SBV regularity results for Hamilton-Jacobi equations

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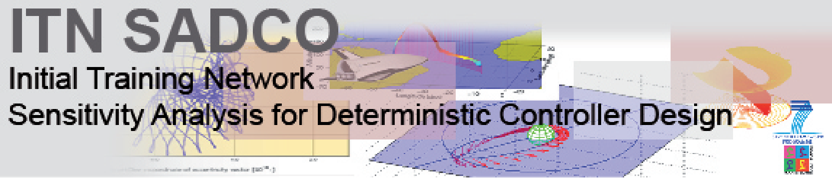

Hamilton-Jacobi equations

$$\partial_t u + H(t, x, D_x u) = 0 \qquad \text{in } \Omega \subset [0, T] \times \mathbb{R}^n \tag{1}$$

- $H : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ is called Hamiltonian
- Ω is an open domain in $\mathbb{R}^{n+1}$

Definition (Crandall, Evans and Lions ∼ 1980)

A locally Lipschitz continuous function $u : \Omega \to \mathbb{R}$ is called a *viscosity solution* of (1) provided that

i) u is a viscosity subsolution of (1): for each $v \in C^\infty(\Omega)$ such that $u - v$ has a maximum at $(t_0, x_0) \in \Omega$,

$$\partial_t v(t_0, x_0) + H(t_0, x_0, D_x v(t_0, x_0)) \leq 0;$$

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When $H(t, x, p)$ is smooth in all variables and convex with respect to p

$\Rightarrow$ the viscosity solution of

$$\partial_t u + H(t, x, D_x u) = 0 \qquad \text{in } \Omega \subset [0, T] \times \mathbb{R}^n$$

is the value function

$$u(t, x) := \min \left\{ u_0(\xi(0)) + \int_0^t L(s, \xi(s), \dot{\xi}(s))ds \;\middle|\; \xi(t) = x,\ \xi \in [C^2([0, t])]^n \right\}$$

- L, the Lagrangian, is the Legendre transform

$$L(t, x, v) = \sup_v \{\langle v, p \rangle - H(t, x, p)\}$$

- u_0 is the initial datum at time $t = 0$, u_0 bounded Lipschitz

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Previous Results

The structure of the non differentiability set of viscosity solutions has been studied by Fleming (1969), Cannarsa and Soner (1987), and others.

When the Hamiltonian is *strictly convex*

$$H_{pp}(t, x, p) \text{ is positive definite}$$

$\Rightarrow$ u is locally *semiconcave* in both variables:

$\forall K \subset\subset \Omega$, $\exists C > 0$ s.t. $(t, x) \mapsto u(t, x) - C(t^2 + |x|^2)$ is concave on K

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Given $f \in BV(\mathbb{R}^n)$, it is possible to decompose the distributional derivative of f into three mutually singular measures:

$$Df = D_a f + D_c f + D_j f.$$

- $D_a f$ is the absolutely continuous part with respect to $\mathcal{H}^n$
- $D_j f$ is the part of the measure which is concentrated on the rectifiable $(n-1)$-dimensional set J, where the function f has "jump" discontinuities
- $D_c f$ is the singular part which satisfies $D_c f(E) = 0$ for every Borel set E with $\mathcal{H}^{n-1}(E) < \infty$.

If $D_c f = 0$, we say that $f \in SBV(\mathbb{R}^n)$

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$$\partial_t u + H(t, x, D_x u) = 0 \qquad \text{in } \Omega \subset [0, T] \times \mathbb{R}^n,$$

H strictly convex in the last variable,
$u(0, \cdot) = u_0(\cdot) \in W^{1,\infty}(\mathbb{R}^n) \cap C^{R+1}(\mathbb{R}^n)$, with $R \geq 1$, then Du is locally SBV

- (n=1)Ambrosio and De Lellis in J. Hyperbolic Differ. Equ. 2004, Bianchini, De Lellis and Robyr in ARMA 2011

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$$c_H^{-1} Id_n(p) \leq H_{pp}(p) \leq c_H Id_n(p)$$

for some $c_H > 0$. Then the set of times

$$S := \{t | \ D_x u(t, \cdot) \notin SBV_{loc}(\Omega_t)\}$$

is at most countable.

