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Mather measures in semiclassical Analysis

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The Mather minimal measures

Definition

Tonelli-Lagrangians, i.e. $L : \mathbb{T}^n_x \times \mathbb{R}^n_\xi \to \mathbb{R}$ is convex in the fibers and superlinear above every compact subset of $\mathbb{T}^n_x$.

Definition

The P-Action minimizing measures, i.e. $\forall\ P \in \mathbb{R}^n$ the compactly supported Borel probability measures $d\mu_P$ on $\mathbb{T}^n \times \mathbb{R}^n$ invariant under Lagrangian flow and satisfing the following variational problem

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} [\, L(x,\xi) - P \cdot \xi \,]\, d\mu_P(x,\xi) = \inf_{d\mu} \int_{\mathbb{T}^n \times \mathbb{R}^n} [\, L(x,\xi) - P \cdot \xi \,]\, d\mu(x,\xi)$$

where the infimum is taken over all invariant compactly supported Borel probability measures $d\mu$.

The Weyl quantization on the Torus

The class of Hörmander's symbols $b \in S^m(\mathbb{T}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, consisting of those functions in $C^\infty(\mathbb{T}^n \times \mathbb{R}^n)$ which are 2π-periodic in x and for which $\forall\ \alpha, \beta \in \mathbb{Z}_+^n\ \exists\ C_{\alpha\beta} > 0$ s.t.

$$|\partial_\eta^\alpha \partial_x^\beta b(x,\eta)| \leq C_{\alpha\beta} \langle \eta \rangle^{m-|\alpha|}$$

where $\langle \eta \rangle := (1 + |\eta|^2)^{1/2}$.

Definition

The Weyl quantization on the torus is given by

$$\mathrm{Op}_\hbar^w(b)\psi(x) = (2\pi)^{-n} \sum_{\eta \in \mathbb{Z}^n} \int_{\mathbb{T}^n} e^{i(x-y)\cdot\eta} b(y, \hbar\eta/2)\psi(2y - x)dy$$

where $\psi \in C^\infty(\mathbb{T}^n; \mathbb{C})$.

The Weyl quantization on the Torus

Definition

The Wigner distribution

$$\langle \psi, \mathrm{Op}_\hbar^w(b)\psi\rangle_{L^2(\mathbb{T}^n)} = \sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} b(x,\eta) W_\hbar\psi(x,\eta)\,dx$$

The Wigner transform

$$W_\hbar\psi(x,\eta) = (2\pi)^{-n} \int_{\mathbb{T}^n} e^{2\frac{i}{\hbar} z\cdot\eta} \psi(x-z)\overline{\psi}(x+z)\,dz$$

Wigner measures on the Torus

Let $\psi_\hbar \in C^\infty(\mathbb{T}^n)$ a family of wave functions with $0 < \hbar \leq 1$.
The classical (or semiclassical) limit of the Wigner Transform

$$W_\hbar \psi_\hbar(x, \eta) \rightharpoonup dw(x, \eta) \qquad (wm)$$

as $\hbar \to 0^+$, in the weak-$\star$ topology defined on Borel measures and possibly passing through subsequences.

Definition

If the asymptotics (wm) works, then $dw(x, \eta)$ is called Wigner measure.

On the link between Wigner and Mather measures

- We look for a class of Wigner measures such that

$$dw = \mathcal{L}_\star d\mu_P$$

- Select wave functions $\psi_\hbar \in C^\infty(\mathbb{T}^n; \mathbb{C})$ in a constructive way

$\Rightarrow$ A class of Mather measures comes from the semiclassical limit of wave functions, not directly by the study of the classical system.

$\Rightarrow$ Remind that any $\mathcal{L}_\star d\mu_P$ is invariant under Hamiltonian flow, hence stationary solution of the Liouville equation in the phase space.

$\Rightarrow$ Remind that Liouville equation naturally arises in semiclassical Analysis (Lions-Paul and many subsequent others ...).

Why invariant measures in semiclassical Analysis are important?

Take Schrödinger equation

$$\begin{aligned} i\hbar\partial_t\psi(t,x) &= -\frac{\hbar^2}{2m}\Delta_x\psi(t,x) + V(x)\psi(t,x) \\ \psi(0,x) &= \psi_\hbar(x). \end{aligned}$$

and look at

$$W_\hbar\psi_\hbar(t,x,\eta) \rightharpoonup dw(t,x,\eta) \quad \text{as } \hbar \to 0^+$$

Then,

$$\partial_t dw(t,x,\eta) - \nabla_x V(x)\cdot\nabla_\eta dw(t,x,\eta) + \frac{1}{m}\eta\cdot\nabla_x dw(t,x,\eta) = 0$$

in the measure sense. Namely, $\forall \phi \in C^\infty([0,T]\times\mathbb{T}^n\times\mathbb{R}^n)$ with $\operatorname{supp}(\mathcal{F}_\eta^{-1}\phi(t,x,\cdot)) \subseteq K$,

$$\int_0^T \sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{\mathbb{T}^n}\Big(\partial_t - \nabla_x V(x)\cdot\nabla_\eta + \frac{1}{m}\eta\cdot\nabla_x\Big)\phi(t,x,\eta)\; dw(t,x,\eta) = 0$$

