Spectral stability of small-amplitude traveling waves via geometric singular perturbation theory

Johannes Wächtler
Konstanz University, Germany

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Overview

- Spectral stability, Evans function techniques
- Spectral stability of small shock waves associated with a degenerate mode
  joint work with H. Freistühler and P. Szmolyan

Spectral stability of shock waves

System of viscous conservation laws

$$u_t + f(u)_x = u_{xx}, \quad u \in \mathbb{R}^n$$

Shock wave

$$u(x,t) = \phi(x - st), \quad \phi(\pm\infty) = u^{\pm}, \quad \xi = x - st$$

Eigenvalue problem for ϕ: Non-autonomous linear system on $\mathbb{C}^{2n}$

$$W' = \underbrace{\begin{pmatrix} Df(\phi) - sI & I \\ \kappa I & 0 \end{pmatrix}}_{A(\phi,\kappa)} W, \quad ' = \frac{d}{d\xi} \qquad \text{(EVP)}$$

$\kappa \in \mathbb{C}$ eigenvalue $:\Leftrightarrow$ $\exists$ non-trivial sol. $W(\xi,\kappa)$, $W(\pm\infty,\kappa) = 0$.

Due to shift invariance of ϕ: $\kappa = 0$ is an eigenvalue.

ϕ **spectrally stable** $:\Leftrightarrow$

(i) There are no eigenvalues in $\mathbb{H} := \{\kappa \in \mathbb{C} \, : \, \mathrm{Re}\,\kappa \geq 0\}$ except at $\kappa = 0$.

(ii) The trivial eigenvalue $\kappa = 0$ is simple.

Spectral stability $\Rightarrow$ nonlinear stability [Zumbrun, Howard 1998]

Tool to find unstable eigenvalues:
Evans function $\mathcal{E}(\kappa)$, analytic function with

$$\kappa \text{ eigenvalue} \iff \mathcal{E}(\kappa) = 0.$$

Evans function techniques

Assume

- $\phi(\xi) \to u^{\pm}$ at exponential rate.
- $\forall\, \kappa \in \mathbb{H} \setminus \{0\}$: $A(u^{\pm}, \kappa)$ is hyperbolic with n-dimensional unstable space $U^{\pm}(\kappa)$ and n-dimensional stable space $S^{\pm}(\kappa)$ (*consistent splitting*).

Need to track the evolutions of $U^{-}(\kappa)$ and $S^{+}(\kappa)$.

$\mathcal{G}_d^{2n}(\mathbb{C})$ Grassmann manifold of d-dimensional subspaces of $\mathbb{C}^{2n}$, $d = 1, \ldots, 2n$.

(EVP) induces a flow on $\mathcal{G}_d^{2n}(\mathbb{C})$:

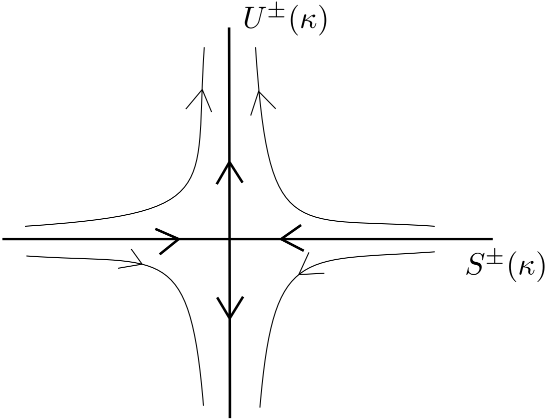
$$X' = \mathcal{A}^d(\phi, \kappa)(X).$$

For fixed $\phi = u_0$: d-dimensional invariant spaces of $A(u_0, \kappa)$ fixed points of the induced system.

