# A Low Mach Number Limit of a Dispersive Navier-Stokes System

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# Outline

- What is our objective?
- Transition regime models.
- Main result.
- Local well-posedness results for the DNS system and the *ghost effect system*.
- Strategy for the rigorous proof of the low Mach number limit of the DNS:
  - Take into account the anti-symmetric structure of the high-order dispersive term.
  - Establish uniform bounds for the solution by considering the interaction of its fast and slow motion parts.
    - A priori estimates for the slow motion $\psi^\epsilon = (\epsilon p^\epsilon, \epsilon u^\epsilon, \theta^\epsilon)$.
    - A priori estimates for the fast motion $(p^\epsilon, u^\epsilon)$.
    - Local decay of the energy of the fast motion.

## Our objective

Establish a low Mach number limit for classical solutions over the whole space of a compressible fluid dynamic system that includes dispersive corrections to the Navier-Stokes equations.

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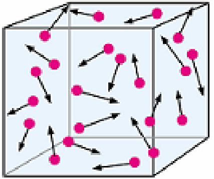
The limiting system is a so-called *ghost effect system* (Sone 2002) which is not derivable from the Navier-Stokes system of gas dynamics but is derivable from kinetic equations.

## The long term objective

This work is part of a research program that aims to **identify** fluid dynamic regimes and to construct a unified model that captures them all. Such a model can also be useful in transition regimes when classical fluid equations are inadequate to describe the dynamics of fluids while computations using kinetic models are expensive.

- Kinetic equations
    - Evolution of systems consisting of a large number of particles.



    - Mathematical formulation: partial differential equations for the particle distribution function $f(t, x, v)$:
        - t: time;
        - x: position;
        - v: velocity;
        - $f \geq 0$.

- Classical examples of kinetic equations:
  - Boltzmann equation (gas dynamics):

$$\partial_t f + v \cdot \nabla_x f = \mathcal{C}[f];$$

    - $f = f(t, x, v)$: particle distribution function;
    - $\mathcal{C}[f]$: collision among particles;
    - Example:

$$C[f] = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \Big( f(v')f(v_1') - f(v)f(v_1) \Big) b(v, v_1, \omega) \mathrm{d}\omega \mathrm{d}v_1 \,.$$

- Classical macroscopic regime:
  - Many collisions, systems close to local equilibrium.
  - Measurement:

$$Kn = \frac{\text{mean free path}}{\text{macroscopic length}}.$$

    - Kn: Knudsen number;
    - Macroscopic regime: small Knudsen number regime.

- Classical example of fluid equations:
  - Navier-Stokes equation

$$\begin{aligned}
\partial_t\rho + \nabla_x\cdot(\rho u) &= 0\,, && \text{conservation of mass}\\
\partial_t(\rho u) + \nabla_x\cdot(\rho u\otimes u) + \nabla_x p &= \nabla_x\cdot\Sigma\,, && \text{conservation of momentum}\\
\partial_t(\rho e) + \nabla_x\cdot(\rho e u) + \nabla_x\cdot(p u) &= \nabla_x\cdot(\Sigma u) - \nabla_x\cdot q\,, && \text{conservation of energy}
\end{aligned}$$

$\rho$: mass density
$\rho u$: momentum density
$\rho e = \frac{1}{2}\rho|u|^2 + \frac{D}{2}\rho\theta$: energy density

$p$: pressure
$\Sigma$: viscous stress tensor
$q$: heat flux

$(p,\Sigma,q) = (p,\Sigma,q)(\rho,u,\theta,\nabla_x u,\nabla_x\theta)$: constitutive relation

- Kinetic equation $\Longrightarrow$ fluid equations?

**Boltzmann equation**

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{Kn} C[f]$$

$$\rho = \langle f \rangle, \quad \rho u = \langle v f \rangle, \quad \rho e = \langle \tfrac{1}{2}|v|^2 f \rangle.$$

$\rightarrow$ $\Downarrow$ $\leftarrow$ $Kn \to 0$

**Navier-Stokes equation**

$$\begin{aligned}
\partial_t \rho + \nabla_x \cdot (\rho u) &= 0\,, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x p &= \nabla_x \cdot \Sigma\,, \\
\partial_t (\rho e) + \nabla_x \cdot (\rho e u) + \nabla_x \cdot (p u) &= \nabla_x \cdot (\Sigma u) - \nabla_x \cdot q\,,
\end{aligned}$$

## Transition Regime

- Plenty collisions but not enough to drive the system very close to local equilibrium;
- Computations using full kinetic equations are expensive.
- Classical macroscopic systems are inaccurate.

