High Order Methods with Adaptive Mesh Refinement for the Solution of the Relativistic MHD equations

Olindo Zanotti

University of Trento

Collaborator: Michael Dumbser, University of Trento

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Outline

- Part I: Astrophysical motivations for considering Resistive Relativistic MHD
- Part II: The mathematics of Resistive Relativistic MHD: a hyperbolic system with stiff source terms
- Part III: A numerical scheme for RRMHD
  - High order reconstruction
  - Using local space-time Discontinuous Galerkin for treating stiffness in the source terms
  - Construction of 1-step time evolution schemes
  - Adaptive Mesh Refinement
- Part IV: High Lundquist number relativistic magnetic reconnection

Why Resistive MHD?

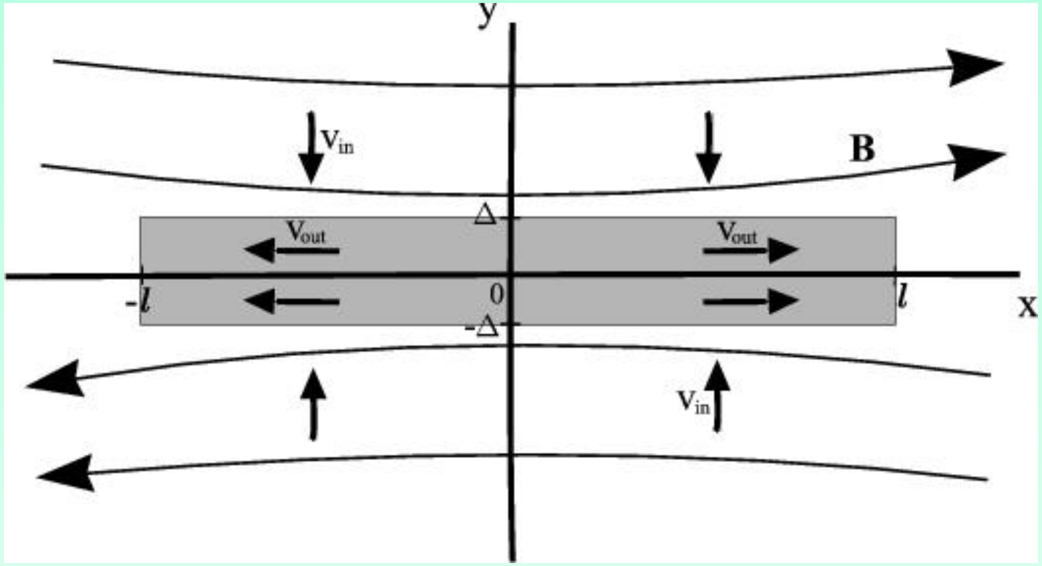
In most cases, the ideal MHD approximation of infinite conductivity is correct:

$$R_m = \frac{v_0 L}{\eta} >> 1, \qquad S = \frac{v_A L}{\eta} >> 1$$

magnetic Reynolds and Lundquist number

ratio between diffusion timescale and advection timescale

In some cases, however, such an approximation is completely wrong and resistivity must be taken into account. This is particularly the case when magnetic reconnection takes place



(Lyubarski 2005)

$$\partial_t \vec{B} - \vec{\nabla} \times (\vec{v} \times \vec{B}) - \eta \nabla^2 \vec{B} = 0$$

$$\vec{J} = \vec{\nabla} \times \vec{B}$$

Where relativistic magnetic reconnection?

✓ Giant flares in gamma-ray repeaters (Lyutikov 2003, 2006)

✓ Current sheets at the Y point in pulsars magnetospheres

✓ Dissipation of alternating fields at the termination shock of a relativistic pulsar wind (Petri & Lyubarski 2007)

✓ Gamma-ray burst jets, where particle acceleration induced by magnetic reconnection is supposed to take place (Drenkhahn & Spruit 2002 , McKinney & Uzdensky 2010)

✓Accretion disc coronae of AGN where magnetic loops emerge from the disc via buoyancy instability (di Matteo 1998, Jaroschek 2004)

The Relativistic plasma

- The energy-momentum tensor of a relativistic plasma is:

$$T^{\mu\nu} = T_m^{\mu\nu} + T_{em}^{\mu\nu} = (h\rho u^\mu u^\nu + pg^{\mu\nu}) + F^{\mu\lambda}F^\nu{}_\lambda - (F^{\kappa\lambda}F_{\kappa\lambda})g^{\mu\nu}/4$$

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2$$ assume 3+1 formalism

- The Faraday EM tensor satisfies Maxwell's equations

$$\nabla_\mu F^{\mu\nu} = -J^\nu \longrightarrow \vec{\nabla}\cdot\vec{E} = 4\pi\rho, \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}$$

$$\nabla_\mu F^{*\mu\nu} = 0 \longrightarrow \vec{\nabla}\cdot\vec{B} = 0, \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\left(F^{*\mu\nu} = \varepsilon^{\mu\nu\kappa\lambda}F_{\kappa\lambda}/2\right)$$

- Unlike ideal MHD, the current is not a derived quantity, and it is provided after specifying Ohm's law

$$\vec{J} = \rho_c\vec{v} + \sigma\Gamma[\vec{E} + \vec{v}\times\vec{B} - (\vec{E}\cdot\vec{v})\vec{v}]$$

- In ideal MHD one simply has $\vec{E} = -\vec{v}\times\vec{B}$

Ohm's law

The electric conductivity is actually a tensor

$$J^\alpha = \rho_e u^\alpha + \sigma^{\alpha\beta} e_\beta$$

$$\sigma^{\alpha\beta} = \sigma(g^{\alpha\beta} + \xi b^\alpha b^\beta + \varsigma \varepsilon^{\alpha\beta\gamma\delta} u_\gamma b_\delta)$$

$$\xi = \left(\frac{e\tau}{m}\right)^2, \quad \varsigma = \left(\frac{e\tau}{m}\right) >> 1, \quad \tau \equiv \text{collision time}$$

$$\Longrightarrow \quad \vec{J} = q\vec{v} + \sigma\Gamma[\vec{E} + \vec{v}\times\vec{B} - (\vec{E}\cdot\vec{v})\vec{v}] + \sigma\Gamma\xi^2(\vec{E}\cdot\vec{B})[\vec{B} - \vec{v}\times\vec{E} - (\vec{B}\cdot\vec{v})\vec{v}]$$

RRMHD	ideal RMHD
$\partial_t D + \partial_i (D v^i) = 0$	$\partial_t D + \partial_i (D v^i) = 0$
$\partial_t S_j + \partial_i Z^i_j = 0$	$\partial_t S_j + \partial_i Z^i_j = 0$
$\partial_t U + \partial_i S^i = 0$	$\partial_t U + \partial_i S^i = 0$
$\partial_t B^i + e^{ijk} \partial_j E_k + \partial_i \Phi = 0$	$\partial_t B^i + e^{ijk} \partial_j E_k + \partial_i \Phi = 0$
$\partial_t E^i - e^{ijk} \partial_j B_k + \partial_i \Psi = -J^i$	$E^i = -e^{ijk} v_j B_k$
$\partial_t \Phi + \partial_i B^i = -\kappa \Phi$	$\partial_t \Phi + \partial_i B^i = -\kappa \Phi$
$\partial_t \Psi + \partial_i E^i = \rho_c - \kappa \Psi$	
$\partial_t q + \partial_i J^i = 0$	

Hyperbolic Divergence Cleaning approach of Dedner et al. JCP, 175, 645, 2002

- where

$$D = \rho \Gamma$$

$$S^i = h \rho \Gamma^2 v^i + \varepsilon^{ijk} E_j B_k$$

$$U = h \rho \Gamma^2 - p + B^2/2 + E^2/2$$

$$Z^i_j = h \rho \Gamma^2 v^i v_j - B^i B_j - E^i E_j + \left(p + B^2/2 + E^2/2\right) \delta^i_j$$