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Ideas and general strategy

Idea (Bressan)

If $D_x u(\bar{t}, \cdot)$ is not SBV for a certain time $\bar{t}$, then at future times $\bar{t} + \delta$ the Cantor part of $D_x^2 u(\bar{t}, \cdot)$ gets transformed into jump singularities

Strategy

Construct a bounded monotone functional $F(t)$, whose jumps are related to the presence of a Cantor part in $|D_x^2 u(t, \cdot)|$

F can have only a countable number of jumps $\Rightarrow |D_x^2 u(t, \cdot)|$ can have positive Cantor part for a countable number of t's only

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In the case $H = H(p)$, the viscosity solution is

$$u(t,x) = \min_{y \in \Omega_0} \left\{ u(0,y) + tL\left(\frac{x-y}{t}\right) \right\}$$

$u(t,\cdot)$ is differentiable iff $y := x - tH_p(D_x u(t,x))$ is the only minimizer

y is the only minimizer for $u(s, x - sH_p(D_x u(t,x)))$ $0 \leq s \leq t$

When $u(t,\cdot)$ is differentiable the curve $\xi(s) := x - sH_p(D_x u(t,x))$ is called *characteristic*

- Characteristics are straight lines
- Characteristics do not cross (No-crossing property)

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What about points where $u(t, x)$ is not differentiable in x?

Definition

Let $u : \Omega \to \mathbb{R}$, for any $(t, x) \in \Omega$ the set

$$D_x^+ u(t, x) = \left\{ p \in \mathbb{R}^n \middle| \limsup_{y \to x} \frac{u(t, y) - u(t, x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\},$$

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The curve $\xi(s) := x - sH_p(p)$ for $p \in D_x^+ u(t,x)$ is called *generalized characteristic*

- Generalized characteristics are straight lines
- If we reduce to a sufficiently small time interval generalized characteristics do not cross (No-crossing property)

$$X_{t,0}(x) := x - tH_p(D_x^+ u(t,x)) \quad \forall x \in \Omega_t$$

$$\chi_{t,0}(x) := x - tH_p(D_x u(t,x)) \quad \forall x \in U_t := \{x \mid D_x^+ u(t,x) \text{ is single-valued}\}$$

$$\boxed{F(t) := \mathcal{H}^n(\chi_{t,0}(U_t)).}$$

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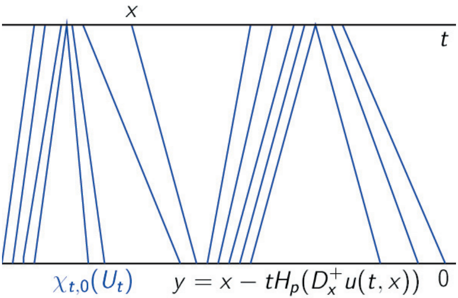
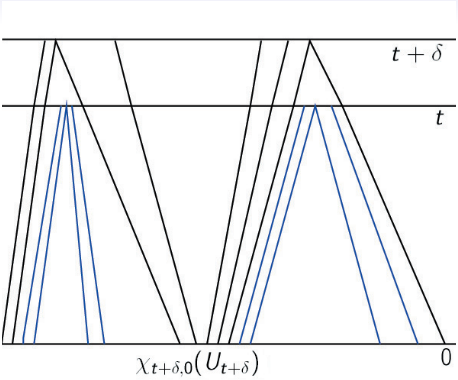


If $|(D_x^2 u(t,\cdot))_c|(\Omega_t) > 0 \Rightarrow \exists A \subset U_t$ s.t.

- $\mathcal{H}^n(A) = 0$
- A is the set where the Cantor part is concentrated
- $$\chi_{t,0}(A) \cap \chi_{t+\delta,0}(U_{t+\delta}) = \emptyset \quad \forall \delta > 0.$$



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SBV Regularity when $H := H(t, x, p)$

(H1) $H \in C^3([0, T] \times \mathbb{R}^n \times \mathbb{R}^n)$ with bounded second derivatives
$\exists a, b, c > 0$ s.t.

- i) $H(t, x, p) \geq -c$
- ii) $H(t, x, 0) \leq c$
- iii) $|H_{px}(t, x, p)| \leq a + b|p|$

(H2) $\exists c_H > 0$ s.t. for any $t \in \mathbb{R}$, $x \in \mathbb{R}^n$

$$c_H^{-1} Id_n(p) \leq H_{pp}(t, x, p) \leq c_H Id_n(p)$$

Theorem (Bianchini and T. [2])

Let u be a viscosity solution of

$$\partial_t u + H(t, x, D_x u) = 0 \qquad in\ \Omega \subset [0, T] \times \mathbb{R}^n,$$

assume (H1), (H2). Then the set of times

$$S := \{t \mid D_x u(t, \cdot) \notin [SBV_{loc}(\Omega_t)]^n\}$$

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SBV Regularity when $H := H(t, x, p)$

(H1) $H \in C^3([0, T] \times \mathbb{R}^n \times \mathbb{R}^n)$ with bounded second derivatives
$\exists a, b, c > 0$ s.t.

i) $H(t, x, p) \geq -c$
ii) $H(t, x, 0) \leq c$
iii) $|H_{px}(t, x, p)| \leq a + b|p|$

