# Hydrodynamic Limit of the Gross-Pitaevskii equation

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# Outline

- Introduction
- Wave Group
- Main Theorem and Proof

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# Gross-Pitaevskii equation

Time scaled Gross-Pitaevskii equation

$$i\varepsilon^{\alpha}\partial_t\psi^{\varepsilon} + \frac{\varepsilon^{2\alpha}}{2}\Delta\psi^{\varepsilon} - \frac{1}{\varepsilon^2}(|\psi^{\varepsilon}|^2 - \rho_0)\psi^{\varepsilon} = 0\,.$$

Madelung transform (1927)

$$\psi^{\varepsilon} = R\exp(iS/\varepsilon^{\alpha})$$

GP becomes

$$\partial_t R + \frac{R}{2}\Delta S + \nabla R\cdot\nabla S = 0\,,$$

$$\partial_t S + \frac{1}{2}|\nabla S|^2 + \frac{R^2-\rho_0}{\varepsilon^2} = \frac{\varepsilon^{2\alpha}}{2}\frac{\Delta R}{R}\,.$$

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Hydrodynamic Variables

$$\rho^\varepsilon = R^2 = |\psi^\varepsilon|^2$$

$$u^\varepsilon = \nabla S = \frac{i\varepsilon^\alpha}{2|\psi^\varepsilon|^2}(\psi^\varepsilon \nabla \overline{\psi^\varepsilon} - \overline{\psi^\varepsilon} \nabla \psi^\varepsilon)$$

$$J^\varepsilon = \rho^\varepsilon u^\varepsilon , \quad \varphi^\varepsilon = \frac{\rho^\varepsilon - \rho_0}{\varepsilon} ,$$

Hydrodynamic structure of GP

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon u^\varepsilon) = 0 , \\ \partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \frac{1}{\varepsilon^2} \nabla (\rho^\varepsilon - \rho_0) = \frac{\varepsilon^{2\alpha}}{2} \nabla \left[ \frac{\Delta \sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}} \right] . \end{cases}$$

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# Hydrodynamic Structure

The hydrodynamic Euler equation $(\rho^\varepsilon, J^\varepsilon)$

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla \cdot J^\varepsilon = 0\,, \\ \partial_t J^\varepsilon + \nabla \cdot (\frac{J^\varepsilon \otimes J^\varepsilon}{\rho^\varepsilon}) + \frac{1}{\varepsilon}\rho_0 \nabla \varphi^\varepsilon + \frac{1}{2}\nabla(\varphi^\varepsilon)^2 = \frac{\varepsilon^{2\alpha}}{4}\nabla \cdot \left[\rho^\varepsilon \nabla^2 \log \rho^\varepsilon\right]. \end{cases}$$

- $J_0^\varepsilon \to J_0 = \rho_0 v_0$, $\varphi_0^\varepsilon \to 0$, and $\nabla \cdot (\rho_0 v_0) = 0$.

Hydrodynamic Limit ($\varepsilon \to 0$ )
Lake equations (anelastic system) with nonconstant density $\rho_0$

$$\begin{cases} \nabla \cdot (\rho_0 u) = 0\,, \\ \partial_t(\rho_0 u) + \nabla \cdot (\rho_0 u \otimes u) + \rho_0 \nabla \pi = 0\,, \\ \rho_0 u(x,0) = \rho_0 v_0\,. \end{cases}$$

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# Review Previous Work

Incompressible limit of the Navier-Stokes or Euler system
- classical solution: S. Klainerman, A. Majda (CPAM, 81).
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# Helmholtz Decomposition

Let $f \in L^2_{1/\rho_0}(\mathbb{T}^n)$, the weighted Helmholtz decomposition

$$f = \mathbb{H}_{\rho_0}[f] \oplus \mathbb{H}^{\perp}_{\rho_0}[f]$$

with

$$\operatorname{div} \mathbb{H}_{\rho_0}[f] = 0, \quad \mathbb{H}^{\perp}_{\rho_0}[f] = \rho_0 \nabla \Psi \, .$$

where $\Psi \in D^{1,2}(\mathbb{T}^n)$ is the unique solution of the problem

$$\int_{\mathbb{T}^n} \rho_0 \nabla \Psi \cdot \nabla \varphi dx = \int_{\mathbb{T}^n} f \cdot \nabla \varphi dx \, , \qquad \forall \varphi \in D^{1,2}(\mathbb{T}^n) \, .$$

