The Limit of the Boltzmann Equation to the Euler Equations for Riemann Problems

Yi WANG (王益)

Institute of Applied Mathematics, AMSS, CAS, Beijing 100190, P. R. China

Joint work with Feimin Huang, Yong Wang and Tong Yang

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Boltzmann equation with slab symmetry

$$f_t + \xi_1 f_x = \frac{1}{\varepsilon} Q(f,f), \quad (f,x,t,\xi) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}^3, \tag{1}$$

- $f(x,t,\xi)$: density distribution function of particles
- $\varepsilon > 0$: Knudsen number $\sim$ the mean free path
- collision operator for hard sphere model

$$\begin{aligned} Q(f,g)(\xi) \equiv \; & \frac{1}{2} \int_{\mathbf{R}^3} \int_{\mathbf{S}^2_+} \Big(f(\xi')g(\xi'_*) + f(\xi'_*)g(\xi') \\ & - f(\xi)g(\xi_*) - f(\xi_*)g(\xi) \Big) |(\xi - \xi_*) \cdot \Omega| \, d\xi_* d\Omega, \end{aligned}$$

where

$$\xi' = \xi - [(\xi - \xi_*) \cdot \Omega] \, \Omega, \quad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \Omega] \, \Omega.$$

Boltzmann equation $\to$ compressible Euler equations as $\varepsilon \to 0$:

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = 0, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = 0, \; i = 2, 3, \\ [\rho(e + \frac{|u|^2}{2})]_t + [\rho u_1 (e + \frac{|u|^2}{2}) + p u_1]_x = 0, \end{cases} \tag{2}$$

where the macroscopic variables are defined by

$$\begin{pmatrix} \rho \\ \rho u_i \\ \rho(e + \frac{|u|^2}{2}) \end{pmatrix} = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \xi_i \\ \frac{|\xi|^2}{2} \end{pmatrix} f(x, t, \xi) d\xi, \tag{3}$$

with the pressure $p = R\rho\theta$ and the internal energy $e = \frac{3}{2} R\theta$.

For a solution $f(t,x,\xi)$ of (1), set

$$f(t,x,\xi) = \mathbf{M}(t,x,\xi) + \mathbf{G}(t,x,\xi),$$

where the local Maxwellian $\mathbf{M}(t,x,\xi) = \mathbf{M}_{[\rho,u,\theta]}(\xi)$ represents the macroscopic component of the solution defined by the five conserved quantities, i.e., the mass density $\rho(t,x)$, the momentum $\rho u(t,x)$, and the total energy $\rho(e + \frac{1}{2}|u|^2)(t,x)$ given in (3), through

$$\mathbf{M} = \mathbf{M}_{[\rho,u,\theta]}(t,x,\xi) = \frac{\rho(t,x)}{\sqrt{(2\pi R\theta(t,x))^3}} e^{-\frac{|\xi - u(t,x)|^2}{2R\theta(t,x)}}. \tag{4}$$

And $\mathbf{G}(t,x,\xi)$ represents the microscopic component.

Justification:

- How to justify this limit is still a challenging open problem going way back to Maxwell, mainly because the singularities in solutions to the Euler equations.
- Hilbert(1912) introduced the Hilbert expansion to show formally that the first order approximation of Boltzmann equation (1) gives Euler equations (2).

Relation to the Hilbert's sixth problem:

" Mathematical treatment of the axioms of physics."

··· Important investigations by physicists on the foundations of mechanics are at hand; I refer to the writings of Mach, Hertz, Boltzmann and Volkmann. It is therefore very desirable that the discussion of the foundations of mechanics be taken up by mathematicians also. Thus Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes, there merely indicated, which lead from the atomistic view to the laws of motion of continua. Conversely one might try to derive the laws of the motion of rigid bodies by a limiting process from a system of axioms depending upon the idea of continuously varying conditions of a material filling all space continuously, these conditions being defined by parameters. For the question as to the equivalence of different systems of axioms is always of great theoretical interest.

⋯Further, the mathematician has the duty to test exactly in each instance whether the new axioms are compatible with the previous ones. The physicist, as his theories develop, often finds himself forced by the results of his experiments to make new hypotheses, while he depends, with respect to the compatibility of the new hypotheses with the old axioms, solely upon these experiments or upon a certain physical intuition, a practice which in the rigorously logical building up of a theory is not admissible. The desired proof of the compatibility of all assumptions seems to me also of importance, because the effort to obtain such proof always forces us most effectually to an exact formulation of the axioms.

Problem considered:

Can we justify the hydrodynamic limit in the setting of Riemann solutions?

Riemann problem (1860's):

• Euler system (2) with initial data

$$(\rho, u, \theta)(x, 0) = \begin{cases} (\rho_-, u_-, \theta_-), & x < 0, \\ (\rho_+, u_+, \theta_+), & x > 0, \end{cases}$$

• General Riemann solution is a superposition of three basic wave patterns: shock, rarefaction wave and contact discontinuity.

• Note that Riemann solution captures the local and global behavior of the solutions.

Previous works on the hydrodynamic limit:

- Smooth solutions:
  - T. Nishida (1978, CMP): analytical solution by using Cauchy-Kovalevskaja theorem;
  - R. Caflisch (1980, CPAM): solution near a local Maxwellian without initial layer;
  - S. Ukai-K. Asona (1983, HMJ): local smooth solution by contraction mapping method;
  - M. Lachowicz (1987, MMAS): the case including initial layer.

- Single wave pattern:
  - S. H. Yu (2005, CPAM): shock wave, for any fixed time $T > 0$ with rate $\varepsilon^{\frac{1}{10}}$;
  - Huang-W-Yang (2010, CMP): contact discontinuity uniformly in time with rate: $\varepsilon^{\frac{1}{4}}$;
  - Z. P. Xin-H. H. Zeng (2010, JDE): rarefaction wave uniformly in time with rate: $\varepsilon^{\frac{1}{5}}|\ln \varepsilon|$.

- Superposition the basic wave patterns:
  - Huang-W-Yang (2010, KRM): Rarefaction wave+contact discontinuity+rarefaction wave uniformly in time with rate $\varepsilon^{\frac{1}{5}}$.
  - Huang-W-Yang (2012, ARMA): For Full CNS, Rarefaction wave+Shock wave, for any fixed time $T > 0$ with rate $\varepsilon^{\frac{1}{5}}$.

Superposition the basic wave patterns

Difficulties:

- Wave interactions
- Different structures of wave patterns:
  - Shock wave: compressible, antiderivative of perturbation;
  - Rarefaction wave: expanding, original perturbation;
  - Contact discontinuity: linearly degenerate, no definite sign.
- How to justify the limit for the generic case: rarefaction wave+ contact discontinuity+shock wave?

Superposition the basic wave patterns

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Recently, we succeed in justifying this limit by introducing two kinds of hyperbolic waves with different solution backgrounds to capture the extra masses carried by the hyperbolic approximation of the rarefaction wave and the diffusion approximation of the contact discontinuity.