## Purpose of developing transition regime models

- Bridge the gap between kinetic equations and classical macroscopic equations.

- Transition regime models

Kinetic equations

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Transition regime models

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Classical macroscopic equations

- Construction and analysis of interior equations:
  - Higher order equations: adding higher order correction terms;
  - Larger moment systems: including more moments of the density function.
- Construction of appropriate boundary conditions:
  - Boundary conditions of macroscopic equations should match the given boundary conditions of the underlying kinetic equations.
  - Boundary layer analysis.

- Thermal induced flow:
  - Maxwell: 1879

$$\rho \partial_t u + \nabla_x p - \nabla_x \cdot \Sigma + \tau \nabla_x \Delta_x \theta = 0.$$

  - Kogan, Galkin and Fridlender: nonlinear thermal stress, 1976.
  - Aoki, Sone, Sugimoto, Takata ...

- Ghost effect regime:
  - small bulk velocity: $\boxed{U = \epsilon u}$
  - small fluctuations in pressure field: $\boxed{p = p_0 + \epsilon p_1}$
  - large variation in temperature/density field: $\boxed{\nabla_x \rho, \nabla_x \theta \sim \mathcal{O}(1)}$
- Classical fluid equations like NS are not accurate in the ghost effect regime.

- Ghost effect system

$$
\begin{aligned}
\nabla_x(\rho\theta) &= 0, \\
\partial_t\rho + \nabla_x\cdot(\rho u) &= 0, \\
\partial_t(\rho u) + \nabla_x\cdot(\rho u\otimes u) + \nabla_x P &= \nabla_x\cdot\Sigma + \nabla_x\cdot\tilde{\Sigma}, \\
\partial_t(\rho\theta) + \nabla_x\cdot(\rho\theta u) &= -\nabla_x\cdot q,
\end{aligned} \tag{1}
$$

- Constitutive relation

$$
\begin{aligned}
\Sigma &= \mu(\theta)\left(\nabla_x u + (\nabla_x u)^\top - \frac{2}{3}(\nabla_x\cdot u)\mathbb{I}\right), \\
\tilde{\Sigma} &= \tau_1(\theta)\left(\nabla_x^2\theta - \frac{1}{3}(\Delta_x\theta)\mathbb{I}\right) + \tau_2(\theta)\left(\nabla_x\theta\otimes\nabla_x\theta - \frac{1}{3}|\nabla_x\theta|^2\mathbb{I}\right), \\
q &= -k(\theta)\nabla_x\theta.
\end{aligned}
$$

- Question: Can one in some sense unify the classical fluid equations and the ghost-effect system?

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- Higher order equations: dispersive Navier-Stokes.
- Reference:
  - Levermore: *Gas Dynamics Beyond Navier-Stokes* .

- Main questions:
  - Well-posedness of the DNS system?
  - Recovery of the ghost effect system in the ghost effect regime?

- Main results:
  - Well-posedness of the DNS system?
    - Result: local well-posedness in Sobolev spaces;
      **Reference:** *Ph.D. Thesis of Weiran Sun (2009).*
  - Recovery of a ghost effect system in the ghost effect regime?
    - Result: DNS converges to a ghost effect system in the low Mach number limit.
      **Reference:** *Levermore, Sun, Trivisa SIMA (2012).*

- Dispersive Navier-Stoke system (DNS):

$$
\begin{aligned}
\partial_t \rho + \nabla_x \cdot (\rho u) &= 0\,, \\
\rho \partial_t u + \rho u \cdot \nabla_x u + \nabla_x p(\rho, \theta) &= \nabla_x \cdot \Sigma + \nabla_x \cdot \tilde{\Sigma} \\
\tfrac{3}{2} \rho \partial_t \theta + \tfrac{3}{2} \rho u \cdot + \nabla_x \theta + p \nabla_x \cdot u &= (\Sigma + \tilde{\Sigma}) : \nabla_x u + \nabla_x \cdot \tilde{q} - \nabla_x \cdot q\,, \\
(\rho, u, \theta)(x, 0) &= (\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}})(x)\,,
\end{aligned}
\tag{2}
$$

- time variable $t \in [0, \infty)$, space variable $x \in \mathbb{R}^3$.
- Density: $\rho$, bulk velocity: $u \in \mathbb{R}^3$, temperature: $\theta$.