Of course all of this holds also in curved spacetime

$$\partial_t(\sqrt{\gamma}\mathbf{F}^0) + \partial_i(\sqrt{\gamma}\mathbf{F}^i) = \mathbf{s}\ ,$$

$$\mathbf{F}^0 = \begin{bmatrix} D \\ S_j \\ U \end{bmatrix}, \quad \mathbf{F}^i = \begin{bmatrix} \alpha v^i D - \beta^i D \\ \alpha W^i_j - \beta^i S_j \\ \alpha S^i - \beta^i U \end{bmatrix}\ ,$$

$$\mathbf{s} = \gamma^{1/2} \begin{bmatrix} 0 \\ \frac{1}{2}\alpha W^{ik}\partial_j \gamma_{ik} + S_i \partial_j \beta^i - U \partial_j \alpha \\ \alpha W^{ik} K_{ik} - S^j \partial_j \alpha \end{bmatrix}\ .$$

$$D = \rho W$$

$$\vec{S} = h\rho W^2 \vec{v} + \vec{E} \times \vec{B}$$

$$U = h\rho W^2 - p + \frac{1}{2}(E^2 + B^2)$$

+ evolution equations for the electromagnetic field

$$\partial_t(\sqrt{\gamma}E^i) - \partial_k(\beta^k \sqrt{\gamma}E^i) - [ijk]\partial_j(\alpha B_k) + \partial_j(\alpha\sqrt{\gamma}\gamma^{ij}\Psi) = \Psi\partial_j(\alpha\sqrt{\gamma}\gamma^{ij}) - \sqrt{\gamma}E^k\partial_k\beta^i - \alpha\sqrt{\gamma}J^i$$

$$\partial_t(\sqrt{\gamma}B^i) - \partial_k(\beta^k \sqrt{\gamma}B^i) + [ijk]\partial_j(\alpha E_k) + \partial_j(\alpha\sqrt{\gamma}\gamma^{ij}\Phi) = \Phi\partial_j(\alpha\sqrt{\gamma}\gamma^{ij}) - \sqrt{\gamma}B^k\partial_k\beta^i$$

$$\partial_t\left(\frac{\sqrt{\gamma}}{\alpha}\Psi\right) + \partial_i(\sqrt{\gamma}E^i) - \partial_i\left(\frac{\sqrt{\gamma}}{\alpha}\beta^i\Psi\right) = -\kappa\sqrt{\gamma}\Psi - \Psi\partial_i\left(\frac{\sqrt{\gamma}}{\alpha}\beta^i\right) + \sqrt{\gamma}\rho_c$$

$$\partial_t\left(\frac{\sqrt{\gamma}}{\alpha}\Phi\right) + \partial_i(\sqrt{\gamma}B^i) - \partial_i\left(\frac{\sqrt{\gamma}}{\alpha}\beta^i\Phi\right) = -\kappa\sqrt{\gamma}\Phi - \Phi\partial_i\left(\frac{\sqrt{\gamma}}{\alpha}\beta^i\right)$$

$$\partial_t(\sqrt{\gamma}\rho_c) + \partial_i(\alpha\sqrt{\gamma}J^i - \beta^i\sqrt{\gamma}\rho_c) = 0\ .$$

RRMHD: a system with potential stiff source terms!

$$\partial_t u + \partial_x f(u) = \frac{1}{\varepsilon} s(u) \qquad \varepsilon = \frac{\tau_a}{\tau_d}$$

τ_a ← Timescale of advection

τ_d ← Timescale of dissipative process

$$\partial_t \vec{E} - \vec{\nabla} \times \vec{B} + \vec{\nabla}\Psi = -\vec{J}$$

$$\vec{J} = q\vec{v} + \sigma\Gamma[\vec{E} + \vec{v} \times \vec{B} - (\vec{E} \cdot \vec{v})\vec{v}]$$

In the limit of $\sigma \longrightarrow \infty$ the source becomes stiff

In the limit of $\sigma \longrightarrow 0$ the electric field evolves with a hyperbolic equation, not a parabolic one like in classical MHD

$$\partial_t \vec{B} - \vec{\nabla} \times (\vec{v} \times \vec{B}) - \frac{1}{\sigma} \nabla^2 \vec{B} = 0$$

Traditional approaches:

1. Strang-splitting, i.e. operator splitting, (Komissarov 2007, Zenitani et al. 2009)
2. Implicit-Explicit Runge Kutta (Palenzuela et al 2009)

Computational strategy

1. Apply a reconstruction operator to the Discontinuous Galerkin scheme at the beginning of each time step
2. Provide the time evolution of the reconstructed polynomials by solving the local space-time Discontinuous Galerkin predictor step
3. Solve the Riemann problem by some flux formula
4. Perform a one-step time update from time level n to n+1 with quadrature (or quadrature free) formulation

Reconstruction: the $P_N P_M$ scheme (I)

Set up a triangulation of space:

$$Q_\Omega = \bigcup_{m=1}^{N_E} Q^{(m)}$$

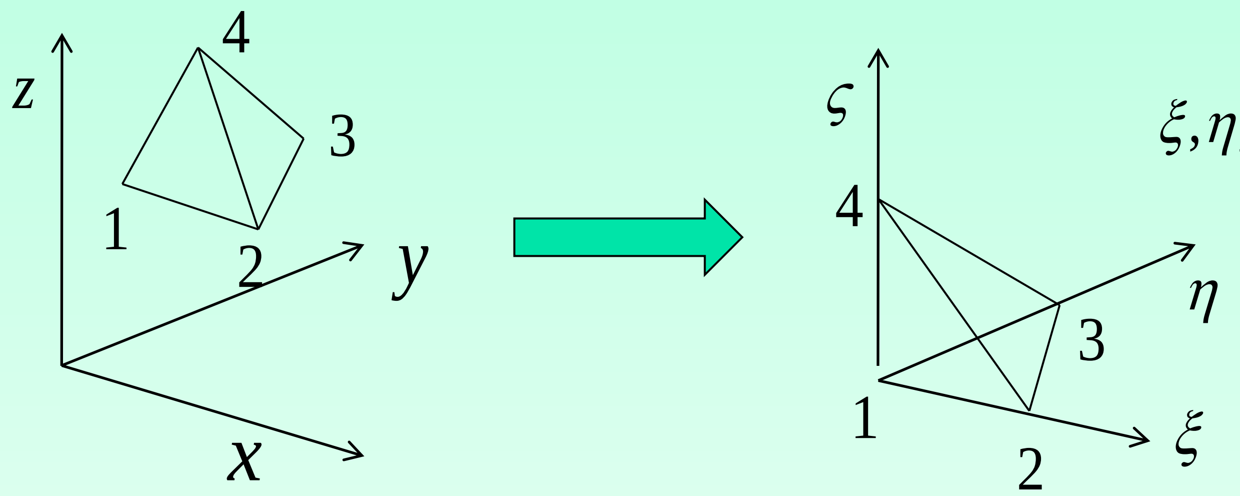
$$\vec{x} = \vec{x}(Q^{(m)}, \vec{\xi})$$

Physical coordinates — reference coordinates

$$x = X_1^{(m)} + (X_2^{(m)} - X_1^{(m)})\xi + (X_3^{(m)} - X_1^{(m)})\eta + (X_4^{(m)} - X_1^{(m)})\varsigma$$

$$y = Y_1^{(m)} + (Y_2^{(m)} - Y_1^{(m)})\xi + (Y_3^{(m)} - Y_1^{(m)})\eta + (Y_4^{(m)} - Y_1^{(m)})\varsigma$$

$$z = Z_1^{(m)} + (Z_2^{(m)} - Z_1^{(m)})\xi + (Z_3^{(m)} - Z_1^{(m)})\eta + (Z_4^{(m)} - Z_1^{(m)})\varsigma$$



$$\xi, \eta, \varsigma \in [0,1]$$

Reconstruction: the $P_N P_M$ scheme (II)