Main Result

- 1 Rarefaction wave+2 Contact discontinuity +3 Shock wave
  - Theorem 1: (Huang-W-Wang-Yang, Preprint)
    Let $(\widetilde{V}, \widetilde{U}, \widetilde{\Theta})(t, x)$ be a Riemann solution to the Euler equations which is a superposition of a 1-rarefaction wave, a 2-contact discontinuity and a 3-shock wave, and $\delta = |(v_+ - v_-, u_+ - u_-, \theta_+ - \theta_-)|$ be the wave strength. There exist a small positive constant δ_0, and a global Maxwellian $\mathbf{M}_\star = \mathbf{M}_{[v_\star, u_\star, \theta_\star]}$ such that if the wave strength satisfies $\delta \le \delta_0$, then in any time interval $[h, T]$ with $0 < h < T$, there exists a positive constant $\varepsilon_0 = \varepsilon_0(\delta, h, T)$, such that if the Knudsen number $\varepsilon \le \varepsilon_0$, then the Boltzmann equation admits a family of smooth solutions $f^{\varepsilon,h}(t, x, \xi)$ satisfying

$$\sup_{(t,x)\in\Sigma_{h,T}} \|f^{\varepsilon,h}(t,x,\xi) - \mathbf{M}_{[\widetilde{V},\widetilde{U},\widetilde{\Theta}]}(t,x,\xi)\|_{L^2_\xi(\frac{1}{\sqrt{\mathbf{M}_\star}})} \le C_{h,T}\,\varepsilon^{\frac{1}{5}}|\ln\varepsilon|,$$

where $\Sigma_{h,T} = \{(t,x) | h \le t \le T, |x| \ge h, |x - s_3 t| \ge h\}$, the norm $\| \cdot \|_{L^2_\xi(\frac{1}{\sqrt{\mathbf{M}_\star}})}$ is $\| \frac{\cdot}{\sqrt{\mathbf{M}_\star}} \|_{L^2_\xi(\mathbf{R}^3)}$ and the positive constant $C_{h,T}$ depends on h and T but is independent of ε. Consequently, when $\varepsilon \to 0+$ and then $h \to 0+$, we have

$$\| f^{\varepsilon,h}(\xi) - \mathbf{M}_{[\widetilde{V},\widetilde{U},\widetilde{\Theta}]}(\xi) \|_{L^2_\xi(\frac{1}{\sqrt{\mathbf{M}_\star}})}(t,x) \to 0, \ \ a.e. \text{ in } \ [0,T] \times \mathbf{R}.$$

Remark 1. The above theorem shows that away from the initial time $t = 0$, the contact discontinuity at $x = 0$ and the shock discontinuity at $x = s_3 t$, for small total wave strength $\delta \leq \delta_0$ and Knudsen number $\varepsilon \leq \varepsilon_0$, there exists a family of smooth solutions $f^{\varepsilon,h}(t, x, \xi)$ of the Boltzmann equation which tends to the Maxwellian $\mathbf{M}_{[\widetilde{V},\widetilde{U},\widetilde{\Theta}]}(t, x, \xi)$ with $(\widetilde{V}, \widetilde{U}, \widetilde{\Theta})(t, x)$ being the Riemann solution to the Euler equations as a superposition of a 1-rarefaction wave, a 2-contact discontinuity and a 3-shock wave when $\varepsilon \to 0$ with a convergence rate $\varepsilon^{\frac{1}{5}}|\ln \varepsilon|$. Note that this superposition of waves is the most generic case for the Riemann problem. Similar results hold for any other superpositions of waves by using the same analysis.

Remark 2. Note that the analysis can also be applied to the vanishing viscosity limit of the one dimensional compressible Navier-Stokes equations. In fact, the vanishing viscosity limit of the one dimensional compressible Navier-Stokes equations in some sense can be viewed as a special case of hydrodynamic limit of Boltzmann equation to the Euler equations by neglecting the microscopic effect.

Main ideas:

- Since the compressibility of the viscous shock wave, the anti-derivative technique is used for all basic waves.
- In the integrated system, since the approximation rarefaction wave is constructed by the hyperbolic equation, thus the viscous term is an error term whose decay rate is not enough for the desired estimates. Here we construct Hyperbolic wave I (Huang-W-Yang, ARMA, 2012) to recover the viscous terms to the inviscid approximation of rarefaction wave pattern. The Hyperbolic wave I is constructed by the linearized system around the approximation rarefaction wave.

- Also in the integrated system, we construct the hyperbolic wave II to remove the error terms due to the viscous contact wave approximation. Note that the construction of the hyperbolic wave II can not be done simply around the contact wave approximation as the hyperbolic wave I for the rarefaction wave. Otherwise, the wave interaction terms thus induced will lead to insufficiently decay in term of the Kundsen number. Instead, it is constructed around the superposition of the approximate 1-rarefaction wave, the hyperbolic wave I, the 2-viscous contact wave and the 3-shock profile as a whole. Moreover, it also takes care of the non-conservative terms in the previous reduced system so that the energy estimates can be carried out for anti-derivative of the perturbation.

Since the problem considered in this paper is one dimensional in the space variable $x \in \mathbf{R}$, in the macroscopic level, it is more convenient to write the Boltzmann equation in the *Lagrangian* coordinates.
Thus (1) and (2) in the Lagrangian coordinates become, respectively,

$$f_t - \frac{u_1}{v} f_x + \frac{\xi_1}{v} f_x = \frac{1}{\varepsilon} Q(f, f), \tag{5}$$

and

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = 0, \\ u_{it} = 0, \ \ i = 2, 3, \\ (\theta + \frac{|u|^2}{2})_t + (pu_1)_x = 0. \end{cases} \tag{6}$$

Moreover, we have

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = \frac{4\varepsilon}{3}(\frac{\mu(\theta)}{v}u_{1x})_x - \int \xi_1^2 \Pi_{1x} d\xi, \\ u_{it} = \varepsilon(\frac{\mu(\theta)}{v}u_{ix})_x - \int \xi_1\xi_i \Pi_{1x} d\xi, \; i = 2, 3, \\ (\theta + \frac{|u|^2}{2})_t + (pu_1)_x = \varepsilon(\frac{\kappa(\theta)}{v}\theta_x)_x + \frac{4\varepsilon}{3}(\frac{\mu(\theta)}{v}u_1u_{1x})_x \\ \quad +\varepsilon\sum_{i=2}^{3}(\frac{\mu(\theta)}{v}u_iu_{ix})_x - \int \frac{1}{2}\xi_1|\xi|^2\Pi_{1x}d\xi. \end{cases} \tag{7}$$

and

$$\mathbf{G}_t - \frac{u_1}{v}\mathbf{G}_x + \frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{M}_x) + \frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{G}_x) = \frac{1}{\varepsilon}(\mathbf{L}_\mathbf{M}\mathbf{G} + Q(\mathbf{G},\mathbf{G})), \tag{8}$$

with

$$\mathbf{G} = \varepsilon\mathbf{L}_\mathbf{M}^{-1}(\frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{M}_x)) + \Pi_1, \tag{9}$$

$$\Pi_1 = \mathbf{L}_\mathbf{M}^{-1}[\varepsilon(\mathbf{G}_t - \frac{u_1}{v}\mathbf{G}_x + \frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{G}_x)) - Q(\mathbf{G},\mathbf{G})], \tag{10}$$