## Constitutive relations

- Ideal gas law: $p(\rho, \theta) = \rho\theta$.
- Viscous stress tensor:

$$\Sigma = \mu(\theta)(\nabla_x u + (\nabla_x u)^\top - \frac{2}{3}(\nabla_x \cdot u)\mathbb{I}), \;\; \mu(\theta) > 0.$$

- Heat flux: $q(\theta) = -\kappa(\theta)\nabla_x\theta, \;\; \kappa(\theta) > 0$.
- The total energy density:

$$\rho e = \frac{1}{2}\rho|u|^2 + \frac{3}{2}\rho\theta.$$

The quantities $\tilde{\Sigma}$ and $\tilde{q}$ denote dispersive corrections to the stress tensor and heat flux respectively and are given by

- Dispersive term in the velocity equation $\nabla_x \cdot \tilde{\Sigma}$ where

$$\begin{aligned}\tilde{\Sigma} &= \tau_1(\rho,\theta)(\nabla_x^2\theta - \frac{1}{3}(\Delta_x\theta)\mathbb{I}) \\ &+ \tau_2(\rho,\theta)(\nabla_x\theta \otimes \nabla_x\theta - \frac{1}{3}|\nabla_x\theta|^2\mathbb{I}) \\ &+ \tau_3(\rho,\theta)(\nabla_x u(\nabla_x u)^\top - (\nabla_x u)^\top \nabla_x u)\end{aligned} \tag{3}$$

- Dispersive term in the temperature equation $\nabla_x \cdot \tilde{q}$ where

$$\begin{aligned}\tilde{q} &= \tau_4(\rho,\theta)(\Delta_x u + \frac{1}{3}\nabla_x\nabla_x \cdot u) \\ &+ \tau_5(\rho,\theta)\nabla_x\theta \cdot (\nabla_x u + (\nabla_x u)^\top - \frac{2}{3}(\nabla_x \cdot u)\mathbb{I}) \\ &+ \tau_6(\rho,\theta)\left(\nabla_x u - (\nabla_x u)^\top\right) \cdot \nabla_x\theta.\end{aligned} \tag{4}$$

- $\tau_1, \ldots, \tau_6$ are $C^\infty$ functions of their variables.

One feature of the DNS system is that it possesses an entropy structure provided the transport coefficients in $\tilde{\Sigma}$ and $\tilde{q}$ satisfy

$$\tau_4 = \frac{\theta}{2}\tau_1, \qquad \frac{\tau_2}{\theta} + \frac{2\tau_5}{\theta^2} = \partial_\theta\left(\frac{\tau_4}{\theta^2}\right), \tag{5}$$

such that

$$\tilde{\Sigma} : \frac{\nabla_x u}{\theta} + \tilde{q} \cdot \frac{\nabla_x \theta}{\theta^2} = \nabla_x \cdot \left(\frac{\tau_1}{2\theta}\nabla_x\theta \cdot \left(\nabla_x u + (\nabla_x u)^\top - \frac{2}{3}(\nabla_x \cdot u)\mathbb{I}\right)\right).$$

- Entropy density

$$\eta = \rho \log\left(\frac{\rho}{\theta^{3/2}}\right).$$

- Entropy equation

$$\begin{aligned}\partial\eta + \nabla_x \cdot \left(\eta u + \frac{q+\tilde{q}}{\theta}\right) = & -\left(\frac{\Sigma}{\theta} : \nabla_x u - \frac{q}{\theta^2} \cdot \nabla_x \theta\right) \\ & -\left(\frac{\tilde{\Sigma}}{\theta} : \nabla_x u - \frac{\tilde{q}}{\theta^2} \cdot \nabla_x \theta\right).\end{aligned}$$

- Total entropy is formally dissipated by the dispersive NS system in the same way as in the NS system over domains without boundaries.