At time t^n the vector of conserved quantities is represented by piecewise polynomials of degree N

$$u_h(\vec{\xi}, t^n) = \sum_{l=1}^{L_N} \Phi_l(\vec{\xi}) \hat{u}_l^n(t^n)$$

Traditional representation   Galerkin

From it, we reconstruct over polynomials of degree M : $M \geq N$

$$w_h(\vec{\xi}, t^n) = \sum_{l=1}^{L_M} \Psi_l(\vec{\xi}) \hat{w}_l^n(t^n)$$

$$L_M = \frac{1}{d!}(M+1)\cdot \ldots (M+d)$$

Number of degrees of freedom

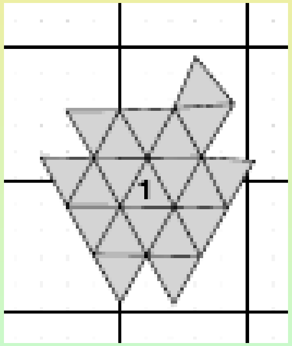
Choose Ψ_l among Legendre polynomials

Ψ_l and Φ_l coincide up to order N, while $\int_{Q_i} \Psi_m \Psi_n d\vec{\xi} = 0, \forall m, n, m \neq n$

Reconstruction: the $P_N P_M$ scheme (III)

Choose a stencil for the reconstruction:

$$S^{(m)} = \bigcup_{k=1}^{n_e} T^{(j(k))}$$



The number of cells in the stencil is taken as

$$n > L_M / L_N$$

$$n = \text{CEILING}(L_M / L_N)$$

The reconstruction is obtained through L2 projection of the unknown polynomials

Weak form of the identity of the reconstructed solution of degree M and the original numerical solution of degree N:

$$\int_{Q_i} \Phi_k w_h d\vec{\xi} = \int_{Q_i} \Phi_k u_h d\vec{\xi} \quad \forall Q_i \in S^{(m)}$$

As many equations as the element of the stencil. For stability reasons, the number of elements in the stencil is chosen larger than n

Computed with Gaussian quadrature

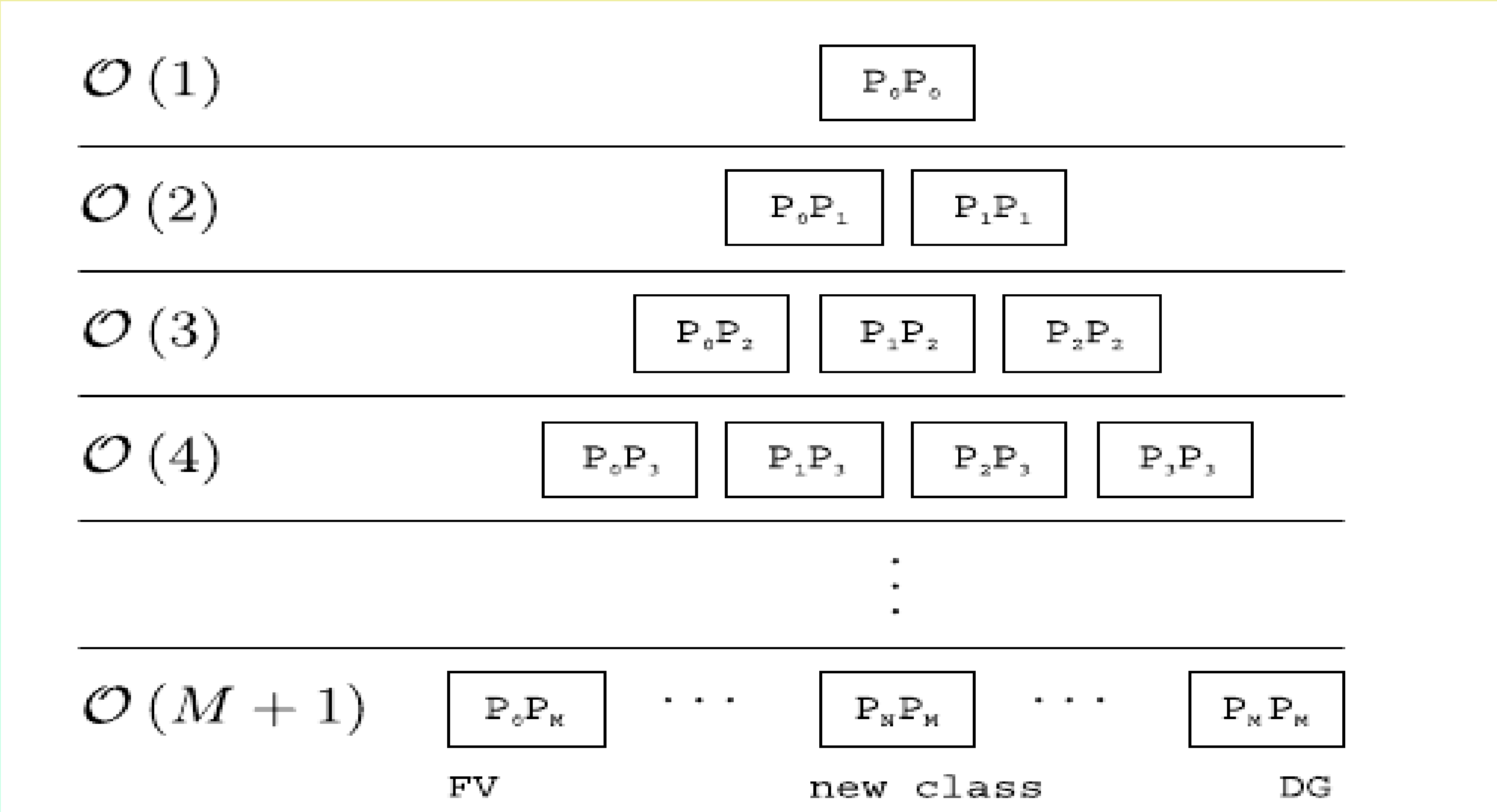
Computed analytically

$$\widehat{w}_l^{(m)} = \widehat{u}_l^{(m)} \quad \text{for} \quad 1 \le l \le L_N$$

Solved using the constrained least squares technique

To recap:

Reconstruction is applied to polynomials of degree N and generates polynomials of degree M



Dumbser et al (2008) JCP, 227, 8209

$N = 0 \quad \Rightarrow \quad \text{classical FV}$

$N = M \quad \Rightarrow \quad \text{classical Galerkin}$

$M > N \quad \Rightarrow \quad \text{hybrid class}$

Time evolution of the reconstructed polynomial

Use the local space-time Discontinuous Galerkin approach

$$\frac{\partial W}{\partial t} + \nabla \cdot F(W) = S(W) \quad \Longrightarrow \quad \frac{\partial W}{\partial \tau} + \nabla_{\xi} \cdot F^{*}(W) = S^{*}$$

$$F^{*} = \Delta t F(W) J^{T}, \qquad S^{*} = \Delta t S(W), \qquad J = \frac{\partial \vec{\xi}}{\partial \vec{x}}$$

Multiply by the space-time test function $\theta_k = \theta_k(\vec{\xi}, \tau)$ and integrate over the space-time reference control volume

$$\left\langle \theta_k, \frac{\partial W_h}{\partial \tau} \right\rangle + \left\langle \theta_k, \nabla_{\xi} \cdot F_h^{*}(W_h) \right\rangle = \left\langle \theta_k, S^{*} \right\rangle$$

$$\langle f, g \rangle_Q = \int_0^1 \int_Q f(\xi, \tau) g(\xi, \tau) d\vec{\xi} d\tau$$

$$[f, g]_Q^{\tau} = \int_Q f(\xi, \tau) g(\xi, \tau) d\vec{\xi}$$

$$W_h = \sum_l \theta_l \hat{W}_l$$

$$F^{*}{}_h = \sum_l \theta_l \hat{F}_l^{*} = \sum_l \theta_l F^{*}(\hat{W}_l)$$

$$S_h^{*} = \sum_l \theta_l \hat{S}_l^{*} = \sum_l \theta_l S^{*}(\hat{W}_l)$$

….after integration by parts in time:

$$\left[\theta_k, W_h\right]^1 - \left[\theta_k, w_h\right]^0 - \left\langle W_h, \frac{\partial \theta_k}{\partial \tau}\right\rangle + \left\langle \theta_k, \nabla_\xi \cdot F_h^*(W_h)\right\rangle = \left\langle \theta_k, S^*\right\rangle$$

