Asymptotic-preserving schemes for unusual long-time asymptotics

Rodolphe Turpault (1)
joint work with C. Berthon (1), P. LeFloch (2) & F. Marche (3)

(1) LMJL, Université de Nantes
(2) CNRS-Université P. et M. Curie
(3) I3M, Université Montpellier 2

06/26/2012

Outline

Framework

Consider the following class of hyperbolic systems with source terms:

$$\partial_t U + \mathrm{div}(\mathbf{F}(U)) = -\gamma R(U), \tag{1}$$

where $U \in \Omega \subset \mathbb{R}^N$, $\gamma = \gamma(U) > 0$ and $R(U)$ fulfills compatibility properties (Chen-Levermore-Liu).
Essentially:

- $\exists Q \in M_{nN}(\mathbb{R})$ with rank $n < N$ / $QR(U) = 0$.
- The equilibrium map $\mathcal{E} : \omega := Q\Omega \to \Omega$ is determined by $Q\mathcal{E}(u) = u$ and $R(\mathcal{E}(u)) = 0$, $u \in \omega$.
- $\boxed{Q\mathbf{F}(\mathcal{E}(u)) = c, \;\; u \in \omega.}$

Framework

Assume $t = \mathcal{O}(\frac{1}{\varepsilon})$ and $\gamma = \mathcal{O}(\frac{1}{\varepsilon^m})$.

Then, when $\varepsilon \to 0$, and if $m = 1$ (1) degenerates into:

$$\partial_t u = \operatorname{div}\left(\mathcal{M}(u)\nabla u\right),$$

where $u = QU$ and $\mathcal{M}(u)$ is a nonlinear diffusion matrix.

If $m > 1$ the asymptotic regime is more complicated.

Example 0: Telegraph Equations

The Telegraph system

$$\partial_t u + a\partial_x u = \sigma(v - u),$$
$$\partial_t v - a\partial_x v = \sigma(u - v).$$

Where $\sigma \in \mathbb{R}^+$. The associated diffusive regime is governed by:

$$\partial_t(u + v) = \partial_x \left(\frac{a^2}{2\sigma} \partial_x(u + v) \right).$$

Example 1: Euler w friction

Isentropic Euler + friction.

$$\partial_t \rho + \partial_x q = 0,$$
$$\partial_t q + \partial_x \left(\frac{q^2}{\rho} + p(\rho) \right) = -\kappa(\rho) q,$$

where $\rho > 0$, $q \in \mathbb{R}$ and $p : \mathbb{R}_+ \to \mathbb{R}_+ \ / \ p'(\rho) > 0$.
The associated diffusive regime is governed by:

$$\partial_t \rho = \partial_x (p^{'}(\rho) \partial_x \rho).$$

Example 2: M1 model for radiative transfer

2D $M1$ model for radiative transfer

$$\begin{aligned} \partial_t E + \partial_x \mathbf{F}_x + \partial_y \mathbf{F}_y &= c\sigma(aT^4 - E), \\ \partial_t \mathbf{F}_x + c^2\partial_x \mathbf{P}_{xx} + c^2\partial_y \mathbf{P}_{xy} &= -c\sigma \mathbf{F}_x, \\ \partial_t \mathbf{F}_y + c^2\partial_y \mathbf{P}_{xy} + c^2\partial_y \mathbf{P}_{yy} &= -c\sigma \mathbf{F}_y, \\ \rho C_v \partial_t T &= c\sigma(E - aT^4), \end{aligned}$$

where $\mathbf{P} = \mathbf{P}(\frac{\|\mathbf{F}\|}{cE})$ is a prescribed function and $\sigma = \sigma(E, F, T)$. The associated asymptotic regime is described by the so-called equilibrium diffusion equation

$$\partial_t(\rho C_v T + aT^4) + \mathrm{div}\left(\frac{4acT^3}{3\sigma}\nabla T\right) = 0.$$

AP numerical schemes

Most numerical schemes do not degenerate correctly. For example, the choice:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}(\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}) - \Delta t \gamma R(U_i^n),$$

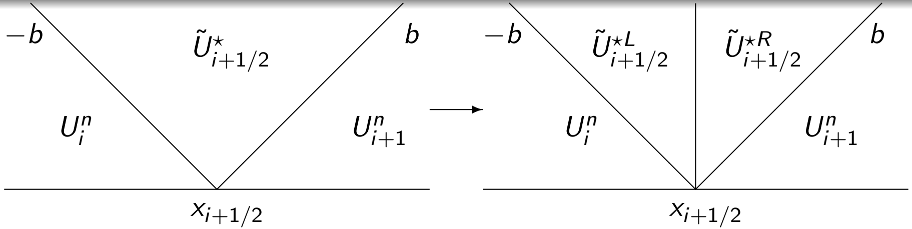
with a Rusanov flux for the Euler equations with friction leads is asymptotically consistant with:

$$\partial_t \rho = \partial_x \left(\frac{b\Delta x}{2} \partial_x \rho \right).$$

Main techniques to build AP schemes:

- control the numerical diffusion (Gosse-Toscani,...)
- design a suitable hydrostatic reconstruction.