The proof of the local well-posedness of the DNS system follows using the classical energy method for *hyperbolic-parabolic systems*.

- Although we have third-order dispersive terms, the leading orders of these terms form an anti-symmetric structure.

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  Therefore they do not hamper the usual $L^2$-$H^s$ estimates.
- The rest of the dispersive terms are of orders up to two. Although this is the same order as the dissipation, they do not introduce extra difficulties because they are of order $O(\epsilon^2)$ while the dissipative terms are of order $O(\epsilon)$.
- Here we do need the viscosity coefficient $\mu(\theta)$ and $\kappa(\theta)$ to be bounded away from zero when $\theta$ is bounded from below.

## Local well-posedness for the ghost effect system

- Classical energy estimates for hyperbolic-parabolic equations.
- Difficulty: There is a third-order term in $\theta$ in the GES and there is no anti-symmetric structure to balance this term. Key observation: The leading order of this term is in the form of a gradient, which can be incorporated into the pressure term. By doing so the rest of the terms can be treated as perturbations. Similar proofs can be found for combustion models and Kazhikhov-Smagulov type models.

- Main Question:
  - Can the dispersive Navier-Stokes system recover the ghost effect system in the ghost effect regime?
- Result
  - The DNS system converges to the ghost effect system in a low Mach number limit.

The *ghost effect system* can be formally derived from kinetic equations using a Hilbert expansion method (cf. Sone (2002)). This is a system *beyond* classical fluid equations that describes the phenomenon in which the temperature field of the fluid has finite variations, and the flow is driven by the gradient of the temperature field.

- Scaling
  - Kundsen number: $\epsilon$.
  - Transport coefficients

$$\mu = \epsilon\hat{\mu}, \qquad \kappa = \epsilon\hat{\kappa}, \qquad \tau_i = \epsilon^2\hat{\tau}_i, \;\; i = 1, \cdots, 6\,.$$

- Ghost effect regime

$$p_\epsilon = e^{\epsilon p^\epsilon}, \qquad U_\epsilon = \epsilon u^\epsilon, \qquad \Theta_\epsilon = e^{\theta^\epsilon}, \qquad \rho_\epsilon = e^{\rho^\epsilon}.$$

- Long time scale: $t = \frac{1}{\epsilon}\tau$.
- Low Mach number limit: Von Kármen relation:

$$\mathrm{Re} \propto \frac{\mathrm{Kn}}{\mathrm{Ma}}.$$

Here *Ma* denotes the Mach number which is typically used to compare a typical flow velocity with a characteristic speed of sound $c$.

- Flunctuation Equations:

$$\begin{cases} \partial \rho_\epsilon + \nabla_x \cdot (\rho_\epsilon u^\epsilon) = 0, \\ e^{-\theta^\epsilon}(\partial_t + u^\epsilon \cdot \nabla_x) u^\epsilon + \frac{1}{\epsilon} \nabla_x p^\epsilon = e^{-\epsilon p^\epsilon}(\nabla_x \cdot \Sigma + \nabla_x \cdot \tilde{\Sigma}), \\ \frac{3}{2}(\partial_t + u^\epsilon \cdot \nabla_x)\theta^\epsilon + \nabla_x \cdot u^\epsilon = \epsilon^2 e^{-\epsilon} p^\epsilon \left( \left( \Sigma + \tilde{\Sigma} \right) : \nabla_x u^\epsilon + \nabla_x \cdot \tilde{q} \right) \\ \qquad (\rho_\epsilon, u^\epsilon, \theta^\epsilon)(x, 0) = (\rho_\epsilon^{in}, u_\epsilon^{in}, \theta_\epsilon^{in})(x). \end{cases}$$

- fast motion: $(u^\epsilon, p^\epsilon)$ vary on the time scale $\mathcal{O}(\epsilon)$;
- slow motion: $(\epsilon p^\epsilon, \epsilon u^\epsilon, \theta^\epsilon)$ vary on the time scale $\mathcal{O}(1)$.