Autonomous end-systems

$$W' = A(u^{\pm}, \kappa)W \quad \text{on } \mathbb{C}^{2n}$$

$U^{\pm}(\kappa)$, $S^{\pm}(\kappa)$ invariant spaces



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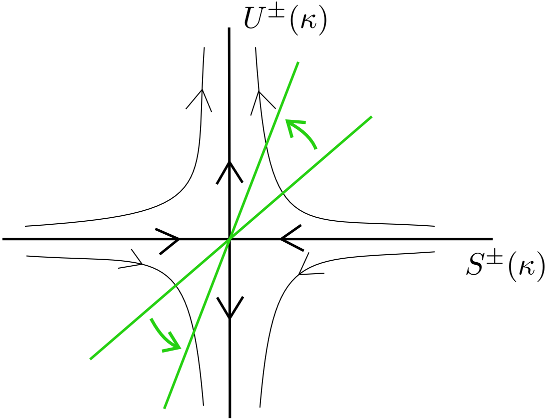
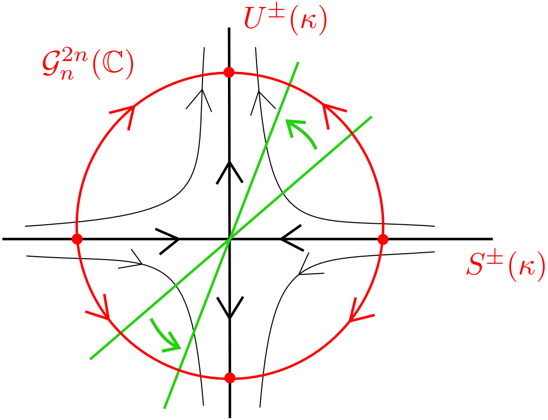


Autonomous end-systems

$$X' = \mathcal{A}^n(u^\pm, \kappa)(X) \quad \text{on } \mathcal{G}_n^{2n}(\mathbb{C})$$

$U^\pm(\kappa)$, $S^\pm(\kappa)$ invariant spaces

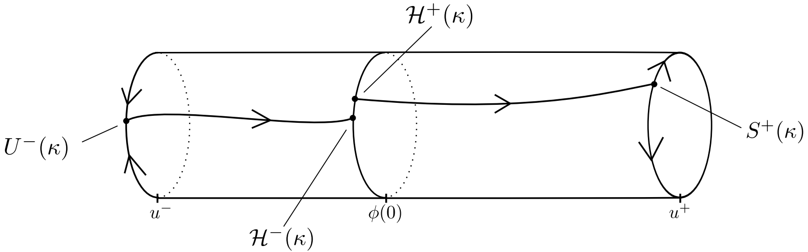


$U^\pm(\kappa)$ hyperbolic attractor, $S^\pm(\kappa)$ hyperbolic repeller

Augmented eigenvalue problem: profile equation + (EVP)

$$u' = f(u) - su - c,$$
$$X' = \mathcal{A}^n(u, \kappa)(X)$$

on $\mathbb{R}^n \times \mathcal{G}_n^{2n}(\mathbb{C})$.



$\mathcal{H}^{\pm} : \mathbb{H} \to \mathcal{G}_n^{2n}(\mathbb{C})$ unstable, stable *Evans bundles* for ϕ (analytic):

$$W(\pm\infty, \kappa) = 0 \iff W(0, \kappa) \in \mathcal{H}^{\pm}(\kappa)$$

for all $\kappa \in \mathbb{H} \setminus \{0\}$ and any sol. $W(\xi, \kappa)$ of (EVP).

$$\kappa \text{ is an eigenvalue.} \iff \mathcal{H}^-(\kappa) \cap \mathcal{H}^+(\kappa) \neq \{0\}$$

Evans function for ϕ:

$$\mathcal{E}(\kappa) := \det\left[\eta_1^-(\kappa), \ldots, \eta_n^-(\kappa), \eta_1^+(\kappa), \ldots, \eta_n^+(\kappa)\right]$$

with $\{\eta_1^\pm(\kappa), \ldots, \eta_n^\pm(\kappa)\}$ analytic bases of $\mathcal{H}^\pm(\kappa)$.

$\mathcal{E}$ analytic function with κ eigenvalue $\iff$ $\mathcal{E}(\kappa) = 0$.

ϕ spectrally stable $\iff$

(i) $\mathcal{E}(\kappa) \neq 0$ for all $\kappa \in \mathbb{H} \setminus \{0\}$

(ii) $\mathcal{E}'(0) \neq 0$

[Alexander, Gardner, Jones 1990; Gardner, Zumbrun 1998, ...]

