Controllability of a scalar conservation law with nonlocal velocity

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Outline

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Semiconductor wafer manufacture system

The semiconductor manufacturing system which has a highly re-entrant character.

- very high volume (number of parts manufactured per unit time) at each process
- very large number of consecutive production steps

Cycle time: 6 weeks   Work steps: 500   WIP: 60,000 wafers

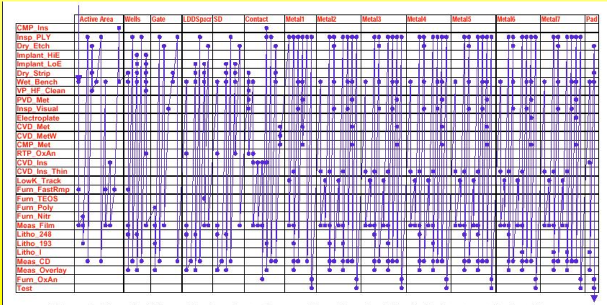


Figure 1. Generic 130-nm Semiconductor Process Flow Showing Massively Reentrant Product Flows

Model

1-D scalar conservation law with nonlocal velocity [Armbruster et al., 2006]

$$\begin{cases} \rho_t(t,x) + (\rho(t,x)\lambda(W(t)))_x = 0, \quad (t,x) \in (0,T) \times (0,1), \\ \rho(0,x) = \rho_0(x), \quad x \in (0,1), \\ \rho(t,0)\lambda(W(t)) = u(t), \quad t \in (0,T). \end{cases} \tag{1}$$

with $W(t) = \int_0^1 \rho(t,x)dx$.
$\lambda(\cdot) \in C^1([0,+\infty);(0,+\infty))$ and is usually decreasing.
A typical special case: $\lambda(W) = \frac{1}{1+W}$.

- The space variable $x \in [0, 1]$ represents the whole manufacture process. $x = 0$ corresponds to the entry of the factory (manufacture line), while $x = 1$ corresponds to the exit of the factory (manufacture line).
- $\rho(t, x) \geq 0$ is the density of product at time t and stage x. $\rho(t, \cdot)$ ($t \geq 0$) is called the state in exact controllability problem.
- The control $u(t)$ of this system is acted on the influx $\rho(t, 0)\lambda(W(t))$, the rate of products entering the factory. The natural output is the outflux $y(t) := \rho(t, 1)\lambda(W(t))$, the rate of products exiting the factory.

Some control problems (I)

- Exact (state) Controllability Problem
  Find control to drive arbitrary initial state ρ_0 to arbitrary final state ρ_1 over some $[0, T]$.
- Time-optimal Transition Problem
  Find time-optimal control to drive given initial state ρ_0 to given final state ρ_1.
- Stabilization Problem
  Find feedback control to obtain asymptotic stability of the closed-loop system. Natural feedback $u(t) = ky(t)$ s.t. $\|\rho(t, \cdot) - \overline{\rho}\| \to 0$ as $t \to \infty$.

Some control problems (II)

- Nodal Profile (Outflux) Controllability Problem
  Find control to drive the system, which starts from arbitrary initial state ρ_0, to reach exactly some given value y_d for the outflux y over $[T_1, T_2]$.
- Demand Tracking Problem
  Find control to minimize the *Error Signal* $\|y - y_d\|$, the difference between the actual outflux $y(t)$ and a given demand forecast $y_d(t)$.
- Backlog Problem
  Find control to minimize the *Accumulated Error Signal* $\|\beta\|$, $\beta(t) = \int_0^t y(s)\,ds - \int_0^t y_d(s)\,ds$.

Related work

- D. Armbruster, D. Marthaler, C. Ringhofer, K. Kempf and T.-C. Jo, *A continuum model for a re-entrant factory*, 2006.
- M. La Marca, D. Armbruster, M. Herty and C. Ringhofer, *Control of continuum models of production systems*, 2010.
- R. Colombo, M. Herty and M. Mercier, *Control of the continuity equation with a non local flow*, 2010.

Definition of weak solution in L^1

Let $T > 0$, $\rho_0 \in L^1(0,1)$ and $u \in L^1(0,T)$ be given. A weak solution of the system (1) is a function $\rho \in C^0([0,T]; L^1(0,1))$ such that, for every $\tau \in [0,T]$ and every $\varphi \in C^1([0,\tau]\times[0,1])$ such that

$$\varphi(\tau,x) = 0, \forall x \in [0,1] \quad \text{and} \quad \varphi(t,1) = 0, \forall t \in [0,\tau],$$

one has

$$\int_0^\tau \int_0^1 \rho(t,x)(\varphi_t(t,x) + \lambda(W(t))\varphi_x(t,x))dxdt + \int_0^\tau u(t)\varphi(t,0)dt + \int_0^1 \rho_0(x)\varphi(0,x)dx = 0.$$

Well-posedness in L^1

Theorem (Coron, Kawski, W, 2010)

If $\rho_0 \in L^1_+(0,1)$ and $u \in L^1_+(0,T)$, then system (1) admits a unique weak solution $\rho \in C^0([0,T]; L^1_+(0,1))$, which is also nonnegative almost everywhere in $Q = (0,T) \times (0,1)$.