Rarefaction Wave

Burgers equation

$$\begin{cases} w_t + ww_x = 0, \\ w(0,x) = w_\sigma(x) = w(\frac{x}{\sigma}) = \frac{w_+ + w_-}{2} + \frac{w_+ - w_-}{2} \tanh \frac{x}{\sigma}, \end{cases} \quad (11)$$

where $\sigma = \varepsilon^{\frac{1}{5}} > 0$. Denote its solution by $w_\sigma^r(t,x)$.
The smooth approximate rarefaction wave profile denoted by $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t,x)$ can be defined by

$$\begin{cases} S^{R_1}(t,x) = s(V^{R_1}(t,x), \Theta^{R_1}(t,x)) = s_+, \\ w_\pm = \lambda_{1\pm} := \lambda_1(v_\pm, \theta_\pm), \\ w_\sigma^r(t,x) = \lambda_1(V^{R_1}(t,x), s_+), \\ U_1^{R_1}(t,x) = u_{1+} - \int_{v_+}^{V^{R_1}(t,x)} \lambda_1(v, s_+) dv, \\ U_i^{R_1}(t,x) \equiv 0, \ i = 2, 3. \end{cases} \quad (12)$$

Note that $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t,x)$ defined above satisfies

$$\begin{cases} V_t^{R_1} - U_{1x}^{R_1} = 0, \\ U_{1t}^{R_1} + P_x^{R_1} = 0, \\ U_{it}^{R_1} = 0, \ i = 2, 3, \\ \mathcal{E}_t^{R_1} + (P^{R_1} U_1^{R_1})_x = 0, \end{cases} \tag{13}$$

where $P^{R_1} = p(V^{R_1}, \Theta^{R_1}) = \frac{2\Theta^{R_1}}{3V^{R_1}}$ and $\mathcal{E}^{R_1} = \Theta^{R_1} + \frac{|U^{R_1}|^2}{2}$. The properties of the rarefaction wave profile can be summarized as follows.

Lemma 2.1[cf. Xin(1993)] The approximate rarefaction waves $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t,x)$ constructed in (12) have the following properties:

(1) $U^{R_1}_{1x}(t,x) > 0$ for $x \in \mathbf{R}$, $t > 0$;

(2) For any $1 \le p \le +\infty$, the following estimates holds,

$$\|(V^{R_1}, U^{R_1}_1, \Theta^{R_1})_x\|_{L^p(dx)} \le C \min\left\{\delta^{R_1}\sigma^{-1+1/p},\ (\delta^{R_1})^{1/p} t^{-1+1/p}\right\},$$
$$\|(V^{R_1}, U^{R_1}_1, \Theta^{R_1})_{xx}\|_{L^p(dx)} \le C \min\left\{\delta^{R_1}\sigma^{-2+1/p},\ \sigma^{-1+1/p} t^{-1}\right\},$$

where the positive constant C depends only on p and the wave strength;

(3) If $x \geq \lambda_{1+}^{R_1} t$, then

$$|(V^{R_1}, U^{R_1}, \Theta^{R_1})(t,x) - (v_+, u_+, \theta_+)| \leq Ce^{-\frac{2|x-\lambda_{1+}t|}{\sigma}},$$

$$|\partial_x^k (V^{R_1}, U^{R_1}, \Theta^{R_1})(t,x)| \leq \frac{C}{\sigma^k} e^{-\frac{2|x-\lambda_{1+}t|}{\sigma}}, \ k = 1, 2;$$

(4) There exist positive constants C and σ_0 such that for $\sigma \in (0, \sigma_0)$ and $t > 0$,

$$\sup_{x \in \mathbf{R}} |(V^{R_1}, U^{R_1}, \mathcal{E}^{R_1})(t,x) - (v^{r_1}, u^{r_1}, E^{r_1})(\frac{x}{t})| \leq \frac{C}{t}[\sigma \ln(1+t) + \sigma|\ln \sigma|].$$

Hyperbolic Wave I

To capture the propagation of the extra "large mass" induced by hyperbolicity of the rarefaction wave profile in the viscous setting. Consider a linear system

$$
\begin{cases}
d_{1t} - d_{2x} = 0, \\
d_{2t} + (p_v^{R_1} d_1 + p_{u_1}^{R_1} d_2 + p_E^{R_1} d_3)_x = \frac{4}{3}\varepsilon\big(\frac{\mu(\Theta^{R_1})U_{1x}^{R_1}}{V^{R_1}}\big)_x, \\
d_{3t} + [(pu_1)_v^{R_1} d_1 + (pu_1)_{u_1}^{R_1} d_2 + (pu_1)_E^{R_1} d_3]_x \\
\qquad = \varepsilon\big(\frac{\kappa(\Theta^{R_1})\Theta_x^{R_1}}{V^{R_1}}\big)_x + \frac{4}{3}\varepsilon\big(\frac{\mu(\Theta^{R_1})U_1^{R_1}U_{1x}^{R_1}}{V^{R_1}}\big)_x,
\end{cases}
\tag{14}
$$

where $p = \frac{R\theta}{v} = p(v,u,E) = \frac{2E-u^2}{3v}$ and $p_v^{R_1} = p_v(V^{R_1}, U^{R_1}, \mathcal{E}^{R_1})$ etc.

Now we set

$$(D_1, D_2, D_3)^t = L^{R_1}(d_1, d_2, d_3)^t, \tag{15}$$

where $L^{R_1} = L^{R_1}(V^{R_1}, U^{R_1}, s_+)$ is the matrix defined by the left eigenvectors $l_i^{R_1}(V^{R_1}, U^{R_1}, s_+), i = 1, 2, 3$.
Then

$$\begin{cases} D_{1t} + (\lambda_1^{R_1} D_1)_x = b_{12}^{R_1} H_1^{R_1} + b_{13}^{R_1} H_2^{R_1} + a_{12}^{R_1} V_x^{R_1} D_2 + a_{13}^{R_1} V_x^{R_1} D_3, \\ D_{2t} = b_{22}^{R_1} H_1^{R_1} + b_{23}^{R_1} H_2^{R_1} + a_{22}^{R_1} V_x^{R_1} D_2 + a_{23}^{R_1} V_x^{R_1} D_3, \\ D_{3t} + (\lambda_3^{R_1} D_3)_x = b_{32}^{R_1} H_1^{R_1} + b_{33}^{R_1} H_2^{R_1} + a_{32}^{R_1} V_x^{R_1} D_2 + a_{33}^{R_1} V_x^{R_1} D_3, \end{cases} \tag{16}$$

where $H_1^{R_1} = \frac{4}{3}\varepsilon(\frac{\mu(\Theta^{R_1})U_{1x}^{R_1}}{V^{R_1}})_x$, $H_2^{R_1} = \varepsilon(\frac{\kappa(\Theta^{R_1})\Theta_x^{R_1}}{V^{R_1}})_x + \frac{4}{3}\varepsilon(\frac{\mu(\Theta^{R_1})U_1^{R_1}U_{1x}^{R_1}}{V^{R_1}})_x$.

Note that the equations of D_2, D_3 are decoupled from D_1 due to the intrinsic property of the rarefaction wave.