$D^{1,2}(\mathbb{T}^n)$ : completion of $C_0^\infty(\mathbb{T}^n)$ w.r.t. $\|\nabla \cdot \|_{L^2_{1/\rho_0}(\mathbb{T}^n)}$.
$L^2_{1/\rho_0}(\mathbb{T}^n)$ : weighted Hilbert space with the scalar product

$$< v, w >_{1/\rho_0} = \int_{\mathbb{T}^n} v \cdot w \frac{dx}{\rho_0} \, .$$

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# Helmholtz Decomposition

(A) Conservation of charge

$$\frac{\partial}{\partial t}\rho^{\varepsilon} + \nabla \cdot J^{\varepsilon} = 0.$$

(B) Conservation of momentum (current)

$$\frac{\partial}{\partial t}J^{\varepsilon} + \frac{1}{2}\varepsilon^{2\alpha}\nabla \cdot \left[ (\nabla\psi^{\varepsilon} \otimes \nabla\overline{\psi^{\varepsilon}} + \nabla\overline{\psi^{\varepsilon}} \otimes \nabla\psi^{\varepsilon}) - \nabla^2(|\psi^{\varepsilon}|^2) \right]$$

$$+\frac{1}{2}\nabla(\varphi^{\varepsilon})^2 + \frac{1}{\varepsilon}\rho_0\nabla\varphi^{\varepsilon} = 0\,.$$

Define

$$J^{\varepsilon} = \mathbb{H}_{\rho_0}[J^{\varepsilon}] + \mathbb{H}_{\rho_0}^{\perp}[J^{\varepsilon}] = \mathbb{H}_{\rho_0}[J^{\varepsilon}] + \rho_0\nabla w^{\varepsilon}\,,$$

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the equation can be rewritten as

$$\begin{cases} \varepsilon\partial_t\varphi^\varepsilon + \mathrm{div}\,(\rho_0\nabla w^\varepsilon) = 0\,, \\ \varepsilon\partial_t(\sqrt{\rho_0}\nabla w^\varepsilon) + \sqrt{\rho_0}\nabla\varphi^\varepsilon = \varepsilon\frac{1}{\sqrt{\rho_0}}F^\varepsilon\,, \end{cases}$$

where

$$F^\varepsilon = -\frac{\varepsilon^{2\alpha}}{2}\mathbb{H}^\perp_{\rho_0}\nabla\cdot\left(\nabla\psi^\varepsilon\otimes\nabla\overline{\psi^\varepsilon} + \nabla\overline{\psi^\varepsilon}\otimes\nabla\psi^\varepsilon\right)$$

$$-\frac{1}{2}\mathbb{H}^\perp_{\rho_0}\nabla(\varphi^\varepsilon)^2 + \frac{\varepsilon^{2\alpha}}{4}\mathbb{H}^\perp_{\rho_0}\nabla\Delta\rho^\varepsilon\,.$$

It is obvious that $\partial_t\varphi^\varepsilon$ and $\partial_t(\sqrt{\rho_0}\nabla w^\varepsilon)$ are of order $O(1/\varepsilon)$ and are highly oscillatory as $\varepsilon\to 0$. So we have to introduce the wave group in order to filter out the fast oscillating wave.

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# Wave Group

Define the wave group $\mathcal{L}(\tau) = e^{\tau L}$, $\tau \in \mathbb{R}$, where $L$ is

$$L \begin{pmatrix} \phi \\ \sqrt{\rho_0} v \end{pmatrix} = - \begin{pmatrix} \operatorname{div}(\rho_0 v) \\ \sqrt{\rho_0} \nabla \phi \end{pmatrix} .$$

- The spectrum of $L$ is equivalent to the spectrum of $-\nabla \cdot (\rho_0 \nabla)$.
- Let$\{\kappa_j, \chi_j\}_{j=1}^{\infty}$ be the spectrum of $-\nabla \cdot (\rho_0 \nabla)$, where $0 < \kappa_1 < \kappa_2 < \cdots$,then the spectrum of $L$ is

$$\left\{ i\sqrt{\kappa_j}, \begin{pmatrix} \chi_j \\ \frac{i}{\sqrt{\kappa_j}} \sqrt{\rho_0} \nabla \chi_j \end{pmatrix} \right\} \text{ and } \left\{ -i\sqrt{\kappa_j}, \begin{pmatrix} \chi_j \\ \frac{-i}{\sqrt{\kappa_j}} \sqrt{\rho_0} \nabla \chi_j \end{pmatrix} \right\}.$$

Let

$$U^\varepsilon = \begin{pmatrix} \varphi^\varepsilon \\ \sqrt{\rho_0} \nabla w^\varepsilon \end{pmatrix}$$

We have

$$\partial_t U^\varepsilon = \frac{1}{\varepsilon} L U^\varepsilon + \frac{1}{\sqrt{\rho_0}} \widehat{F}^\varepsilon .$$

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# Wave Group

$\mathcal{L}(\tau) \equiv e^{\tau L}$ : the evolution group associated with $L$.