Small shock waves

Consider a strictly hyperbolic system

$$u_t + f(u)_x = u_{xx}, \quad u \in \mathbb{R}^n.$$

$\lambda_1(u) < \cdots < \lambda_n(u)$ eigenvalues of $Df(u)$;
right eigenvectors $r_j(u)$, $j = 1, \ldots, n$.

- Small shock waves: $|u^{\pm} - u_*| \ll 1$
- Genuinely nonlinear mode

$$\nabla \lambda_k(u_*) \cdot r_k(u_*) \neq 0 :$$

Limit equation

$$u_t + (u^2)_x = u_{xx}, \quad u \in \mathbb{R}.$$

[Freistühler, Szmolyan 2002; Plaza, Zumbrun 2004]

Small shock waves associated with a degenerate mode

Let $\Sigma \subset \mathbb{R}^n$ be a smooth hypersurface with

$$\forall u \in \Sigma : \quad \nabla\lambda_k(u) \cdot r_k(u) = 0, \quad (r_k(u) \cdot \nabla)^2 \lambda_k(u) \neq 0.$$

(Σ transversal to the vector field r_k, outside Σ: $\nabla\lambda_k \cdot r_k \neq 0$)

Family of small shock waves close to Σ,

$$\phi_\varepsilon(x - s_\varepsilon t), \quad \phi_\varepsilon(\pm\infty) = u_\varepsilon^\pm, \quad 0 < \varepsilon \leq \varepsilon_0,$$

with $\lambda_k(u_\varepsilon^-) > s_\varepsilon > \lambda_k(u_\varepsilon^+)$ (non-characteristic) and end states

$$\begin{aligned} u_\varepsilon^- &= u_* - \varepsilon r_k(u_*), \\ u_\varepsilon^+ &= u_* + \varepsilon\alpha r_k(u_*) + \varepsilon^2 w(u_*, \varepsilon, \alpha), \end{aligned}$$

$u_* \in \Sigma$, $\alpha \in (-1, \frac{1}{2})$ fixed, and a vector $w(u_*, \varepsilon, \alpha) \perp r_k(u_*)$.

Theorem

Let $\mathcal{H}_\varepsilon^\pm \colon \mathbb{H} \to \mathcal{G}_n^{2n}(\mathbb{C})$ *be the Evans bundles of* ϕ_ε *and let* $\mathcal{H}_0^\pm \colon \mathbb{H} \to \mathcal{G}_1^2(\mathbb{C})$ *be the Evans bundles of the shock wave*

$$\phi_0, \quad \phi_0(-\infty) = -1,\, \phi_0(+\infty) = \alpha,$$

of the scalar viscous conservation law

$$u_t + (u^3)_x = u_{xx}, \quad u \in \mathbb{R}.$$

Define $H_\varepsilon^\pm \colon \mathbb{H} \to \mathcal{G}_n^{2n}(\mathbb{C})$ *by* $H_\varepsilon^\pm(\kappa) = \mathcal{H}_\varepsilon^\pm(\varepsilon^4\kappa)$. *It holds:*

(i) $H_\varepsilon^\pm$ *converge for* $\varepsilon \to 0$ *as analytic functions,* $\lim_{\varepsilon\to 0} H_\varepsilon^\pm = H_0^\pm$; *in suitable coordinates of* $\mathbb{C}^{2n}$:

$$H_0^-(\kappa) = \mathcal{H}_0^-(\kappa) \oplus (\mathbb{C} \times \{0\})^{n-1},$$
$$H_0^+(\kappa) = \mathcal{H}_0^+(\kappa) \oplus (\{0\} \times \mathbb{C})^{n-1}.$$

(ii) *There exist* $R > 0$, $\varepsilon_0 > 0$ *s. t.*

$$\forall\, \varepsilon \in [0, \varepsilon_0], |\kappa| \geq R : \quad H_\varepsilon^-(\kappa) \cap H_\varepsilon^+(\kappa) = \{0\}.$$

- Scaled Evans functions

$$E_\varepsilon(\kappa) = \mathcal{E}_\varepsilon(\varepsilon^4\kappa)$$

  of ϕ_ε converge to Evans function E_0 of ϕ_0 (as analytic functions).
- As ϕ_0 is stable: Theorem implies spectral stability of ϕ_ε.
- Nonlinear stability of ϕ_ε via energy methods: [Fries 2000]
- Systems with a degenerate mode appear in applications (MHD, elasticity, ...).