Boundary condition to the above linear hyperbolic system (16) in the domain $(t, x) \in [h, T] \times \mathbf{R}$:

$$D_1(t = h, x) = 0, \qquad D_2(t = T, x) = D_3(t = T, x) = 0. \tag{17}$$

Lemma 2.2 There exists a positive constant $C_{h,T}$ independent of ε such that

(1)

$$\|\frac{\partial^k}{\partial x^k} d_i(t,\cdot)\|^2_{L^2(dx)} \leq C_{h,T}\,\frac{\varepsilon^2}{\sigma^{2k+1}}, \quad i=1,2,3,\ k=0,1,2,3.$$

(2) If $x > \lambda_{1+}t$, then we have

$$|d_i(x,t)| \leq C_{h,T}\,\frac{1}{\sigma}e^{-\frac{|x-\lambda_{1+}t|}{\sigma}},$$
$$|d_{ix}(x,t)| \leq C_{h,T}\,\frac{1}{\sigma^2}e^{-\frac{|x-\lambda_{1+}t|}{\sigma}},\ i=1,2,3.$$

Viscous Contact Wave

We construct the viscous contact wave $(V^{CD}, U^{CD}, \Theta^{CD})(t,x)$ satisfies the following system

$$\begin{cases} V_t^{CD} - U_{1x}^{CD} = 0, \\ U_{1t}^{CD} + P_x^{CD} = \frac{4\varepsilon}{3}(\frac{\mu(\Theta^{CD})}{V^{CD}} U_{1x}^{CD})_x - \int \xi_1^2 \Pi_{11x}^{CD} d\xi + Q_1^{CD}, \\ U_{it}^{CD} = \varepsilon(\frac{\mu(\Theta^{CD})}{V^{CD}} U_{ix}^{CD})_x - \int \xi_1 \xi_i \Pi_{11x}^{CD} d\xi + Q_i^{CD}, i = 2, 3, \\ \mathcal{E}_t^{CD} + (P^{CD} U_1^{CD})_x = \varepsilon(\frac{\kappa(\Theta^{CD})}{V^{CD}} \Theta_x^{CD})_x + \frac{4\varepsilon}{3}(\frac{\mu(\Theta^{CD}) U_1^{CD} U_{1x}^{CD}}{V^{CD}})_x \\ \qquad + \sum_{i=2}^{3} \varepsilon(\frac{\mu(\Theta^{CD}) U_i^{CD} U_{ix}^{CD}}{V^{CD}})_x - \int \xi_1 \frac{|\xi|^2}{2} \Pi_{11x}^{CD} d\xi + Q_4^{CD}, \end{cases} \tag{18}$$

where

$$\mathbf{G}^{CD}(t,x,\xi)=\frac{3\varepsilon}{2V^{CD}\Theta^{CD}}\mathbf{L}^{-1}_{\mathbf{M}^{CD}}\{\mathbf{P}^{CD}_1[\xi_1(\frac{|\xi-U^{CD}|^2}{2\Theta^{CD}}\Theta^{CD}_x+\xi\cdot U^{CD}_x)\mathbf{M}^{CD}]\},\tag{19}$$

$$\Pi^{CD}_{11}=\mathbf{L}^{-1}_{\mathbf{M}^{CD}}\big[\varepsilon(-\frac{U^{CD}_1}{V^{CD}}\mathbf{G}^{CD}_x+\frac{1}{V^{CD}}\mathbf{P}^{CD}_1(\xi_1\mathbf{G}^{CD}_x))-Q(\mathbf{G}^{CD},\mathbf{G}^{CD})\big],\tag{20}$$

and $Q^{CD}_i=O(1)\delta^{CD}\varepsilon(1+t)^{-2}e^{-\frac{cx^2}{\varepsilon(1+t)}}, \qquad \text{as } x\to\pm\infty,\ \ i=1,2,3,4.$

Shock Wave

The shock profile can be written as $F^{S_3}(x - s_3 t, \xi)$ that satisfies

$$\begin{cases} -\bar{s}_3(F^{S_3})' + \xi_1(F^{S_3})' = \dfrac{1}{\varepsilon} Q(F^{S_3}, F^{S_3}), \\ F^{S_3}(\pm\infty, \xi) = \mathbf{M}_\pm(\xi) := \mathbf{M}_{[v_\pm, u_\pm, \theta_\pm]}(\xi), \end{cases} \tag{21}$$

where $' = \frac{d}{d\eta}, \eta = x - s_3 t$, and $(v_\pm, u_\pm, \theta_\pm)$ satisfy Rankine-Hugoniot condition and Lax entropy condition and s_3 is 3-shock wave speed.

Then the corresponding fluid terms and non-fluid term satisfy

$$\begin{cases} V_t^{S_3} - U_{1x}^{S_3} = 0, \\ U_{1t}^{S_3} + P_x^{S_3} = \frac{4}{3}\varepsilon(\frac{\mu(\Theta^{S_3})U_{1x}^{S_3}}{V^{S_3}})_x - \int \xi_1^2 \Pi_{1x}^{S_3} d\xi, \\ U_{it}^{S_3} = \varepsilon(\frac{\mu(\Theta^{S_3})U_{ix}^{S_3}}{V^{S_3}})_x - \int \xi_1\xi_i \Pi_{1x}^{S_3} d\xi, \ i = 2, 3, \\ \mathcal{E}_t^{S_3} + (P^{S_3}U_1^{S_3})_x = \varepsilon(\frac{\kappa(\Theta^{S_3})\Theta_x^{S_3}}{V^{S_3}})_x + \frac{4}{3}\varepsilon(\frac{\mu(\Theta^{S_3})U_1^{S_3}U_{1x}^{S_3}}{V^{S_3}})_x \\ \qquad + \varepsilon\sum_{i=2}^{3}(\frac{\mu(\Theta^{S_3})U_i^{S_3}U_{ix}^{S_3}}{V^{S_3}})_x - \int \xi_1 \frac{|\xi|^2}{2}\Pi_{1x}^{S_3} d\xi, \end{cases} \tag{22}$$

and

$$\mathbf{G}_t^{S_3} - \frac{U_1^{S_3}}{V^{S_3}}\mathbf{G}_x^{S_3} + \frac{1}{V^{S_3}}\mathbf{P}_1^{S_3}(\xi_1\mathbf{M}_x^{S_3}) + \frac{1}{V^{S_3}}\mathbf{P}_1^{S_3}(\xi_1\mathbf{G}_x^{S_3})$$
$$= \frac{1}{\varepsilon}\left[\mathbf{L}_{\mathbf{M}^{S_3}}\mathbf{G}^{S_3} + Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_3})\right],$$

where $\mathbf{L}_{\mathbf{M}^{S_3}}$ is the linearized collision operator of $Q(F^{S_3}, F^{S_3})$ with respect to the local Maxwellian $\mathbf{M}^{S_3}$:

$$\mathbf{L}_{\mathbf{M}^{S_3}}g = 2Q(\mathbf{M}^{S_3}, g) = Q(\mathbf{M}^{S_3}, g) + Q(g, \mathbf{M}^{S_3}),$$

and

$$\begin{cases} \mathbf{G}^{S_3} = \varepsilon\mathbf{L}_{\mathbf{M}^{S_3}}^{-1}\left[\frac{1}{V^{S_3}}\mathbf{P}_1^{S_3}(\xi_1\mathbf{M}_x^{S_3})\right] + \Pi_1^{S_3}, \\ \Pi_1^{S_3} = \mathbf{L}_{\mathbf{M}^{S_3}}^{-1}\left[\varepsilon\left(\mathbf{G}_t^{S_3} - \frac{U_1^{S_3}}{V^{S_3}}\mathbf{G}_x^{S_3} + \frac{1}{V^{S_3}}\mathbf{P}_1^{S_3}(\xi_1\mathbf{G}_x^{S_3})\right) - Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_3})\right]. \end{cases} \quad (23)$$

Now we recall the properties of the shock profile obtained by Liu-Yu(2011).