This polynomial is obtained by the high order $P_N P_M$ reconstruction. At this point we use the expansions over the polynomial basis

$$\underbrace{\left(\left[\theta_k, \theta_l\right]^1 - \left\langle \theta_l, \frac{\partial \theta_k}{\partial \tau}\right\rangle\right)}_{K_1} \hat{W}_l - \underbrace{\left[\theta_k, \Psi_l\right]^0}_{F_0} \hat{w}_l^n + \underbrace{\left\langle \theta_k, \nabla_\xi \theta_l\right\rangle}_{K_\xi} \hat{F}_l^* = \underbrace{\left\langle \theta_k, \theta_l\right\rangle}_{M} \hat{S}_l^*$$

The unknown

$$\Rightarrow \quad \boxed{\hat{W}_l^{i+1} - (K_1)^{-1} M \hat{S}_l^{*,i+1} = (K_1)^{-1} F_0 \hat{w}_l^n - (K_1)^{-1} K_\xi \hat{F}_l^{*,i}}$$

….the source term is taken implicitly as:

$$\hat{S}_l^{*,i+1} \approx \hat{S}_l^{*,i} + \Delta t \frac{\partial S}{\partial W}\left(\hat{W}_l^{i+1} - \hat{W}_l^{i}\right)$$

$$\frac{\partial S}{\partial W} = \frac{\partial S}{\partial V}\left(\frac{\partial W}{\partial V}\right)^{-1}$$

In practice, the relevant source term will involve the derivative of the electric current with respect to the primitive variables!

We solve the algebraic equation by first looking for a first guess at first order:

$$K_1 = 1$$

$$K_\xi = 0$$

$$M = 1$$

$$\overline{W} - S^*(\overline{W}) = \hat{w}_l^n$$

The resulting cell average $\overline{W}$ is used as initial guess in the full equation

$$\hat{W}_l^{i+1} - (K_1)^{-1} M \hat{S}_l^{*,i+1} = (K_1)^{-1} F_0 \hat{w}_l^n - (K_1)^{-1} K_\xi \hat{F}_l^{*,i}$$

Corrector: one step time update

Once we have $W_h(\xi,\tau) = \sum_l \theta_l(\xi,\tau)\hat{W}_l$

we update in time through

$$\overline{W}_i^{\,n+1} = \overline{W}_i^{\,n} - \frac{\Delta t}{\Delta x}\left(f_{i+1/2} - f_{i-1/2}\right) + \Delta t \overline{S}_i$$

One step time update!

$$\overline{W}_i^{\,n} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} W(x,t^n)\,dx$$

with

$$f_{i+1/2} = \int_0^1 f_h(W_i(1,\tau), W_{i+1}(0,\tau))\,d\tau$$

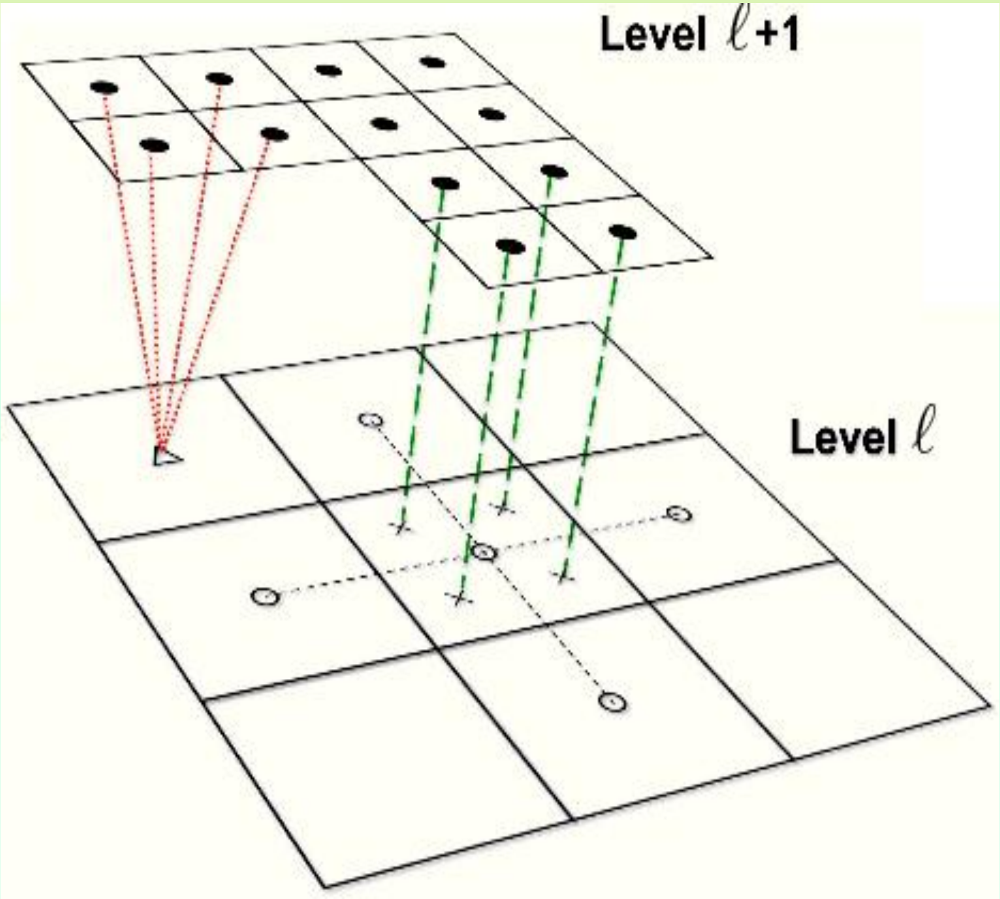
$$\overline{S}_i = \int_0^1\int_0^1 S(W_i(\xi,\tau))\,d\xi\,d\tau$$