- $\mathcal{L}(\tau)$ is unitary in Hilbert space $L^2(\mathbb{T}^n) \times (L^2)^n(\mathbb{T}^n)$.
- $\mathcal{L}(\tau)$ is uniform bound in $H^s(\mathbb{T}^n) \times (H^s)^n(\mathbb{T}^n)$, for all $\tau$ and $s$.

We also define

$$V^\varepsilon = \mathcal{L}\Big(\frac{-t}{\varepsilon}\Big) U^\varepsilon ,$$

by applying the operator $\mathcal{L}(\frac{-t}{\varepsilon})$, $V^\varepsilon$ satisfies

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# Wave Group

Define $\mathcal{L}(\frac{t}{\varepsilon})V^{\varepsilon} = (\mathcal{L}_1(\frac{t}{\varepsilon})V^{\varepsilon}, \mathcal{L}_2(\frac{t}{\varepsilon})V^{\varepsilon})^t$

$$F^{\varepsilon} = \text{div}\left(\frac{1}{\rho_0}\mathbb{H}_{\rho_0}[J^{\varepsilon}] \otimes \mathbb{H}_{\rho_0}[J^{\varepsilon}]\right)$$
$$+\text{div}\left(\frac{1}{\sqrt{\rho_0}}\mathbb{H}_{\rho_0}[J^{\varepsilon}] \otimes \mathcal{L}_2(\tfrac{t}{\varepsilon})V + \frac{1}{\sqrt{\rho_0}}\mathcal{L}_2(\tfrac{t}{\varepsilon})V \otimes \mathbb{H}_{\rho_0}[J^{\varepsilon}]\right)$$
$$+\text{div}\left(\mathcal{L}_2(\tfrac{t}{\varepsilon})V \otimes \mathcal{L}_2(\tfrac{t}{\varepsilon})V\right) + \frac{1}{2}\nabla\left(|\mathcal{L}_1(\tfrac{t}{\varepsilon})V|^2\right)$$
$$+\frac{\varepsilon^{2\alpha}}{4}\nabla\Delta\rho^{\varepsilon}\,.$$

- The resonances may occurs on red part, and it happens if and only if $\pm\sqrt{\kappa_i} \pm \sqrt{\kappa_j} \pm \sqrt{\kappa_k} = 0$, for some $i, j, k$.

- The red part $= \frac{1}{2}\rho_0\nabla\left(\left|\frac{\mathcal{L}_2(\frac{t}{\varepsilon})V}{\sqrt{\rho_0}}\right|^2\right)$.

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$+\operatorname{div}\left(\mathcal{L}_2(\frac{t}{\varepsilon})V\otimes\mathcal{L}_2(\frac{t}{\varepsilon})V\right) + \frac{1}{2}\nabla\left(|\mathcal{L}_1(\frac{t}{\varepsilon})V|^2\right)$
$+\frac{\varepsilon^{2\alpha}}{4}\nabla\Delta\rho^\varepsilon$ .

- The resonances may occurs on red part, and it happens if and only if $\pm\sqrt{\kappa_i}\pm\sqrt{\kappa_j}\pm\sqrt{\kappa_k}=0$, for some $i,j,k$.

- The red part $= \frac{1}{2}\rho_0\nabla\left(\left|\frac{\mathcal{L}_2(\frac{t}{\varepsilon})V}{\sqrt{\rho_0}}\right|^2\right)$.

# Wave Group

Define $\mathcal{L}(\frac{t}{\varepsilon})V^\varepsilon = (\mathcal{L}_1(\frac{t}{\varepsilon})V^\varepsilon, \mathcal{L}_2(\frac{t}{\varepsilon})V^\varepsilon)^t$

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# Assumptions of the initial conditions

(A1) $\psi_0^\varepsilon \in H^{\frac{n}{2}+3}(\mathbb{T}^n, \mathbb{C})$, $\rho_0 \geq c > 0$.

(A2) $J_0^\varepsilon \to J_0 = \rho_0 v_0 + \rho_0 \nabla w_0$ in $L^2(\mathbb{T}^n)$, where $\nabla \cdot (\rho_0 v_0) = 0$.