Construction of the scheme in 1D



Idea: suitable modification of an approximate Riemann solver

$$U^{\star L,R} = \underline{\alpha}\tilde{U}^{\star} + (\mathbb{I}_{\mathrm{d}} - \underline{\alpha})(U_{L,R} - \bar{R}(U_{L,R})),$$

$$\underline{\alpha} = \left(\mathbb{I}_{\mathrm{d}} + \frac{\gamma\Delta x}{2b}(\mathbb{I}_{\mathrm{d}} + \underline{\sigma})\right)^{-1}, \quad \bar{R}(U) = (\mathbb{I}_{\mathrm{d}} + \underline{\sigma})^{-1}R(U).$$

Construction in 1D

The updated approximated solution at time t^{n+1} is naturally defined as

$$U_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} U_{\Delta x}^n(x, t^n + \Delta t) dx.$$

A straightforward computation leads to

$$\begin{aligned}
\frac{1}{\Delta t}(U_i^{n+1} - U_i^n) + \frac{1}{\Delta x}(\underline{\alpha}_{i+1/2}\mathcal{F}_{i+1/2} - \underline{\alpha}_{i-1/2}\mathcal{F}_{i-1/2}) \\
= \frac{1}{\Delta x}(\underline{\alpha}_{i+1/2} - \underline{\alpha}_{i-1/2})F(U_i^n) - \frac{b}{\Delta x}(\mathbb{I}_\mathrm{d} - \underline{\alpha}_{i-1/2})\bar{R}_{i-1/2}(U_i^n) \\
- \frac{b}{\Delta x}(\mathbb{I}_\mathrm{d} - \underline{\alpha}_{i+1/2})\bar{R}_{i+1/2}(U_i^n).
\end{aligned}$$

Properties of the scheme

Theorem

Assume that $\underline{\alpha}_{i+1/2}$ are defined, then the scheme is consistant. Moreover, if $\forall i\ U_i^n \in \Omega$ then $\forall i\ U_i^{n+1} \in \Omega$ provided the transport scheme conserves Ω.

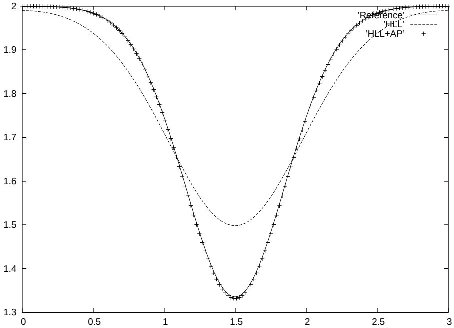
Theorem

The scheme is asymptotic-preserving if $\underline{\sigma}_{i+1/2}$ is chosen so that the following relation holds

$$Q(\mathbb{I}_{\mathrm{d}} + \underline{\sigma}_{i+1/2})^{-1} = \frac{1}{b^2}\mathcal{M}_{i+1/2}Q,$$

where $\mathcal{M}_{i+1/2}$ is a discretization of the diffusion matrix $\mathcal{M}(u)$ at the interface $x_{i+1/2}$.

Numerical Example



Reference (full line) and computed solutions with (+) and without (dotted line) AP correction.

Toy model

Very simplified coupling between the Euler and M1 models.

$$\partial_t \rho + \partial_x q = 0,$$

$$\partial_t q + \partial_x \left(\frac{q^2}{\rho} + p(\rho) \right) = -\kappa q + \sigma f,$$

$$\partial_t e + \partial_x f = 0,$$

$$\partial_t f + \partial_x \chi \left(\frac{f}{e} \right) e = -\sigma f,$$

where $\chi(\xi) = \frac{3+4\xi^2}{5+2\sqrt{4-3\xi^2}}$ or $\chi(\xi) = \frac{1}{3}$.

The asymptotic regime of this system is given by:

$$\begin{aligned} \partial_t \rho - \frac{1}{\kappa} \partial_x^2 p(\rho) - \frac{1}{3\kappa} \partial_x^2 e &= 0, \\ \partial_t e - \frac{1}{3\sigma} \partial_x^2 e &= 0. \end{aligned} \qquad (2)$$

Toy model

The AP correction has to be a nondiagonal matrix, for example:

$$\underline{\sigma}_{i+1/2} = \begin{pmatrix} \frac{\kappa}{\gamma \Delta^n_{i+1/2}} - 1 & 0 & -\frac{\sigma}{\gamma \Delta^n_{i+1/2}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3\sigma}{\gamma} - 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\Delta^n_{i+1/2} = \frac{p^n_{i+1} - p^n_i}{\rho^n_{i+1} - \rho^n_i}$.