← Computed with Gaussian quadrature

Adaptive Mesh Refinement: I

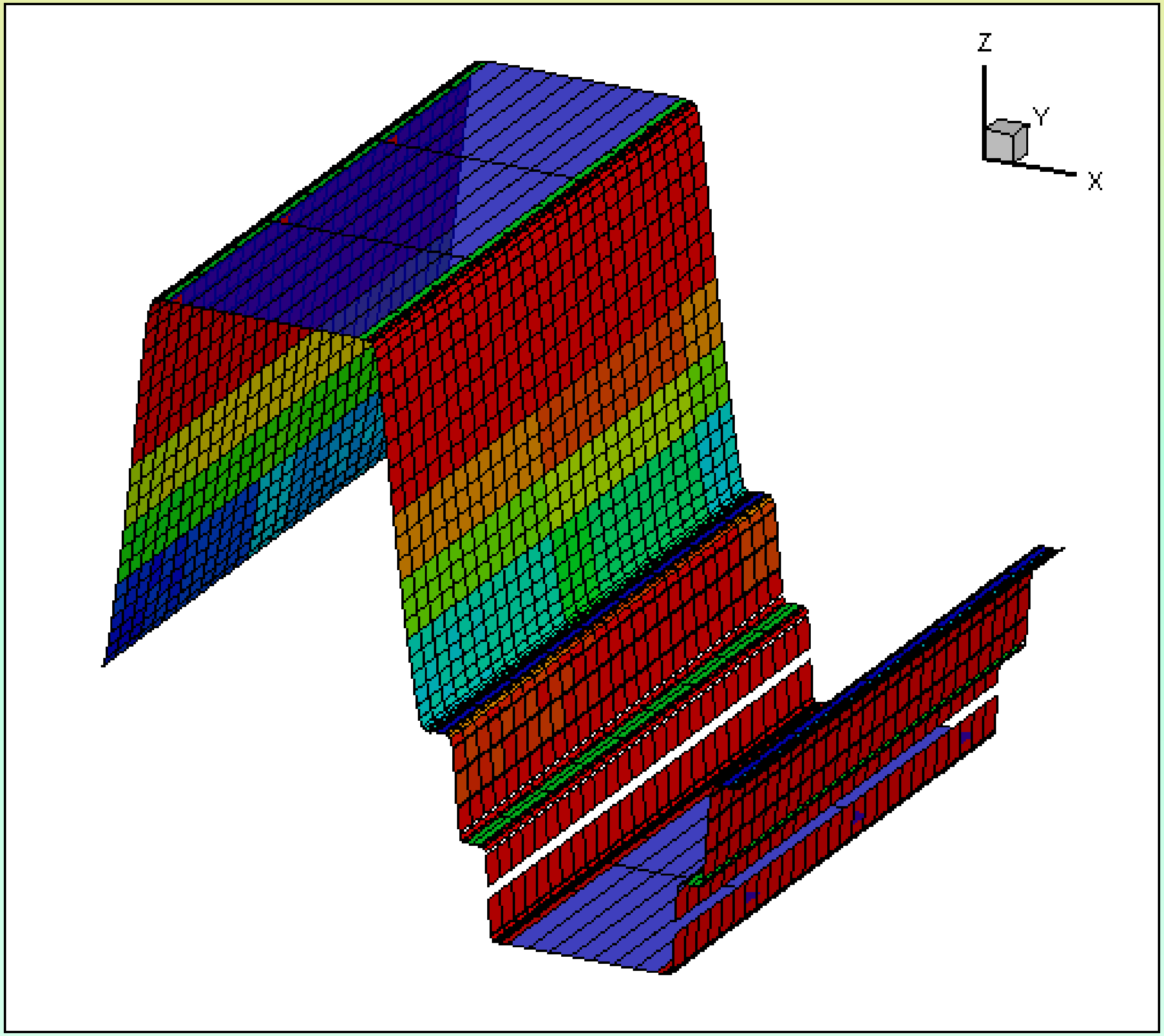
We followed the standard approach by Berger-Oliger-Colella, adopting a set of hierarchical RECTANGULAR grids.

```
CALL ComputeFluxes(reflev)
CALL MPIExchange_duh
CALL Update(reflev)
       !
CALL EvolveVirtual(reflev)
CALL Average(reflev)
       !
IF(reflev.EQ.0) THEN
    CALL ComputeTimeStep(amax)
    DO ireflev = 1, CurrentRefLevel
         CALL TimeEvolution(ireflev)
         CALL Project(ireflev)
         CALL MPIExchange_qh
    ENDDO
ENDIF
```



Mignone et al. ApJ (2012)

Adaptive Mesh Refinement: II



Relativistic magnetic reconnection

Initial conditions (relativistic Harris model)

$$\rho_0 = p_0 = 1$$

$$\sigma_m = \frac{B_0^2}{2\rho_0}$$

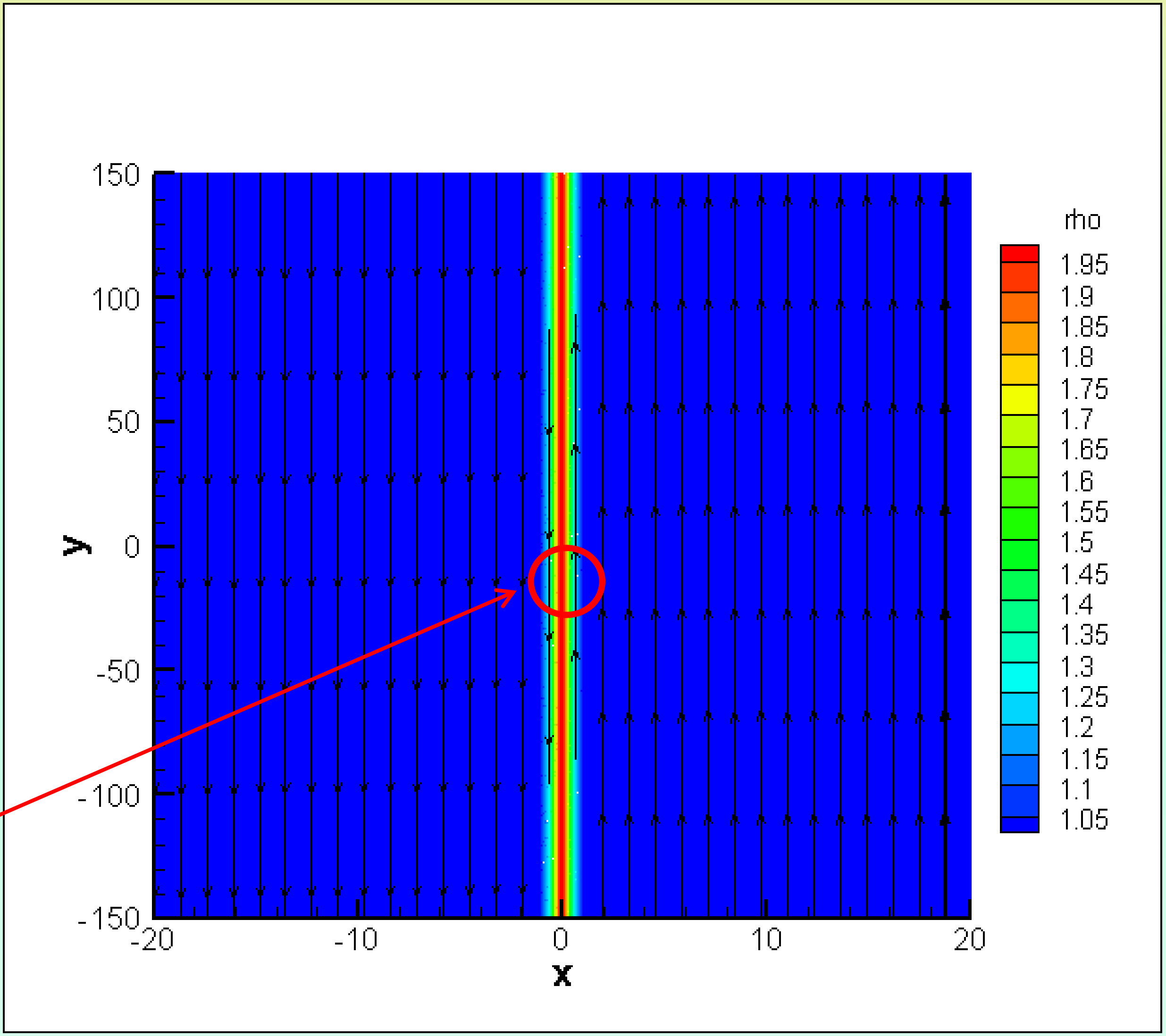
$$\rho = \rho_0 + \sigma_m \frac{1}{\cosh^2(2x)}$$