# Function Space

- Define the operator $\Lambda_\epsilon$ and norm $\|w\|_{H_\epsilon^{s+1}}$:

$$\Lambda_\epsilon = (I - \epsilon^2 \Delta_x)^{1/2},$$

$$\|w\|_{H_\epsilon^{s+1}} = \epsilon^{s+1}\|w\|_{H^{s+1}} + \|w\|_{H^s} \lesssim \|\Lambda_\epsilon^{s+1} w\|_{L^2}.$$

- Define the norms:

$$|||(p,u,\theta)|||_{\epsilon,s,t} := \sup_{[0,t]}\Big(\|(p,u)\|_{H^s} + \|\Lambda_\epsilon^{s+1}(\epsilon p, \epsilon u, \theta)\|_{H^{s+1}}\Big) + \alpha_0\left(\int_0^t \Big(\|\nabla_x u\|_{H^s}^2 + \|\nabla_x \Lambda_\epsilon^{s+1}(\epsilon u, \theta)\|_{H^{s+1}}^2\Big)(\tau)\mathrm{d}\tau\right)^{1/2},$$

$$|||(p^{in}, u^{in}, \theta^{in})|||_{\epsilon,s,0} := \|(p^{in}, u^{in})\|_{H^s} + \|\Lambda_\epsilon^{s+1}(\epsilon p^{in}, \epsilon u^{in}, \theta^{in})\|_{H^{s+1}},$$

# Main Theorem

**Theorem (Levermore-Sun-Trivisa)**

*Let $s \geq 6$. Suppose that there exist positive constants $M_0, \underline{\theta}, c_0, \sigma$ such that the initial data of the fluctuation equations satisfy*

$$|||(p_\epsilon^{in}, u_\epsilon^{in}, \theta_\epsilon^{in} - \underline{\theta})|||_{\epsilon,s,0} \leq M_0 \,,$$

$$|\theta_\epsilon^{in} - \underline{\theta}| \leq c_0 |x|^{-1-\sigma} \,, \qquad |\nabla_x \theta_\epsilon^{in}| \leq c_0 |x|^{-2-\sigma} \,.$$

$$(u_\epsilon^{in}, \theta_\epsilon^{in} - \underline{\theta}) \to (u^{in}, \theta^{in} - \underline{\theta}) \quad in \quad H^s(\mathbb{R}^3) \,.$$

*Then there exists a $T > 0$ such that for any $\epsilon \in (0, 1]$, the Cauchy problem for the fluctuation equations has a unique solution*

$$(p^\epsilon, u^\epsilon, \theta^\epsilon - \underline{\theta}) \in C^0([0, T]; H^{2s+1}(\mathbb{R}^3)) \cap L^\infty(0, T; H^{2s+2}(\mathbb{R}^3)) \cap L^2(0, T; H^{2s+3}(\mathbb{R}^3)) \,.$$

# Main Theorem Continued

**Theorem (Continued)**

*Furthermore, there exists a positive constant $M$ depending only on $T, M_0$ such that*

$$|||(p^\epsilon, u^\epsilon, \theta^\epsilon - \underline{\theta})|||_{\epsilon,s,t} \le M\,.$$

*The sequence of solutions $(p^\epsilon, u^\epsilon, \theta^\epsilon)$ converges weakly $*$ in $L^\infty(0,T;H^s(\mathbb{R}^3))$ and strongly in $L^2(0,T;H^{s'}_{loc}(\mathbb{R}^3))$ for all $s' < s$ to the limit $(0, u, \theta)$, where $(u, \Theta) = (u, e^\theta)$ satisfies*

$$\begin{aligned} \varrho\Theta &= 1\,, \\ \partial_t\varrho + \nabla_x\cdot(\varrho u) &= 0\,, \\ \partial_t(\varrho u) + \nabla_x\cdot(\varrho u\otimes u) + \nabla_x P^* &= \nabla_x\cdot\Sigma + \nabla_x\cdot\tilde{\Sigma}\,, \\ \nabla_x\cdot\left(\tfrac{5}{2}u - \kappa(\Theta)\nabla_x\Theta\right) &= 0\,, \end{aligned}$$

