa, Structure of level sets and Sard-type properties of Lipschitz maps, preprint (2011), to appear on *Annali della Scuola Normale Superiore, Classe di Scienze.*

[3] G. Alberti, S. Bianchini and G. Crippa, On the L^p-differentiability of certain classes of functions, preprint (2012).

[4] F. Bouchut and G. Crippa, Lagrangian flows for vector fields with gradient given by a singular integral, in preparation.

[5] G. Crippa and C. De Lellis, Estimates and regularity results for the DiPerna-Lions flow, *J. Reine Angew. Math.*, **616** (2008), pp. 15-46.

Based on joint works with: Giovanni Alberti (*Università di Pisa*), Stefano Bianchini (*SISSA Trieste*), François Bouchut (*CNRS & Université Paris-Est-Marne-la-Vallée*) and Camillo De Lellis (*Universität Zürich*).