$$p = p_0 + \sigma_m \frac{1}{\cosh^2(2x)}$$

$$B_y = B_0 \tanh(2x)$$

$$v_x = v_y = 0$$

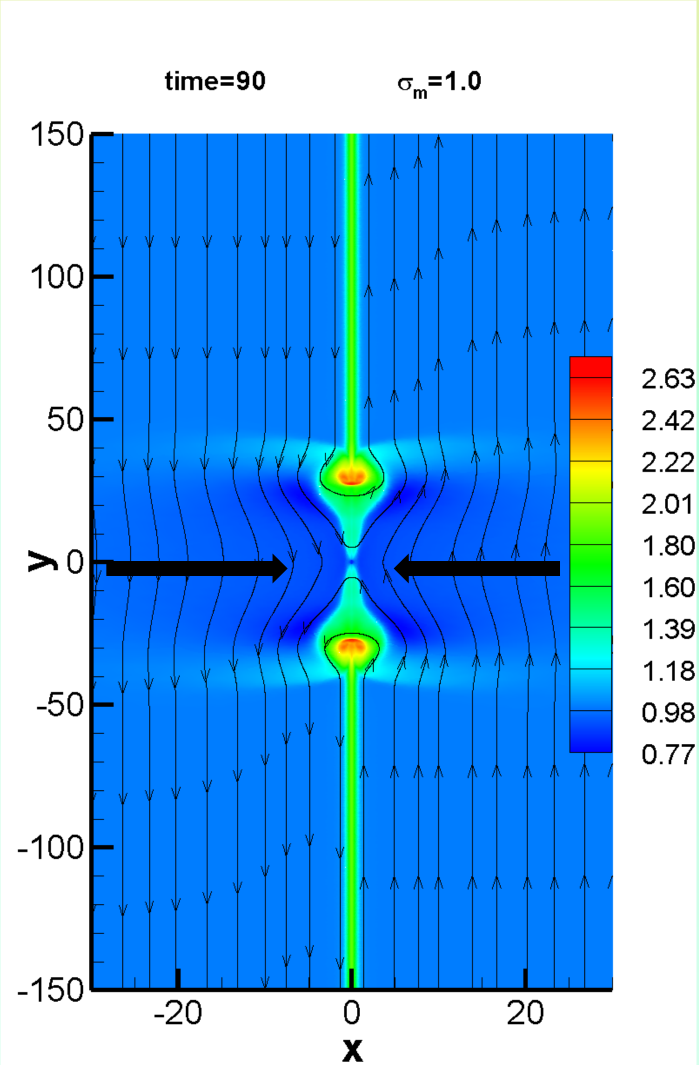
Anomalous resistivity

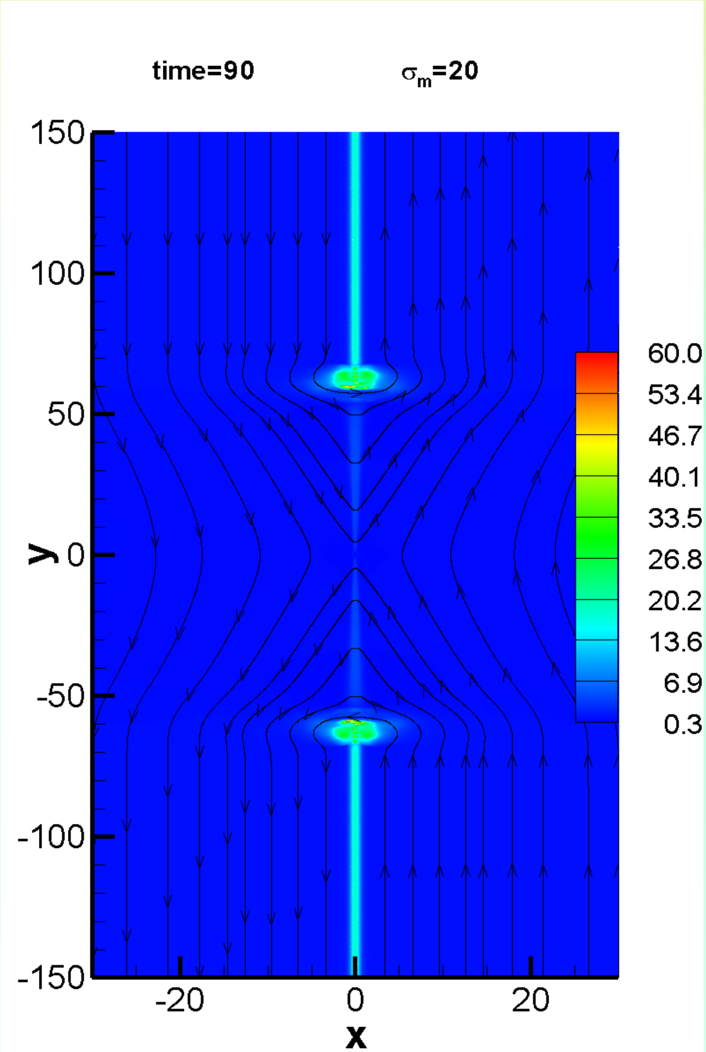


Zanotti & Dumbser MNRAS (2012)

Dynamical evolution

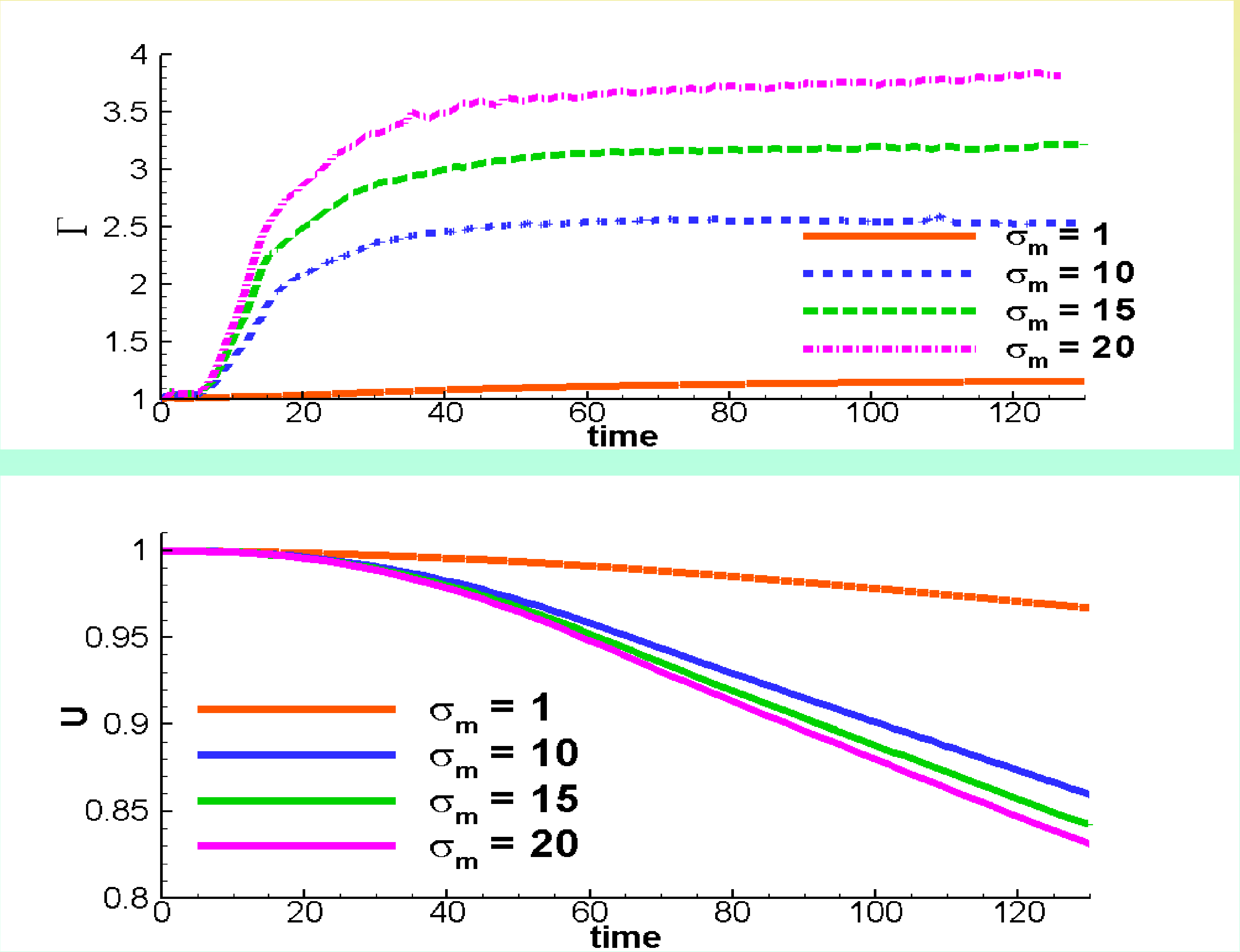
Two blobs of matter are ejected along the y direction while the magnetic field dissipates in a X type topology