Sketch of the proof

General idea

- Problem has a slow-fast structure.
- Use geometric singular perturbation theory [Fenichel 1979] to construct the Evans bundles directly.
  - Find scalings to separate the distinct time scales.
  - Bundles split into slow and fast components. Use GSPT to study the limits of the Evans bundles $\mathcal{H}_\varepsilon^\pm$ for $\varepsilon \to 0$.

Assume without loss of generality

$$u_* = 0, \quad f(0) = 0,$$
$$Df(0) = \operatorname{diag}(\lambda_1^0, \ldots, \lambda_n^0) \text{ with } \lambda_j^0 = \lambda_j(0), \quad \lambda_k^0 = 0.$$

Reduction of the profile equation

ϕ_ε governed by

$$u' = f(u) - su - c.$$

Scaling $\quad u = \varepsilon\bar{u}, \quad s = \varepsilon^2\bar{s}, \quad c = \varepsilon^3\bar{c} \quad$ yields

$$\bar{u}' = \varepsilon^{-1}f(\varepsilon\bar{u}) - \varepsilon^2\bar{s}\bar{u} - \varepsilon^2\bar{c}.$$

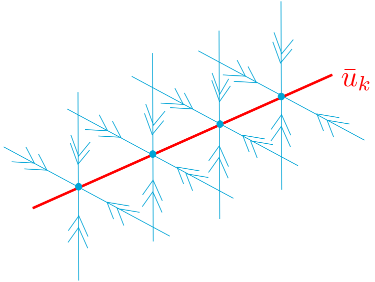
Slow-fast system in standard form:

$$\text{fast} \quad \bar{u}_j' = \lambda_j^0\bar{u}_j + \mathcal{O}(\varepsilon), \quad j \neq k,$$

$$\text{slow} \quad \bar{u}_k' = \mathcal{O}(\varepsilon)$$

As $\lambda_1^0 < \cdots < \lambda_{k-1}^0 < 0 < \lambda_{k+1}^0 < \cdots < \lambda_n^0$:

For $\varepsilon = 0$:
$M_0 = \{\bar{u}_j = 0, j \neq k\}$
normally hyperbolic
critical manifold



GSPT (Fenichel):

M_0 perturbs smoothly to one-dimensional invariant slow manifolds

$$M_\varepsilon = \{\bar{u} \in \mathbb{R}^n \ : \ \bar{u}_j = \varepsilon h_j(\bar{u}_k, \varepsilon), \, j \neq k\}, \quad \varepsilon \in [0, \varepsilon_1],$$

with $h_j(\bar{u}_k, \varepsilon) = -\frac{1}{2\lambda_j^0}\frac{\partial^2 f_j}{\partial u_k^2}(0)\bar{u}_k^2 + \mathcal{O}(\varepsilon), \quad j \neq k.$

Flow on M_ε:

$$\bar{u}_k' = \varepsilon a \bar{u}_k^2 + \varepsilon^2 \left(b\bar{u}_k^3 - \bar{s}^0 \bar{u}_k - \bar{c}_k^0 + \mathcal{O}(\varepsilon) \right)$$

with

$$a = \frac{1}{2}\frac{\partial^2 f_k}{\partial u_k^2}(0) = \frac{1}{2}\nabla\lambda_k(0) \cdot r_k(0) = 0$$

$$b = \frac{1}{6}\frac{\partial^3 f_k}{\partial u_k^3}(0) - \frac{1}{2}\sum_{j \neq k} \frac{1}{\lambda_j^0}\frac{\partial^2 f_k}{\partial u_k \partial u_j}(0)\frac{\partial^2 f_j}{\partial u_k^2}(0) = \frac{1}{6}(r_k \cdot \nabla)^2 \lambda_{k|u=0} \neq 0.$$

Assume from now on $b = 1$ and set $\tau = \bar{u}_k$.