A semiclassical effective Hamiltonian

An inf-sup formula

$$\bar{H}_\hbar(P) := \inf_v \sup_\varphi \int_{\mathbb{T}^n} H(x, P + \nabla_x v(x))\varphi(x)^2 - \frac{1}{2}\hbar^2 |\nabla_x \varphi(x)|^2 \, dx$$

where $v, \varphi \in C^\infty(\mathbb{T}^n; \mathbb{R})$ fulfill

$$\int_{\mathbb{T}^n} v(x)dx = 0, \quad \int_{\mathbb{T}^n} \varphi(x)dx = 1.$$

Introduced and studied by L.C. Evans (Calc. Var. 2009) with $\varepsilon = \hbar^2$.

$\Rightarrow$ Here we are interested in the

$$|\bar{H}_\hbar(P) - \bar{H}(P)| \to 0$$

as $\hbar \to 0^+$ where $\bar{H}(P)$ is the effective Hamiltonian.

Some variational properties

Look for $(v_\hbar, \varphi_\hbar)$ minimizing phases and maximizing amplitudes.

$$\bar{H}_\hbar(P) := \inf_v \sup_\varphi \int_{\mathbb{T}^n} H(x, P + \nabla_x v(x))\varphi(x)^2 - \frac{1}{2}\hbar^2 |\nabla_x \varphi(x)|^2 \, dx$$

- $\exists$ unique $\varphi_\hbar(P, v) \in C^\infty(\mathbb{T}^n; \mathbb{R}^+)$.
  (easy result)
- $\exists$ $v_\hbar \in W^{1,2}(\mathbb{T}^n; \mathbb{R})$ global minimizer.
  (we used weakly lower semicontinuity)
- $\exists$ $\hat{v}_\hbar \in C^\infty(\mathbb{T}^n; \mathbb{R})$ which is $O(\hbar^{1/2})$ approx. global minimizer.
  (we used an Ekeland's result on approx. critical points)

Two important features

We discover the approximated Hamilton-Jacobi equation

$$H(x, P + \nabla_x \hat{v}_\hbar(P,x)) = \bar{H}(P) + O(\hbar^{1/2})$$

and the approximated coupled continuity equation

$$\Big| \int_{\mathbb{T}^n} \phi(x) \mathrm{div}_x \Big(\nabla_\eta H(x, P + \nabla_x \hat{v}_\hbar(P,x)) \varphi_\hbar^2(P, \hat{v}_\hbar, x) \Big) dx \Big| \leq C\hbar^{1/2}$$

for all test function $\phi \in C^\infty(\mathbb{T}^n; \mathbb{R})$.
In the classical asymptotics

$$H(x, P + \nabla_x v(P,x)) = \bar{H}(P)$$

$$\int_{\mathbb{T}^n} \nabla_x \phi(x) \cdot \nabla_\eta H(x, P + \nabla_x v(P,x)) d\sigma_P(x) = 0$$

Main Results

Fix the setting

$$L(x,\xi) := \frac{1}{2}|\xi|^2 + V(x), \quad H(x,\eta) := \frac{1}{2}|\eta|^2 + V(x),$$

and select wave functions as

$$\psi_{\hbar,P}(x) := \varphi_\hbar(P_\hbar, \hat{v}_\hbar, x) e^{\frac{i}{\hbar}(P_\hbar \cdot x + \hat{v}_\hbar(P_\hbar, x))}$$

where $P \in \mathbb{R}^n$, $P_\hbar \in \hbar\mathbb{Z}^n$ fulfill $|P_\hbar - P| \to 0$ as $\hbar \to 0$.

$\Rightarrow$ Now study the Wigner transform of $\psi_{\hbar,P}$.

Main Results

Theorem

Bernardi-Parmeggiani-Z., Annales Henri Poincarè, online first March 2012.

$$W_\hbar \psi_{\hbar,P} \rightharpoonup dw_P \quad \text{as } \hbar \to 0^+$$

where

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} f(x,\eta) dw_P(x,\eta) = \int_{\mathbb{T}^n} f(x, P + \nabla_x v(P,x)) d\sigma_P(x).$$

- We get a subclass of "monokinetic measures".
- They are invariant under Hamiltonian dynamics and supported on the graph of a Lipschitz solution of stationary H-J equation.

Theorem

Let $\hat{H}_\hbar := -\frac{1}{2}\hbar^2\Delta_x + V(x)$. *Then,*

$$\|\hat{H}_\hbar \psi_{\hbar,P} - \bar{H}(P_\hbar)\psi_{\hbar,P}\|_{L^2} \leq C(P)\hbar^{1/2}.$$

- Quite easy to obtain.
- Not surprising, many constructions gives quasimodes with the same order (for example coherent states).
- A generalization of the above results is one of my current objectives.
- A full knowledge about the Aubry-Mather theory in semiclassical Analysis needs more work!

Thank you for the attention!