Lemma 2.3 Assume that $(v_-, u_-, \theta_-) \in S_3(v_+, u_+, \theta_+)$, and the shock wave strength small enough, then there exists a unique shock profile $F^{S_3}(\eta, \xi)$ with $\eta = x - s_3 t$ up to a shift, to the Boltzmann equation in Lagrangian coordinate. Moreover, there are positive constants $c_\pm$ and C such that for $\eta \in \mathbf{R}$,

$$\begin{cases} s_3 V^{S_3}_\eta = -U^{S_3}_{1\eta} > 0, \\ U^{S_3}_i \equiv 0, \ \int \xi_1 \xi_i \Pi^{S_3}_1 d\xi \equiv 0, \ \ i = 2, 3, \\ (|V^{S_3} - v_\pm|, |U^{S_3}_1 - u_{1\pm}|, |\Theta^{S_3} - \theta_\pm|) \leq C\delta^{S_3} e^{-\frac{c_\pm \delta^{S_3}|\eta|}{\varepsilon}}, \quad \text{as } \ \eta \to \pm\infty, \\ \Big(\int \frac{\nu(|\xi|)|\mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi\Big)^{\frac{1}{2}} \leq C(\delta^{S_3})^2 e^{-c_\pm \frac{\delta^{S_3}|\eta|}{\varepsilon}}, \quad \text{as } \ \eta \to \pm\infty. \end{cases}$$

Furthermore, we have

$$V_\eta^{S_3} \sim U_{1\eta}^{S_3} \sim \Theta_\eta^{S_3} \sim \frac{1}{\varepsilon}\Big(\int \frac{\nu(|\xi|)|\mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi\Big)^{\frac{1}{2}},$$

and

$$|\partial_\eta^k(V^{S_3}, U_1^{S_3}, \Theta^{S_3})| \leq C\frac{(\delta^{S_3})^{k-1}}{\varepsilon^{k-1}}|(V_\eta^{S_3}, U_{1\eta}^{S_3}, \Theta_\eta^{S_3})|, \quad k \geq 2,$$

$$\Big(\int \frac{\nu(|\xi|)|\partial_\eta^k \mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi\Big)^{\frac{1}{2}} \leq C\frac{(\delta^{S_3})^k}{\varepsilon^k}\Big(\int \frac{\nu(|\xi|)|\mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi\Big)^{\frac{1}{2}}, \quad k \geq 1,$$

and

$$|\int \xi_1 \varphi_i(\xi) \Pi_{1\eta}^{S_3} d\xi| \leq C\delta^{S_3}|U_{1\eta}^{S_3}|, \quad i = 1, 2, 3, 4,$$

with $\varphi_i(\xi)$ being the collision invariants.

Hyperbolic Wave II

Now we are going to construct the second family of hyperbolic wave. By the presentation so far, we can define an approximate superposition wave $(\bar{V}, \bar{U}, \bar{\mathcal{E}})(t,x)$ by

$$\begin{pmatrix} \bar{V} \\ \bar{U}_1 \\ \bar{\mathcal{E}} \end{pmatrix}(t,x) = \begin{pmatrix} V^{R_1} + d_1 + V^{CD} + V^{S_3} \\ U_1^{R_1} + d_2 + U_1^{CD} + U_1^{S_3} \\ \mathcal{E}^{R_1} + d_3 + \mathcal{E}^{CD} + \mathcal{E}^{S_3} \end{pmatrix}(t,x) - \begin{pmatrix} v_* + v^* \\ u_{1*} + u_1^* \\ E_* + E^* \end{pmatrix},$$

$$\bar{U}_i = U_i^{CD}, i = 2, 3, \tag{24}$$

where $\bar{\mathcal{E}} = \bar{\Theta} + \frac{|\bar{U}|^2}{2}$, $(V^{R_1}, U_1^{R_1}, \mathcal{E}^{R_1})(t,x)$ is the 1-rarefaction wave defined in (12) with the right state (v_+, u_{1+}, E_+) replaced by (v_*, u_{1*}, E_*), $(V^{CD}, U_1^{CD}, \mathcal{E}^{CD})(t,x)$ is the viscous contact wave defined in (18) with the states (v_-, u_{1-}, E_-) and (v_+, u_{1+}, E_+) replaced by (v_*, u_{1*}, E_*) and (v^*, u_1^*, E^*) respectively,

and $(V^{S_3}, U_1^{S_3}, \mathcal{E}^{S_3})(t,x)$ is the fluid part of 3-shock profile of Boltzmann equation defined in (22) with the left state (v_-, u_{1-}, E_-) replaced by (v^*, u_1^*, E^*).
Moreover, we can check that this profile satisfies

$$\begin{cases} \bar{V}_t - \bar{U}_{1x} = 0, \\ \bar{U}_{1t} + \bar{P}_x = \frac{4}{3}\varepsilon(\frac{\mu(\bar{\Theta})\bar{U}_{1x}}{\bar{V}})_x - \int \xi_1^2 \Pi_{11x}^{CD} d\xi - \int \xi_1^2 \Pi_{1x}^{S_3} d\xi + \bar{Q}_{1x} + Q_1^{CD}, \\ \bar{U}_{it} = \varepsilon(\frac{\mu(\bar{\Theta})\bar{U}_{1x}}{\bar{V}})_x - \int \xi_1\xi_i \Pi_{11x}^{CD} d\xi - \int \xi_1\xi_i \Pi_{1x}^{S_3} d\xi + \bar{Q}_{ix} + Q_i^{CD}, \quad i = 2,3, \\ \bar{\mathcal{E}}_t + (\bar{P}\bar{U}_1)_x = \varepsilon(\frac{\kappa(\bar{\Theta})\bar{\Theta}_x}{\bar{V}})_x + \frac{4}{3}\varepsilon(\frac{\mu(\bar{\Theta})\bar{U}_1\bar{U}_{1x}}{\bar{V}})_x + \sum_{i=2}^{3}\varepsilon(\frac{\mu(\bar{\Theta})\bar{U}_i\bar{U}_{ix}}{\bar{V}})_x \\ \qquad - \int \xi_1 \frac{|\xi|^2}{2} \Pi_{11x}^{CD} d\xi - \int \xi_1 \frac{|\xi|^2}{2} \Pi_{1x}^{S_3} d\xi + \bar{Q}_{4x} + Q_4^{CD}, \end{cases} \tag{25}$$

where

$$|(\bar{Q}_{11},\bar{Q}_2,\bar{Q}_3,\bar{Q}_{41})| = C_{h,T}\, e^{-\frac{C_h|x|}{\sigma}} e^{-\frac{C_h}{\sigma}} + O(1)\Big[|(d_1,d_2,d_3)|^2 + \varepsilon|(d_{2x},d_{3x})| + \varepsilon|(U_{1x}^{R_1},\Theta_x^{R_1})||(d_1,d_2,d_3)|\Big],$$

with $\sigma = \varepsilon^{\frac{1}{5}}$ and for some positive constants $C_{h,T}$ and C_h independent of ε. In order to remove the non-conservative error terms $Q_i^{CD}, (i=1,2,3,4)$ coming from the definition of the viscous contact wave, we now introduce the following hyperbolic wave $\vec{b} \triangleq (b_1,b_{21},b_{22},b_{23},b_3)$ and $\vec{b_2} = (b_{12},b_{22},b_{23})$:

$$\begin{cases} b_{1t} - b_{21x} = 0, \\ b_{21t} + [\bar{P}_v b_1 + \bar{P}_{u_1} b_{21} + \bar{P}_{u_2} b_{22} + \bar{P}_{u_3} b_{23} + \bar{P}_E b_3]_x = -Q_1^{CD}, \\ b_{22t} = -Q_2^{CD}, \\ b_{23t} = -Q_3^{CD}, \\ b_{3t} + [(\bar{P}\bar{U}_1)_v b_1 + (\bar{P}\bar{U}_1)_{u_1} b_{21} + (\bar{P}\bar{U}_1)_{u_2} b_{22} + (\bar{P}\bar{U}_1)_{u_3} b_{23} + (\bar{P}\bar{U}_1)_E b_3]_x \\ \qquad = -Q_4^{CD}, \end{cases}$$