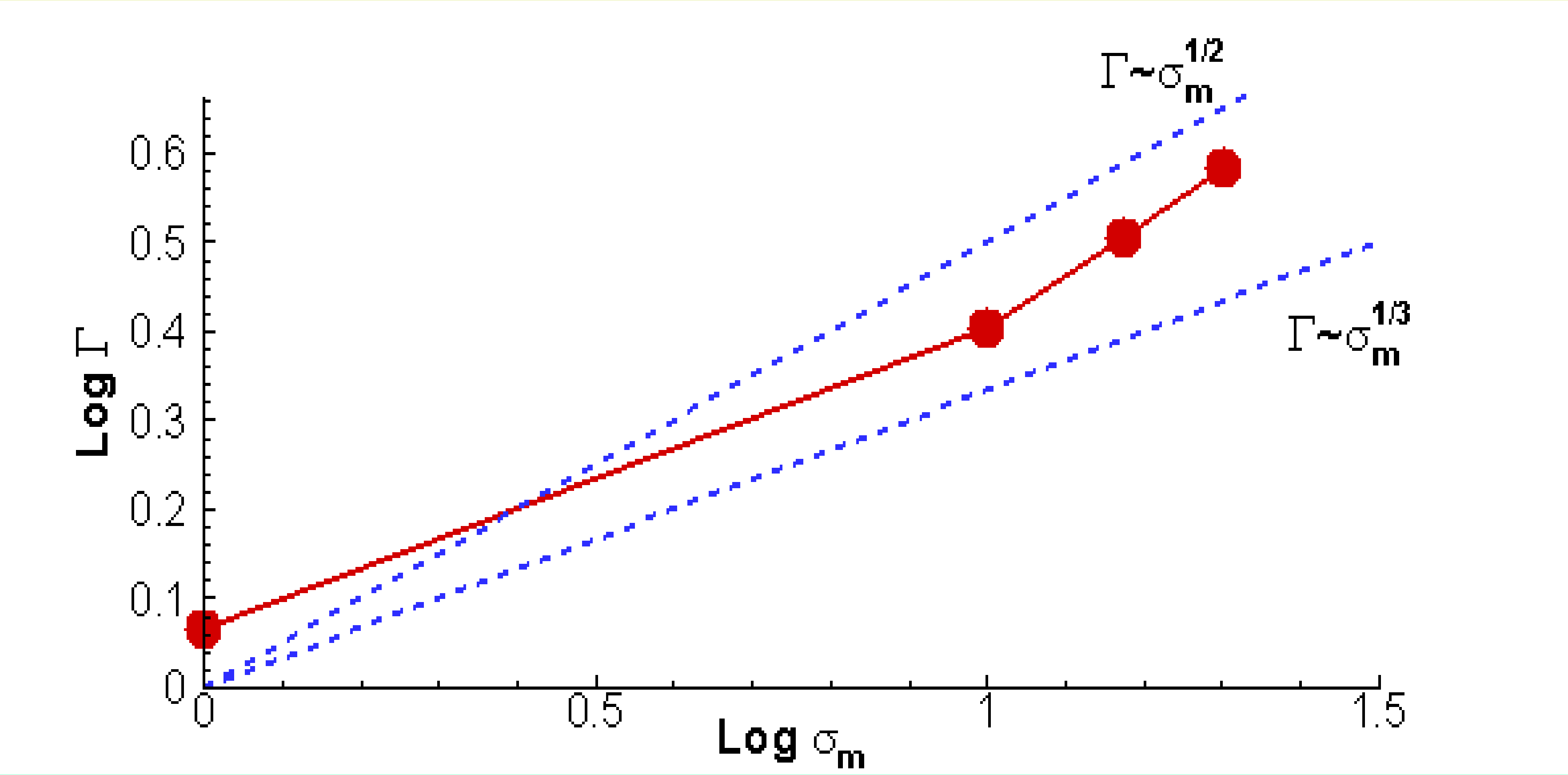


Zanotti & Dumbser MNRAS (2012)

Plasma acceleration



Plasma acceleration



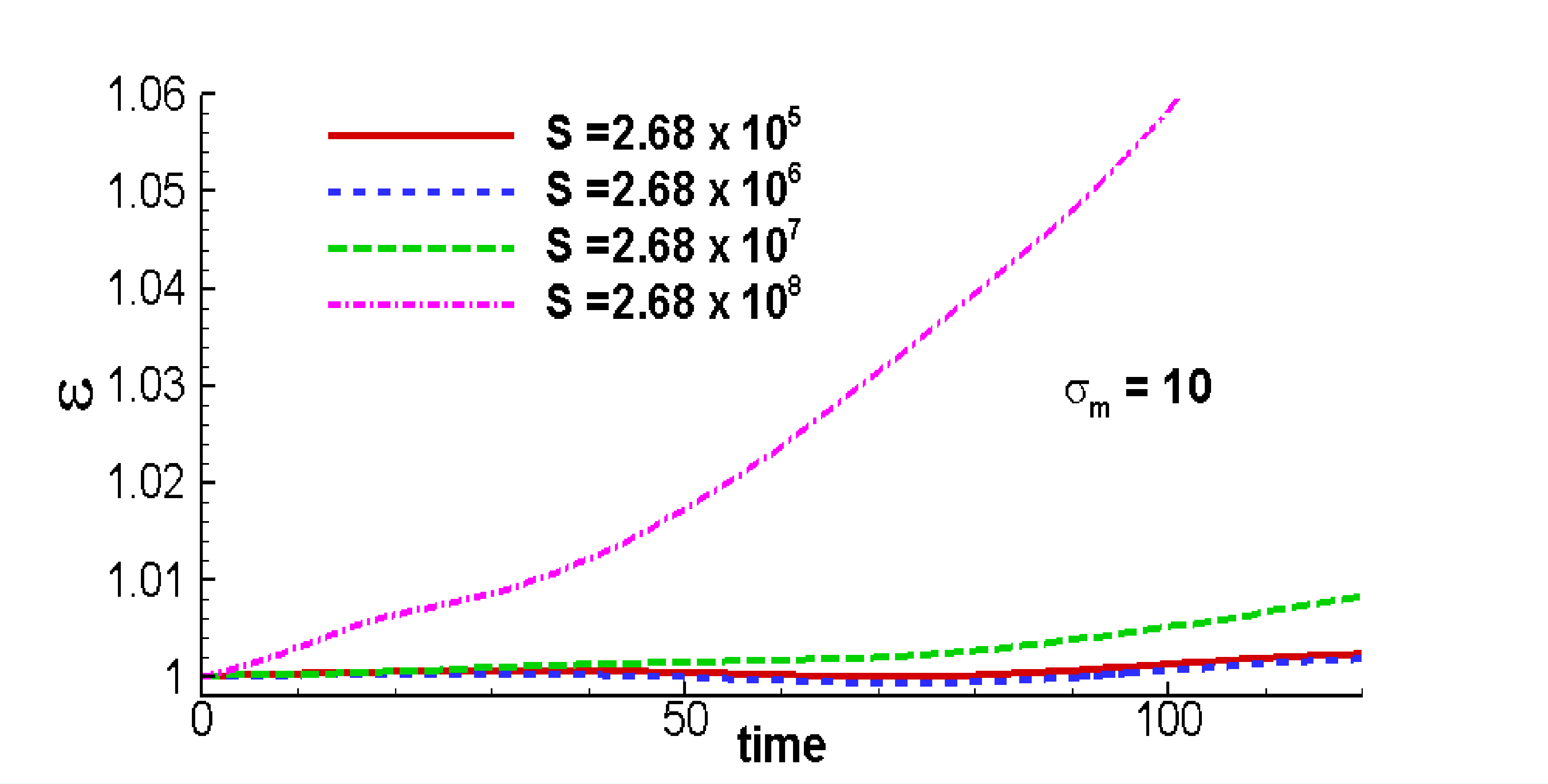
Zanotti & Dumbser MNRAS (2012)

At high magnetizations the dependence is in agreement with the theoretical prediction:

$$\Gamma \propto \sigma_m^{1/2}$$

High Lundquist number magnetic reconnection

What happens when the background resistivity is decreased? (the resistivity jump between the background and the anomalous spot is increased?)