Flow on M_ε governed by

$$\tau' = \varepsilon^2 \left(\tau^3 - \bar{s}^0\tau - \bar{c}_k^0 + \mathcal{O}(\varepsilon)\right), \quad \tau \in [-1, \alpha].$$

Fixed points: $\tau^- = -1$ (repelling), $\tau^+ = \alpha$ (attracting).

Parametrization of the profile by τ (center-manifold coordinate)

Dividing out a factor of ε^2:
Regular perturbation of the profile equation for ϕ_0

Analysis of the eigenvalue problem

Couple the eigenvalue problem with the reduced profile equation to obtain an autonomous system on $[-1, \alpha] \times \mathbb{C}^{2n}$:

$$\begin{aligned} \tau' &= \varepsilon^2 \left(\tau^3 - \bar{s}^0 \tau - \bar{c}_k^0 + \mathcal{O}(\varepsilon) \right), \\ W' &= A_{\varepsilon,\kappa}[\tau] W, \end{aligned} \tag{P}$$

with

$$A_{\varepsilon,\kappa}[\tau] = \begin{pmatrix} Df(\varepsilon \bar{u}) - \varepsilon^2 \bar{s} I & I \\ \kappa I & 0 \end{pmatrix} \in \mathbb{C}^{2n \times 2n}.$$

For $\varepsilon \geq 0$, $\kappa \in \mathbb{H} \setminus \{0\}$: $A_{\varepsilon,\kappa}[\tau]$ is hyperbolic.

GSPT:
$\forall R > 0 : \exists \bar{\varepsilon} > 0$: No eigenvalues in $\{|\kappa| > R\}$ for $\varepsilon < \bar{\varepsilon}$.

Argument breaks down in $\kappa = 0$: $A_{0,0}[\tau]$ is not hyperbolic, $n + 1$ eigenvalues vanish.
Point $(\varepsilon, \kappa) = (0, 0)$ not accessible

Blow-up of $(\varepsilon, \kappa) = (0, 0)$

Scaling regimes

$$
\begin{aligned}
\text{I.} \quad & \varepsilon = r_1 \varepsilon_1,\ \kappa = r_1^2 e^{i\varphi} \\
\text{II.} \quad & \kappa = \varepsilon^2 \zeta_2 \\
\text{III.} \quad & \varepsilon = \delta_3 r_3,\ \kappa = \delta_3^4 r_3^2 e^{i\varphi} \\
\text{IV.} \quad & \kappa = \varepsilon^4 \zeta_4
\end{aligned}
$$

In each regime: (P) has three separated time scales.

GSPT is applicable (slow manifolds, ...)

Bundles split into fast, fast-slow and slow subbundles.

There are no eigenvalues for ϕ_ε if $\varepsilon \ll 1$ and

I. $\varepsilon^2 R_1 \le |\kappa| \le R_0$ with certain $0 < R_1 < R_0$

II. $\varepsilon^2 R_2 \le |\kappa| \le \varepsilon^2 R_1$ with arbitrary $0 < R_2 < R_1$

III. $\varepsilon^4 R_3 \le |\kappa| \le \varepsilon^2 R_2$ with certain $0 < R_3 < R_2$

$\Rightarrow$ $\exists \bar{R} > 0$: No eigenvalues in $\{\kappa : |\kappa| \ge \varepsilon^4 \bar{R}\}$ if $\varepsilon \ll 1$.

In regime IV:
Scaled Evans bundles $H_\varepsilon^\pm(\kappa) = \mathcal{H}_\varepsilon^\pm(\varepsilon^4\kappa)$ converge for $\varepsilon \to 0$ to

$$H_0^-(\kappa) = U_0^f \oplus \hat{\mathcal{H}}_0^-(\kappa) \oplus U_0^{ss}, \quad H_0^+(\kappa) = S_0^f \oplus \hat{\mathcal{H}}_0^+(\kappa) \oplus S_0^{ss},$$

uniformly on compact subsets of $\mathbb{H}$.

□