Proof sketch.

1. Existence and uniqueness of local solution
   Characteristic method + fixed point argument by contraction mapping principle.
   Explicit expression of ρ in terms of $\xi(t) := \int_0^t \lambda(W(\tau))d\tau$.
2. Existence of global solution
   Uniform a priori estimate on $\|\rho(t,\cdot)\|_{L^1(0,1)}$.

□

Regularity

Remark (Hidden regularity)

$\rho \in C^0([0,T]; L^1(0,1)) \subset L^1(0,1; L^1(0,T))$
$\rightsquigarrow \rho \in C^0([0,1]; L^1(0,T))$.

Remark (L^p Regularity)

$\rho_0 \in L^p_+(0,1)$ and $u \in L^p_+(0,T)$ $(1 \leq p < \infty)$
$\Longrightarrow \rho \in C^0([0,T]; L^p_+(0,1)) \cap C^0([0,1]; L^p_+(0,T))$.

Definition of exact (state) controllability

Exact (State) Controllability

State $\rho(t,\cdot) \in L^p_+(0,1)$.
Does there exists Control $u(t) \in L^p_+(0,T)$ to drive the hyperbolic system (1): $\rho_0 \in L^p_+(0,1) \rightsquigarrow \rho_1 \in L^p_+(0,1)$ on $[0,T]$, i.e.,

$$\rho(T,x) = \rho_1(x), \quad x \in (0,1). \tag{2}$$

Classical results can not be applied

- F. Ancona, A. Marson, *On the attainable set for scalar nonlinear conservation laws with boundary control*, 1998.
- A. Bressan, G. M. Coclite, *On the boundary control of systems of conservation laws*, 2002.
- V. Perollaz, *Exact controllability of scalar conservation laws with an additional control and in the context of entropy solutions*, 2011.
- T. Li, B. Rao, *Exact boundary controllability for quasi-linear hyperbolic systems*, 2003.
- T. Li, *Controllability and observability for quasilinear hyperbolic systems*, 2010.

Local controllability result

Theorem (Coron, W, 2012)

Let $\overline{\rho} \geq 0$ *be the given constant equilibrium and* $T_0 := \frac{1}{\lambda(\overline{\rho})}$.
Then, for any $T > T_0$, *any* $\varepsilon > 0$ *and any* $p \in [1, +\infty)$, *there exists* $\nu > 0$ *such that, for any* ρ_0, *any* $\rho_1 \in L^p_+(0,1)$ *with*

$$\|\rho_0(\cdot) - \overline{\rho}\|_{L^p(0,1)} \leq \nu, \ \|\rho_1(\cdot) - \overline{\rho}\|_{L^p(0,1)} \leq \nu,$$

there exists $u \in L^p_+((0,T))$ *with*

$$\|u(\cdot) - \overline{\rho}\lambda(\overline{\rho})\|_{L^p(0,T)} \leq \varepsilon,$$

such that the weak solution $\rho \in C^0([0,T]; L^p(0,1))$ *to system* (1) *satisfies the final condition* (2) *and*

$$\|\rho(t,\cdot) - \overline{\rho}\|_{L^p(0,1)} \leq \varepsilon, \quad \forall t \in [0,T].$$

Proof of Local controllability

Proof sketch.

1. Controllability for linear control system
   $\xi \Longrightarrow u \Longrightarrow \rho$
2. Fixed point argument by contraction mapping principle
   $\xi \Longrightarrow u \Longrightarrow \rho \Longrightarrow F(\xi) = \xi$

□

Proof of Local controllability

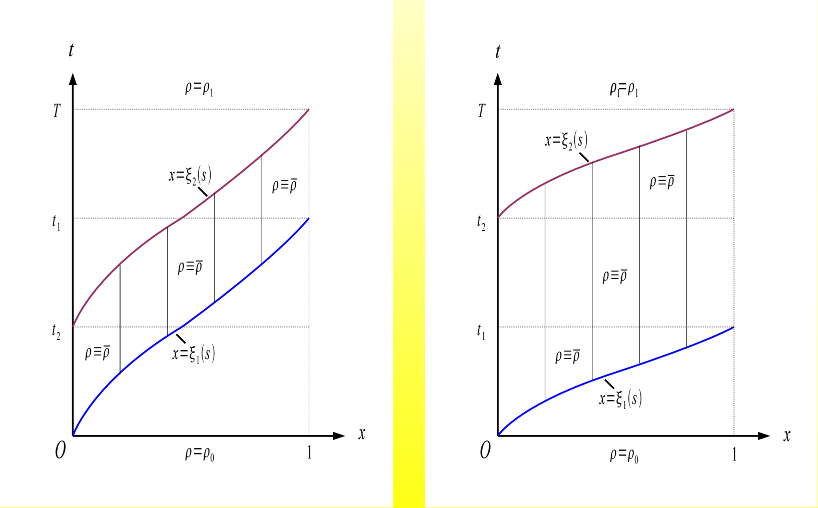


Figure: Construction of ρ for linear system

Global controllability result

Theorem (Coron, W, 2012)

For any $p \in [1, +\infty)$, any $\rho_0 \in L^p_+(0,1)$ and any $\rho_1 \in L^p_+(0,1)$, there exists $T_1 > 0$ (depending on ρ_0 and ρ_1) such that for any $T \geq T_1$, there exists $u \in L^p_+(0,T)$ such that the weak solution $\rho \in C^0([0,T]; L^p(0,1))$ to system (1) satisfies the final condition (2).