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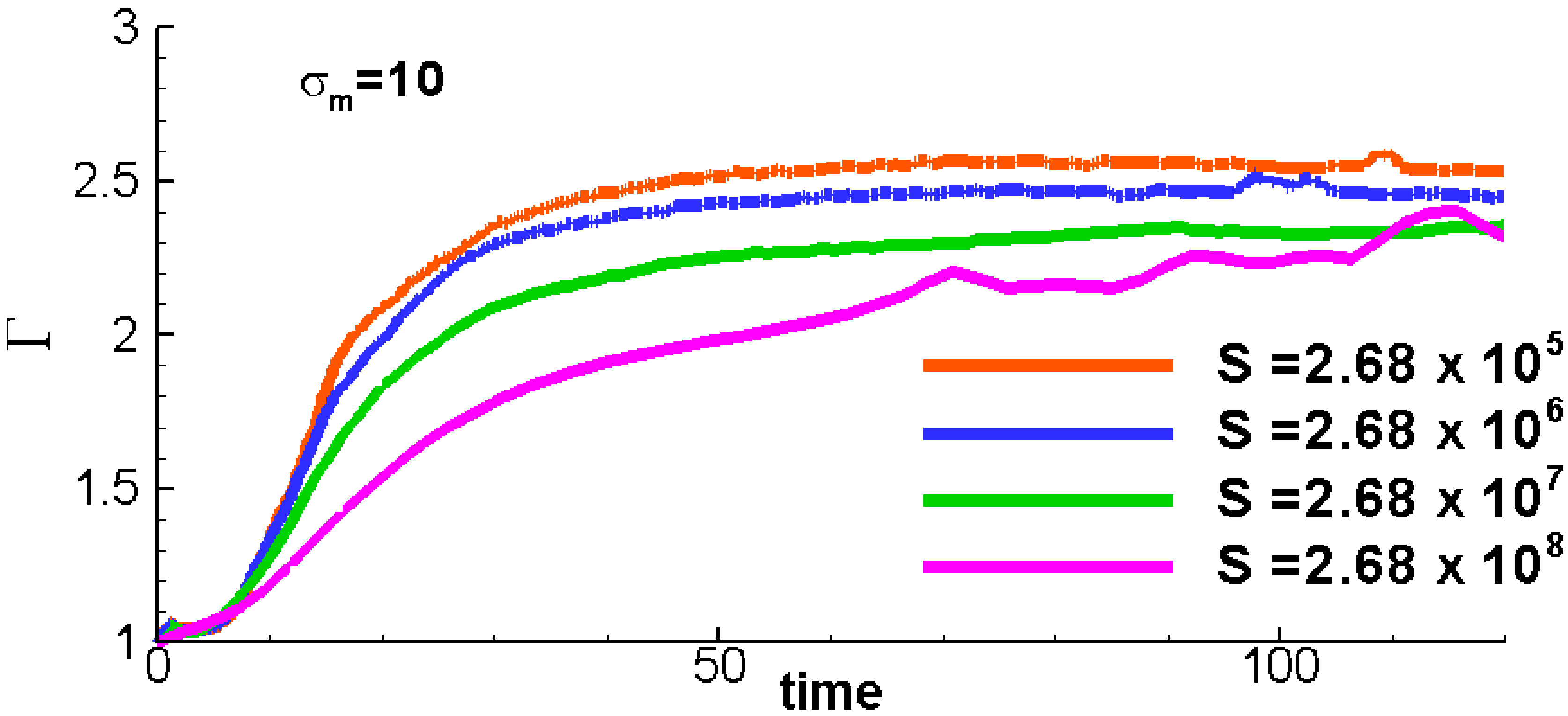

A tearing instability is produced…..

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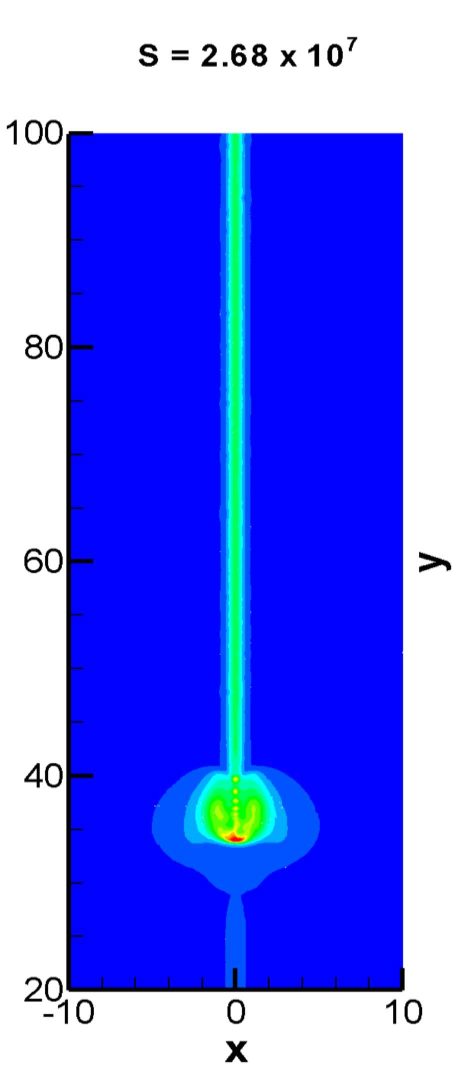
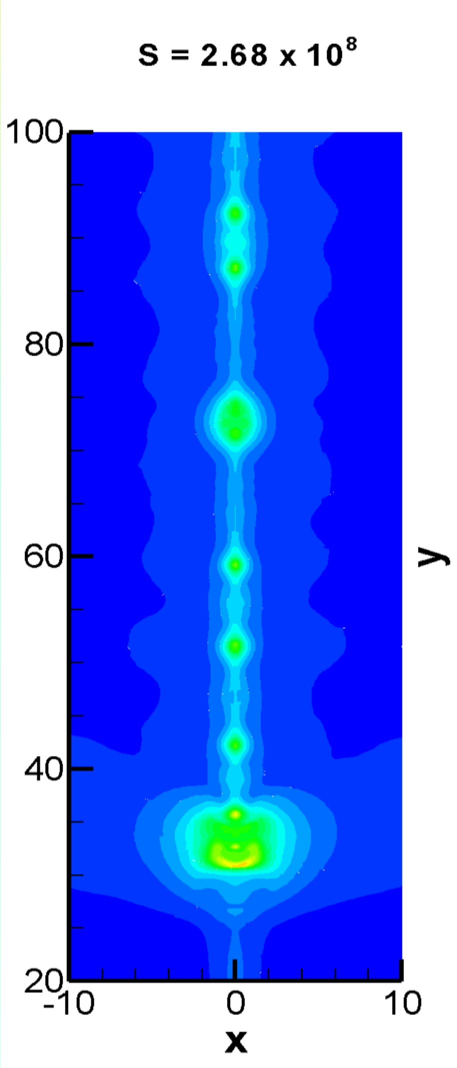
High Lundquist number magnetic reconnection





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Relativistic regime:

$$S > S_c \sim 10^8$$

Newtonian regime (Samtaney et al 2009):

$$S > S_c \sim 10^4$$

Conclusions

Future directions: inclusion of more physical Ohm's laws and generalization to curved spacetimes in general relativity.

- This is a promising period for numerical relativistic MHD. Having included resistivity allows for a larger class of problems to be studied.
  - Disc accretion onto Black Holes
  - Modelling of current sheets around rotating neutron stars

- References:

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