# Main Theorem Continued

## Theorem (Continued)

*The initial of the limiting ghost effect system satisfies*

$$(u, \theta)|_{t=0} = (v^{in}, \theta^{in}),$$

*where*

$$(u_\epsilon^{in}, \theta_\epsilon^{in} - \underline{\theta}) \to (v^{in}, \theta^{in} - \underline{\theta}) \quad in \quad H^s(\mathbb{R}^3),$$

$$\nabla_x \cdot \left(\tfrac{5}{2} v^{in} - \kappa(\Theta^{in}) \nabla_x \Theta^{in}\right) = 0,$$

$$\Pi\left(\frac{1}{\Theta^{in}} u^{in}\right) = \Pi\left(\frac{1}{\Theta^{in}} v^{in}\right),$$

*with $\Pi$ being the projection operator onto the divergence free part in the Hodge decomposition.*

## Strategy

Our analysis builds on the framework of Métivier and Schochet (2001) and Alazard (2005).

- Métivier and Schochet proved the incompressible limit for the non-isentropic Euler equations for classical solutions with general initial data.
- Alazard proved the low Mach number limit for the compressible Navier-Stokes for classical solutions with general initial data.

• Due to the presence of the high-order dispersive terms, our result does not follow from their analysis. We need to take into account of the anti-symmetric structure of those terms.

Let $u^\epsilon = (p^\epsilon, u^\epsilon, \theta^\epsilon)$ be the solution to the DNS system and $\psi^\epsilon = (\epsilon p^\epsilon, \epsilon u^\epsilon, \theta^\epsilon)$, then DNS can be reformulated as

$$A_1(\psi^\epsilon)(\partial_t + u \cdot \nabla_x)u^\epsilon + \frac{1}{\epsilon}A_2(\psi^\epsilon)u^\epsilon = A_3(\psi^\epsilon)u^\epsilon + \mathcal{R}\,,$$

where

- $A_1(\psi^\epsilon)$ is a diagonal matrix,
- $A_2(\psi^\epsilon)u^\epsilon$ are formed by certain combinations of the singular terms with the leading orders from the dispersive terms,
- $A_3(\psi^\epsilon)u^\epsilon$ are the dissipative terms, and
- $\mathcal{R}$ includes the rest of the dispersive terms.

# Remark:

In the case of compressible Navier-Stokes $A_2(\psi^\epsilon)$ is anti-symmetric. Here it is not readily anti-symmetric but it is anti-symmetrizable by a certain symmetrizer matrix composed of symmetric positive operators.

We establish uniform bounds for the solution by considering the interactions of its fast and slow motion parts.

## Theorem

*For each fixed* $\epsilon > 0$*, let* $(p^\epsilon, u^\epsilon, \theta^\epsilon) \in C([0,T]; H^s(\mathbb{R}^3))$ *be the solution to the scaled DNS system. Let*

$$\Omega = |\!|\!|(p^\epsilon, u^\epsilon, \theta^\epsilon - \underline{\theta})|\!|\!|_{\epsilon,s,T}\,, \qquad \Omega_0 = |\!|\!|(p^{\rm in}_\epsilon, u^{\rm in}_\epsilon, \theta^{\rm in}_\epsilon - \underline{\theta})|\!|\!|_{\epsilon,s,0}\,.$$

*Then there exists an increasing function* $C(\cdot)$ *such that*

$$|\!|\!|(p^\epsilon, u^\epsilon, \theta^\epsilon - \underline{\theta})|\!|\!|_{\epsilon,s,T} \leq C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)}\,,$$

*which further indicates that there exists* $T_0 > 0$ *independent of* $\epsilon$ *such that* $|||(p^\epsilon, u^\epsilon, \theta^\epsilon - \underline{\theta})|||_{\epsilon,s,T}$ *are uniformly bounded in* $\epsilon$ *over* $[0, T_0]$.

## Future program

- Global well-posed ness of DNS.
- Boundary conditions for transition regime models.
- Well-posedness of transition regime models on bounded domains.
- Comparison of transition regime models with classical systems.