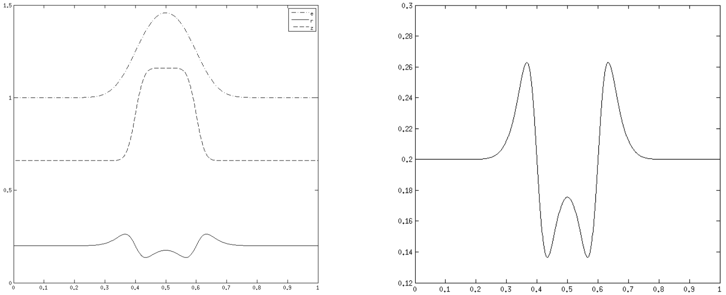
The linear case

When $p(\rho) = C\rho$, the asymptotic system may be rewritten as:

$$\partial_t \zeta - \frac{C}{\kappa}\partial_x^2 \zeta = 0,$$
$$\partial_t e - \frac{1}{3\sigma}\partial_x^2 e = 0,$$

where $\zeta = (3C\sigma - \kappa)\rho + \sigma e$.

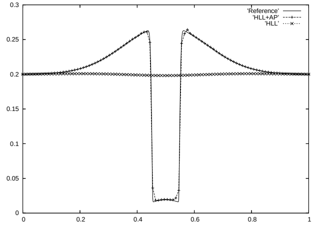
The linear case

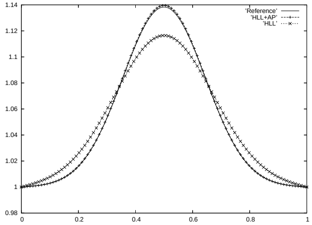


Toy model - linear case. ρ, ε and $10^{-3}\zeta$ at $t = 5$ for $\kappa = 2e3$, $\sigma = 1e3$.

Nonlinear case

Now, $p(\rho) = C\rho^2$.





Reference (full line) and computed solutions (dashed and dotted lines) with and without AP correction. (l): ρ (r): e.

SW + friction

Shallow-Water with Manning-type friction:

$$\partial_t h + \partial_x q = 0,$$
$$\partial_t q + \partial_x\left(\frac{q^2}{h} + g\frac{h^2}{2}\right) = -\frac{\kappa}{h^\eta}|q|q$$

Here $m = 2$ and the asymptotic limit is:

$$\partial_t h = \partial_x \left(\sqrt{\frac{g}{\kappa}} h^{\frac{\eta+1}{2}} \frac{\partial_x h}{\sqrt{|\partial_x h|}} \right).$$

SW + friction

Numerical example:

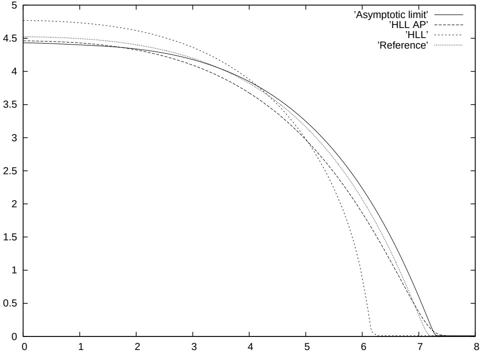


Fig.: h at $t = 10$, $\Delta x = 5e - 2$, $\kappa = 1e3$, $\eta = 1$.

SW + friction:

Numerical example:

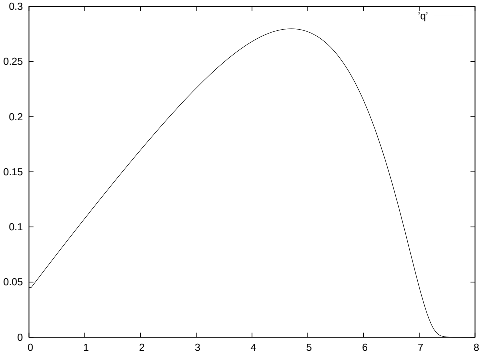


Fig.: q at $t = 10$, $\Delta x = 5e - 2$, $\kappa = 1e3$, $\eta = 1$.

Perspectives

- Investigate even more complex situations,
- Investigate the convergence rate of the numerical schemes,
- Investigate the impact of the choice of $\underline{\alpha}$,
- ...