Diagonalization:

$$\vec{B} \triangleq (B_1, B_{21}, B_{22}, B_{23}, B_3)^t = \bar{L} \cdot (b_1, b_{21}, b_{22}, b_{23}, b_3), \tag{26}$$

where $\bar{L} = \bar{L}(\bar{V}, \bar{U}, \bar{\mathcal{E}})$ is the matrix defined by the left eigenvectors $\bar{l}_i = \bar{l}_1(\bar{V}, \bar{U}, \bar{\mathcal{E}}), i = 1, 2, 3, 4, 5$. So we obtain a diagonalized system

$$\begin{cases} B_{1t} + (\bar{\lambda}_1 B_1)_x = \bar{l}_1 \cdot \vec{Q}^{CD} + \sum\limits_{i=1,3} (\bar{l}_{1t} + \bar{\lambda}_i \bar{l}_{1x}) \cdot \bar{r}_i B_i + \bar{l}_{1t} \cdot \sum\limits_{j=1}^{3} \bar{r}_{2j} B_{2j}, \\ B_{21t} \;\; = \bar{l}_{21} \cdot \vec{Q}^{CD} + \sum\limits_{i=1,3} (\bar{l}_{21t} + \bar{\lambda}_i \bar{l}_{21x}) \cdot \bar{r}_i B_i + \bar{l}_{21t} \cdot \sum\limits_{j=1}^{3} \bar{r}_{2j} B_{2j}, \\ B_{22t} \;\; = \bar{l}_{22} \cdot \vec{Q}^{CD}, \\ B_{23t} \;\; = \bar{l}_{23} \cdot \vec{Q}^{CD}, \\ B_{3t} + (\bar{\lambda}_3 B_3)_x = \bar{l}_3 \cdot \vec{Q}^{CD} + \sum\limits_{i=1,3} (\bar{l}_{3t} + \bar{\lambda}_i \bar{l}_{3x}) \cdot \bar{r}_i B_i + \bar{l}_{3t} \cdot \sum\limits_{j=1}^{3} \bar{r}_{2j} B_{2j}. \end{cases} \tag{27}$$

Now we impose the following boundary condition to the linear hyperbolic system (27) on the domain $(t, x) \in [h, T] \times \mathbf{R}$:

$$(B_1, B_{21}, B_{22}, B_{23}, B_3)(t = T, x) = 0. \tag{28}$$

We can solve the linear diagonalized hyperbolic system (27) under the condition (28) to have the following lemma.
Lemma 2.4 There exists a positive constant δ_0 such that if the wave strength $\delta \le \delta_0$, then there exists a positive constant $C_{h,T}$ which is independent of ε, such that

$$\begin{aligned} &\|\frac{\partial^k}{\partial x^k}(b_1, b_{21}, b_{22}, b_{23}, b_3)(t, \cdot)\|^2_{L^2(dx)} \\ &\qquad + \int_h^T \|\sqrt{|U^{S_3}_{1x}|}\frac{\partial^k}{\partial x^k}(b_1, b_{21}, b_{22}, b_{23}, b_3)(t, \cdot)\|^2_{L^2(dx)} dt \\ &\le C_{h,T}\, \varepsilon^{\frac{5}{2}-2k},\ k = 0, 1, 2, 3. \end{aligned} \tag{29}$$

Superposition Of Wave

With the above preparation, finally, the approximate superposition wave $(V, U, \mathcal{E})(t, x)$ can be defined by

$$\begin{pmatrix} V \\ U_i \\ \mathcal{E} \end{pmatrix}(t,x) = \begin{pmatrix} \bar{V} + b_1 \\ \bar{U}_i + b_{2i} \\ \bar{\mathcal{E}} + b_3 \end{pmatrix}(t,x), \quad i = 1,2,3, \tag{30}$$

where $\mathcal{E} = \Theta + \frac{|U|^2}{2}$.

Then the approximate wave pattern $(V, U, \mathcal{E}, \Theta)(t,x)$ satisfies

$$
\begin{cases}
V_t - U_{1x} = 0, \\
U_{1t} + P_x = \frac{4}{3}\varepsilon(\frac{\mu(\Theta)U_{1x}}{V})_x - \int \xi_1^2 \Pi_{11x}^{CD} d\xi - \int \xi_1^2 \Pi_{1x}^{S_3} d\xi + \bar{Q}_{1x} + Q_{1x}, \\
U_{it} = \varepsilon(\frac{\mu(\Theta)U_{ix}}{V})_x - \int \xi_1\xi_i \Pi_{11x}^{CD} d\xi - \int \xi_1\xi_i \Pi_{1x}^{S_3} d\xi + \bar{Q}_{ix} + Q_{ix}, \quad i = 2,3, \\
\mathcal{E}_t + (PU_1)_x = \varepsilon(\frac{\kappa(\Theta)\Theta_x}{V})_x + \frac{4}{3}\varepsilon(\frac{\mu(\Theta)U_1U_{1x}}{V})_x + \sum_{i=2}^{3}\varepsilon(\frac{\mu(\Theta)U_iU_{ix}}{V})_x \\
\qquad - \int \xi_1 \frac{|\xi|^2}{2} \Pi_{11x}^{CD} d\xi - \int \xi_1 \frac{|\xi|^2}{2} \Pi_{1x}^{S_3} d\xi + \bar{Q}_{4x} + Q_{4x},
\end{cases}
\tag{31}
$$

where $P = p(V,\Theta)$ and

$$
\begin{aligned}
Q_1 &= \Big[P - \bar{P} - (\bar{P}_v b_1 + \bar{P}_u \cdot b_2 + \bar{P}_E b_3)\Big] - \frac{4}{3}\varepsilon\Big[\frac{\mu(\Theta)U_{1x}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_{1x}}{\bar{V}}\Big], \\
&:= Q_{11} + Q_{12}, \\
Q_i &= -\varepsilon\Big[\frac{\mu(\Theta)U_{ix}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_{ix}}{\bar{V}}\Big], \quad i = 2,3, \\
Q_4 &= \Big[PU_1 - \bar{P}\bar{U}_1 - ((\bar{P}\bar{U}_1)_v b_1 + (\bar{P}\bar{U}_1)_u \cdot b_2 + (\bar{P}\bar{U}_1)_E b_3)\Big] \\
&\quad -\varepsilon\Big[(\frac{\kappa(\Theta)\Theta_x}{V} - \frac{\kappa(\bar{\Theta})\bar{\Theta}_x}{\bar{V}}) + \frac{4}{3}(\frac{\mu(\Theta)U_1U_{1x}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_1\bar{U}_{1x}}{\bar{V}}) \\
&\qquad + \sum_{i=2}^{3}(\frac{\mu(\Theta)U_iU_{ix}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_i\bar{U}_{ix}}{\bar{V}})\Big] \\
&:= Q_{41} + Q_{42}.
\end{aligned} \tag{32}
$$