Remark

Global controllability $\iff$ No restriction on the distance between ρ_0 and ρ_1.

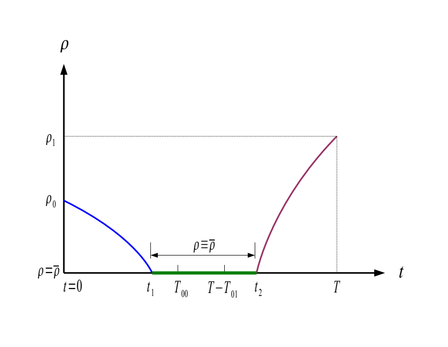
Proof of Global controllability

Proof sketch.

1. Drive the state from ρ_0 at $t = 0$ to some equilibrium $\overline{\rho}$ at $t = T_{00}$.
   Input control $u(t)$ can be induced by natural state control $\rho(t, 0) \equiv \overline{\rho}$.
2. Drive the system from $\overline{\rho}$ at $t = T - T_{01}$ to ρ_1 at $t = T$ by using the reversibility of the hyperbolic system $(t, x, \rho(t, x)) \to (T - t, 1 - x, \rho(T - t, 1 - x))$.

□

Proof of Global controllability



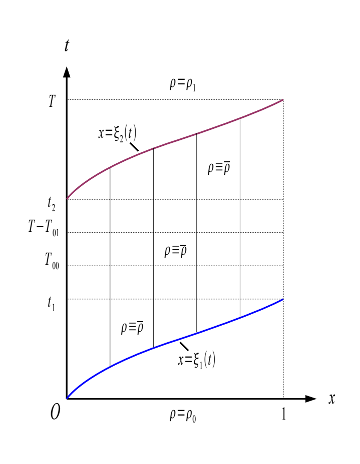


Figure: Idea of proof and Construction of ρ for linear system

Nodal profile controllability

Where does Nodal Profile Controllability come?

- M. Gugat, M. Herty, V. Schleper, *Flow control in gas networks: exact controllability to a given demand*, 2011.
- T. Li, *Exact boundary controllability of nodal profile for quasilinear hyperbolic systems*, 2011.

Definition of nodal profile controllability

Definition

For any given initial data ρ_0, boundary data y_d and any T_1, T with $0 < T_1 < T$, does there exists control $u : (0, T) \mapsto [0, +\infty)$ such that the solution ρ to the system (1) satisfies also the nodal profile condition:

$$\rho(t, 1)\lambda(W(t)) = y_d(t), \quad t \in (T_1, T). \tag{3}$$

Nodal profile controllability result

Theorem (Coron, W, 2012)

Let $\overline{\rho} \geq 0$ *be the given constant equilibrium and let* $T_0 := \frac{1}{\lambda(\overline{\rho})}$. *For any* $p \in [1, +\infty)$, *and any* T_1, T *with* $T_0 < T_1 < T$, *there exists* $\nu > 0$ *such that the following holds: For any* $\rho_0 \in L^p_+(0,1)$ *and any* $y_d \in L^p_+(T_1, T)$

$$\|\rho_0(\cdot) - \overline{\rho}\|_{L^p(0,1)} \leq \nu, \ \|y(\cdot) - \overline{\rho}\lambda(\overline{\rho})\|_{L^p(T_1,T)} \leq \nu,$$

there exists $u \in L^p_+(0,T)$ *such that the weak solution* $\rho \in C^0([0,T]; L^p(0,1))$ *to the system* (1) *satisfies the out-flux condition* (3).

Proof of nodal profile controllability

Proof sketch.

Nodal profile controllability for linear system
Fixed point argument. □

Proof of nodal profile controllability

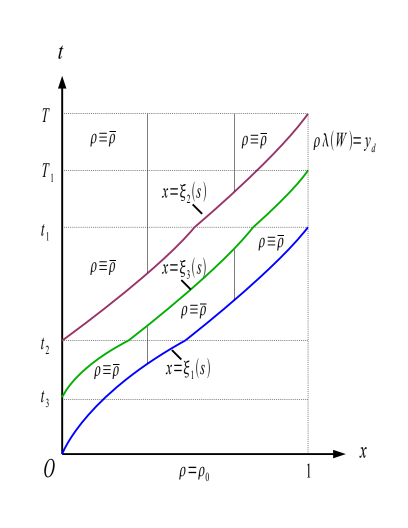


Figure: Construction of ρ for linear system

Future works

- Other application
  Particle synthesis process, Follicle ovulation
- Generalization
  Networks, Coupled systems, Higher dimension
- Stabilization with various types of feedback controls
- Optimal control problems with constrains

Thank you for your attention!