(A3) $\varphi_0^\varepsilon \to \varphi_0$ in $L^2(\mathbb{T}^n)$ and $\varepsilon^\alpha \nabla \sqrt{\rho_0^\varepsilon} \to 0$ in $L^2(\mathbb{T}^n)$.

(A4) Let$\{\kappa_j\}_{j=1}^\infty$ be the spectrum of the elliptic operator $-\nabla \cdot (\rho_0 \nabla)$, then $\pm\sqrt{\kappa_i} \pm \sqrt{\kappa_j} \pm \sqrt{\kappa_k} \neq 0$, for all $i, j, k$.

The divergence free part : lake equation (anelastic system)

$$\begin{cases} \nabla \cdot (\rho_0 u) = 0\,, \\ \partial_t(\rho_0 u) + \nabla \cdot (\rho_0 u \otimes u) + \rho_0 \nabla \pi = 0\,, \\ \rho_0 u(x,0) = \rho_0 v\,. \end{cases}$$

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The oscillating part :

$$\begin{cases} \partial_t V + \mathcal{Q}_1(u, V) = 0\,, \\ V(x,0) = (\varphi_0, \frac{1}{\sqrt{\rho_0}} J_0)\,. \end{cases}$$

$$\mathcal{Q}_1(u, V) = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \mathcal{L}(-s)\mathbb{K}(u, V)ds\,,$$

$$\mathbb{K}(u, V) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{\rho_0}} \mathbb{H}^{\perp}_{\rho_0} \mathrm{div}\,(\sqrt{\rho_0} u \otimes \mathcal{L}_2(s)V + \sqrt{\rho_0}\mathcal{L}_2(s)V \otimes u) \end{pmatrix}.$$

# Limiting Equations (Oscillating part)

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# Outline

- QHD Model
- Wave Group
- Main Theorem and Proof

## Theorem

*Let $\psi^{\varepsilon}$ be the solution of the Schrödinger equations and $\psi_0$ satisfy the assumption of the initial conditions $(A1) - (A4)$, then there exist $T_* > 0$ such that*

$$\rho^{\varepsilon} \longrightarrow \rho_0 \quad \textit{strongly in} \quad L^{\infty}\left([0, T]; L^2(\mathbb{T}^n)\right),$$

$$J^{\varepsilon} \rightharpoonup \rho_0 u \quad \textit{weakly} * \textit{in} \quad L^{\infty}\left([0, T]; L^{4/3}(\mathbb{T}^n)\right),$$

*where $u$ satisfy the lake equations.*

# Proof

Step 1: Construct Energy Equations.

- For GP

$$\frac{d}{dt}\int_{\mathbb{T}^n}\frac{1}{2}\varepsilon^{2\alpha}|\nabla\psi^\varepsilon|^2+\frac{1}{2\varepsilon^2}\left(|\psi^\varepsilon|^2-\rho_0\right)^2dx=0\,.$$

- For limit system

$$\frac{d}{dt}\int_{\mathbb{T}^n}\frac{1}{2}(\rho_0|u|^2+|V|^2)dx=0\,.$$

Step 2: Compactness of $\{V^\varepsilon\}_\varepsilon$ (Lion-Aubin Lemma).

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Step 3: Construct the modulated energy functional

$$H^\varepsilon(t) = \frac{1}{2}\int_{\mathbb{T}^n}\left|\left(\varepsilon^\alpha\nabla - i\left[v + \tfrac{1}{\sqrt{\rho_0}}\mathcal{L}_2(\tfrac{t}{\varepsilon})V\right]\right)\psi^\varepsilon\right|^2 dx$$

$$+\frac{1}{2}\int_{\mathbb{T}^n}|\varphi^\varepsilon - \mathcal{L}_1(\tfrac{t}{\varepsilon})V|^2 dx\,.$$

- Assumption of initial conditions: $H^\varepsilon(0) \to 0$ as $\varepsilon \to 0$.

Step 4: Prove $H^\varepsilon(t) \to 0$ as $\varepsilon \to 0$.

- Consider the evolution on $H^\varepsilon$ and use Gronwall inequality.

Vlasov-Poisson: Y. Brenier (CPDE, 00), N. Masmoudi (CPDE, 01), H.K., Daniel (CPDE, 11).
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# Conclusion

• If there is no resonance, we perform the mathematical derivation of the lake equation (anelastic system) from the classical solution of the GP for general initial data and nonconstant density.

• If density $\rho_0 = 1$, the non-resonance assumption can be removed.

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**THANK YOU**