Straightforward calculation shows that

$$(Q_{11}, Q_{41}) = O(1)|\vec{b}|^2. \tag{33}$$

Reformulation of the Problem

We now reformulate the system by introducing a scaling for the independent variables. Set

$$y = \frac{x}{\varepsilon}, \;\; \tau = \frac{t}{\varepsilon}. \tag{34}$$

In the following, we will also use the notations $(v, u, \theta)(\tau, y), \mathbf{G}(\tau, y, \xi), \Pi_1(\tau, y, \xi)$ and $(V, U, \Theta)(\tau, y)$, etc., in the scaled independent variables. Set the perturbation around the superposition wave $(V, U, \Theta)(\tau, y)$ by

$$\begin{aligned} &(\phi, \psi, \omega, \zeta)(\tau, y) = (v - V, u - U, E - \mathcal{E}, \theta - \Theta)(\tau, y), \\ &\widetilde{\mathbf{G}}(\tau, y, \xi) = \mathbf{G}(\tau, y, \xi) - \mathbf{G}^{S_3}(\tau, y, \xi), \\ &\widetilde{f}(\tau, y, \xi) = f(\tau, y, \xi) - F^{S_3}(\tau, y, \xi). \end{aligned} \tag{35}$$

Under this scaling, the hydrodynamic limit problem is reduced to a time asymptotic stability problem for the Boltzmann equation.

In particular, we can choose the initial value as

$$(\phi,\psi,\omega)(\tau=\frac{h}{\varepsilon},y)=(0,0,0),\quad \widetilde{\mathbf{G}}(\tau=\frac{h}{\varepsilon},y,\xi)=0. \tag{36}$$

Introduce the anti-derivative variables

$$(\Phi,\Psi,\bar{W})(\tau,y)=\int_{-\infty}^{y}(\phi,\psi,\omega)(\tau,y')dy'.$$

Then $(\Phi,\Psi,\bar{W})(\tau,y)$ satisfies that

$$\begin{cases} \Phi_\tau-\Psi_{1y}=0,\\ \Psi_{1\tau}+(p-P)=\frac{4}{3}\Big(\frac{\mu(\theta)u_{1y}}{v}-\frac{\mu(\Theta)U_{1y}}{V}\Big)-\int\xi_1^2(\Pi_1-\Pi_{11}^{CD}-\Pi_1^{S_3})d\xi-\bar{Q}_1-Q_1,\\ \Psi_{i\tau}=\Big(\frac{\mu(\theta)u_{iy}}{v}-\frac{\mu(\Theta)U_{iy}}{V}\Big)-\int\xi_1\xi_i(\Pi_1-\Pi_{11}^{CD}-\Pi_1^{S_3})d\xi-\bar{Q}_i-Q_i,\quad i=2,3,\\ \bar{W}_\tau+(pu_1-PU_1)=\Big(\frac{\kappa(\theta)\theta_y}{v}-\frac{\kappa(\Theta)\Theta_y}{V}\Big)+\frac{4}{3}\Big(\frac{\mu(\theta)u_1u_{1y}}{v}-\frac{\mu(\Theta)U_1U_{1y}}{V}\Big)\\ \quad+\sum_{i=2}^{3}\Big(\frac{\mu(\theta)u_iu_{iy}}{v}-\frac{\mu(\Theta)U_iU_{iy}}{V}\Big)-\int\xi_1\frac{|\xi|^2}{2}(\Pi_1-\Pi_{11}^{CD}-\Pi_1^{S_3})d\xi-\bar{Q}_4-Q_4. \end{cases} \tag{37}$$

To precisely capture the dissiaption of heat conduction, we introduce another variable related to the absolute temperature

$$W=\bar{W}-U\cdot\Psi=\bar{W}-\sum_{i=1}^{3}U_i\Psi_i,$$

then

$$\zeta=W_y-(\frac{|\Psi_y|^2}{2}-U_y\cdot\Psi). \tag{38}$$

For the non-fluid component $\widetilde{\mathbf{G}}(\tau,y,\xi)$, we have

$$\begin{aligned}\widetilde{\mathbf{G}}_\tau-\mathbf{L}_{\mathbf{M}}\widetilde{\mathbf{G}} &= \frac{u_1}{v}\widetilde{\mathbf{G}}_y-\frac{1}{v}\mathbf{P}_1(\xi_1\widetilde{\mathbf{G}}_y)-\Big[\frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{M}_y)-\frac{1}{V^{S_3}}\mathbf{P}_1^{S_3}(\xi_1\mathbf{M}_y^{S_3})\Big]\\ &+2Q(\widetilde{\mathbf{G}},\mathbf{G}^{S_3})+Q(\widetilde{\mathbf{G}},\widetilde{\mathbf{G}})+J_1,\end{aligned} \tag{39}$$

where

$$J_1=\big(\mathbf{L}_{\mathbf{M}}-\mathbf{L}_{\mathbf{M}^{S_3}}\big)\mathbf{G}^{S_3}+(\frac{u}{v}-\frac{U_1^{S_3}}{V^{S_3}})\mathbf{G}_y^{S_3}-\Big[\frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{G}_y^{S_3})-\frac{1}{V^{S_3}}\mathbf{P}_1^{S_3}(\xi_1\mathbf{G}_y^{S_3})\Big]. \tag{40}$$

Let

$$\mathbf{G}^{R_1}(\tau, y, \xi) = \frac{3}{2v\theta}\mathbf{L}_{\mathbf{M}}^{-1}\{\mathbf{P}_1[\xi_1(\frac{|\xi - u|^2}{2\theta}\Theta_y^{R_1} + \xi \cdot U_y^{R_1})\mathbf{M}]\}, \tag{41}$$

and

$$\widetilde{\mathbf{G}}_1(\tau, y, \xi) = \widetilde{\mathbf{G}}(\tau, y, \xi) - \mathbf{G}^{R_1}(\tau, y, \xi) - \mathbf{G}^{CD}(\tau, y, \xi), \tag{42}$$

where $\mathbf{G}^{CD}(\tau, y, \xi)$ is defined in (19).
From scaling transformation (34), we have

$$f_\tau - \frac{u_1}{v}f_y + \frac{\xi_1}{v}f_y = Q(f, f). \tag{43}$$

Thus, we have the equation for $\widetilde{f}$ defined in (35)

$$\widetilde{f}_\tau - \frac{u_1}{v}\widetilde{f}_y + \frac{\xi_1}{v}\widetilde{f}_y = \mathbf{L}_{\mathbf{M}}\widetilde{\mathbf{G}} + Q(\widetilde{\mathbf{G}}, \widetilde{\mathbf{G}}) + J_F, \tag{44}$$

with

$$J_F = (\frac{u_1}{v} - \frac{U_1^{S_3}}{V^{S_3}})F_y^{S_3} - (\frac{1}{v} - \frac{1}{V^{S_3}})\xi_1 F_y^{S_3} + 2Q(\mathbf{M} - \mathbf{M}^{S_3}, \mathbf{G}^{S_3}) + 2Q(\widetilde{\mathbf{G}}, \mathbf{G}^{S_3}). \tag{45}$$

Energy Estimates

Note that to prove the main result Theorem 1, it is sufficient to prove the following theorem on the Boltzmann equation in the scaled independent variables based on the construction of the approximate wave pattern.

Theorem 2: There exist a small positive constants δ_1 and a global Maxwellian $\mathbf{M}_\star = \mathbf{M}_{[v_\star, u_\star, \theta_\star]}$ such that if the wave strength δ satisfies $\delta \le \delta_1$, then on the time interval $[\frac{h}{\varepsilon}, \frac{T}{\varepsilon}]$ for any $0 < h < T$, there is a positive constant $\varepsilon_1(\delta, h, T)$. If the Knudsen number $\varepsilon \le \varepsilon_1$, then the Boltzmann equation admits a family of smooth solution $f^{\varepsilon,h}(\tau, y, \xi)$ satisfying

$$\sup_{\tau \in [\frac{h}{\varepsilon}, \frac{T}{\varepsilon}]} \sup_{y \in \mathbf{R}} \| f^{\varepsilon,h}(\tau, y, \xi) - \mathbf{M}_{[V,U,\Theta]}(\tau, y, \xi) \|_{L^2_\xi(\frac{1}{\sqrt{\mathbf{M}_\star}})} \le C\varepsilon^{\frac{1}{5}}. \tag{46}$$

Since the local existence of solution is standard. To prove the existence on the time interval $[\frac{h}{\varepsilon}, \frac{T}{\varepsilon}]$, we only need to close the following a priori estimate by the continuity argument. Set

$$\mathcal{N}(\tau) = \sup_{\frac{h}{\varepsilon} \le \tau' \le \tau} \Big\{ \|(\Phi, \Psi, W)(\tau', \cdot)\|^2 + \|(\phi, \psi, \zeta)(\tau', \cdot)\|_1^2 + \int\int \frac{|\widetilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star} d\xi dy + \sum_{|\alpha'|=1} \int\int \frac{|\partial^{\alpha'} \widetilde{\mathbf{G}}|^2}{\mathbf{M}_\star} d\xi dy + \sum_{|\alpha|=2} \int\int \frac{|\partial^{\alpha} \widetilde{f}|^2}{\mathbf{M}_\star} d\xi dy \Big\} \le \chi^2 = \varepsilon^{\frac{1}{10}}, \quad \forall \tau \in [\frac{h}{\varepsilon}, \frac{T}{\varepsilon}], \tag{47}$$

where $\partial^\alpha, \partial^{\alpha'}$ denote the derivatives with respect to y and τ, and $\mathbf{M}_\star$ is a global Maxwellian to be chosen.

To close the a priori estimate (47) and to prove Theorem 2, we need the following energy estimates given in Proposition 1 and Proposition 2. First, the lower order estimates are given in the following Proposition.

Proposition 1: Under the assumptions of Theorem 2, there exist positive constants C and $C_{h,T}$ independent of ε such that

$$\sup_{\frac{h}{\varepsilon}\le\tau_1\le\tau}\Big[\|(\Phi,\Psi,W,\Phi_y)(\tau_1,\cdot)\|^2+\int\int\frac{|\widetilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star}(\tau_1,y,\xi)d\xi dy\Big]$$
$$+\int_{\frac{h}{\varepsilon}}^{\tau}\Big[\|\sqrt{|U_{1y}^{S_3}|}(\Psi,W)\|^2+\|(\Phi_y,\Psi_y,W_y,\zeta,\Psi_\tau,W_\tau)\|^2\Big]d\tau$$
$$+\int_{\frac{h}{\varepsilon}}^{\tau}\int\int\frac{\nu(|\xi|)}{\mathbf{M}_\star}|\widetilde{\mathbf{G}}_1|^2 d\xi dy d\tau$$
$$\le C_{h,T}\,\varepsilon\int_{\frac{h}{\varepsilon}}^{\tau}\|(\Psi,W)\|^2 d\tau+C\sum_{|\alpha'|=1}\int_{\frac{h}{\varepsilon}}^{\tau}\|\partial^{\alpha'}(\phi,\psi,\zeta)\|^2 d\tau$$
$$+C\sum_{|\alpha'|=1}\int_{\frac{h}{\varepsilon}}^{\tau}\int\int\frac{\nu(|\xi|)}{\mathbf{M}_\star}|\partial^{\alpha'}\widetilde{\mathbf{G}}|^2 d\xi dy d\tau+C_{h,T}\,\varepsilon^{\frac{2}{5}}.$$

For the higher order energy estimates, we have

Proposition 2: Under the assumptions of Theorem 2, there exist positive constants C and $C_{h,T}$ independent of ε such that

$$
\begin{aligned}
&\sup_{\frac{h}{\varepsilon}\le\tau_1\le\tau}\Bigg[\|(\phi,\psi,\zeta,\phi_y,\psi_y,\zeta_y)(\tau_1,\cdot)\|^2+\sum_{|\alpha'|=1}\int\int\frac{|\partial^{\alpha'}\widetilde{\mathbf{G}}|^2}{\mathbf{M}_\star}(\tau_1,y,\xi)d\xi dy \\
&\qquad+\sum_{|\alpha|=2}\int\int\frac{|\partial^{\alpha}\widetilde{f}|^2}{2\mathbf{M}_\star}(\tau_1,y,\xi)d\xi dy\Bigg] \\
&+\int_{\frac{h}{\varepsilon}}^{\tau}\sum_{1\le|\alpha|\le2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2d\tau+\sum_{1\le|\alpha|\le2}\int_{\frac{h}{\varepsilon}}^{\tau}\int\int\frac{\nu(|\xi|)}{\mathbf{M}_\star}|\partial^\alpha\widetilde{\mathbf{G}}|^2d\xi dy d\tau \\
&\le C(\delta+C_{h,T}\chi)\int_{\frac{h}{\varepsilon}}^{\tau}\int\int\frac{\nu(|\xi|)}{\mathbf{M}_\star}|\widetilde{\mathbf{G}}_1|^2d\xi dy d\tau \\
&\qquad+C(\delta+C_{h,T}\chi)\int_{\frac{h}{\varepsilon}}^{\tau}\|(\phi,\psi,\zeta)\|^2d\tau+C_{h,T}\,\varepsilon^{\frac{1}{2}}.
\end{aligned}
$$

By combining the above lower and higher order estimates given in Proposition 1 and Proposition 2 and choosing the wave strength δ, the bound on the a priori estimate χ and the Knudsen number ε to be suitably small, we obtain

$$
\begin{aligned}
&\mathcal{N}(\tau) + \int_{\frac{h}{\varepsilon}}^{\tau} \Big[\sum_{0\le|\alpha|\le 2} \|\partial^{\alpha}(\phi,\psi,\zeta)\|^2 + \|\sqrt{|U_{1y}^{S_3}|}(\Psi, W)\|^2 \Big] d\tau \\
&+ \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)|\widetilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star} d\xi dy d\tau \\
&+ \sum_{1\le|\alpha|\le 2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)|\partial^{\alpha}\widetilde{\mathbf{G}}|^2}{\mathbf{M}_\star}(\tau, y, \xi) d\xi dy d\tau \le C_{h,T}\, \varepsilon^{\frac{2}{5}}.
\end{aligned}
$$

Therefore, we close the a priori assumption (47) and then complete the proof of Theorem 2.

Thank you!