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The viscosity solution is

$$u(t,x) = \min \left\{ u_0(\xi(0)) + \int_0^t L(s, \xi(s), \dot{\xi}(s)) ds \;\middle|\; \xi(t) = x,\ \xi \in [C^2([0,t])]^n \right\}$$

i) $u(t,\cdot)$ is differentiable iff there is a unique minimizer ξ

ii) for every $0 \le s < t$ a minimizer ξ is the unique minimizer for $u(s, \xi(s))$

ii) for every minimizer ξ there exists a dual arc or co-state

$$p(s) := L_v(s, \xi(s), \dot{\xi}(s)) \qquad s \in [0,t],$$

s.t. ξ, p solve the following system

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Unique minimizers ξ are called *characteristics*

- Characteristics are no more straight lines
- Characteristics do not cross (No-crossing property)

Generalized characteristics are defined as C^2 curves such that they and their dual arc p are solutions of

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For t in $[\tau, \tau + \varepsilon]$ $\varepsilon > 0$ small enough

- Generalized characteristics can be approximated with straight lines
- Generalized characteristics do not cross in (τ, t) (No-crossing property)

For $\tau > 0$

$$X_{t,\tau}(x) := \{\xi(\tau) | \ \xi \text{ is a solution of (2) with } \xi(t) = x, p(t) = p, \text{ where } p \in D_x^+ u(t,x) \ \} \quad \forall x \in \Omega_t,$$

$$\chi_{t,\tau}(x) := X_{t,\tau}(x) \quad \forall x \in U_t,$$

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Kind of SBV regularity in the convex case

$$\partial_t u + H(D_x u) = 0 \quad \text{in } \Omega \subset [0, T] \times \mathbb{R}^n$$

- H is $C^2(\mathbb{R}^n)$, convex
- $\lim_{|p| \to \infty} \frac{H(p)}{|p|} = +\infty$

$\Rightarrow$ L is strictly convex but no more regular

$\Rightarrow$ $u(t, \cdot)$ is no more semiconcave, it is only locally Lipschitz, and $D_x u(t, \cdot)$ looses its BV regularity

IDEA

Study the vector field

$$d(t, x) := H_p(D_x u(t, x))$$

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$d(t, x)$ is defined only on the set of points (t, x) where $u(t, x)$ is differentiable in x

$\mathrm{div} d(t, \cdot)$ is a locally finite Radon measure

$\Rightarrow$ in the one-d case $d(t, x) = H_p(D_x u(t, x))$ is BV

Theorem (Bianchini and T. [1])

In the one-d case the vector field $d(t, \cdot)$ belongs to $SBV(\Omega_t)$, out of a countable number of $t \in [0, T]$.

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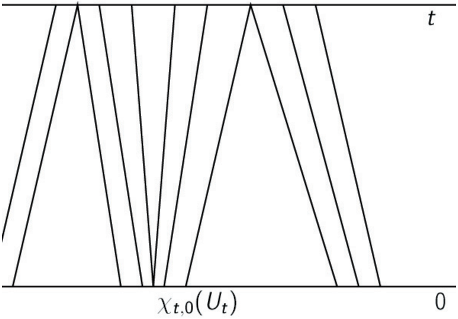

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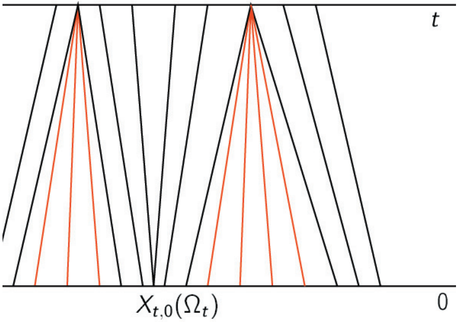

$\Rightarrow$ The general strategy applies.

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We can reduce to the set

$$U := \{(t, x) | \; u(t, x) \text{ is differentiable in } x\}.$$

IDEA

Add some hypotheses so that

- Where $L(d(t, x))$ is C^2 we can reduce locally to the uniformly convex case $\Rightarrow$ SBV-regularity holds
- Where $L(d(t, x))$ is not twice differentiable we can reduce step by step to dimension one.

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$$\Sigma_{\pi_n} := \{(t,x) \in U | \ d(t,x) \in V_{\pi_n}\} \quad \text{and} \quad \Sigma^c_{\pi_n} := U \setminus \Sigma_{\pi_n}.$$

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For $j = n, \ldots, 3$

$\forall (j-1)$-dimensional plane π_{j-1} in Π_{π_j}, let $L_{\pi_{j-1}} : \pi_{j-1} \to \mathbb{R}$ be the $(j-1)$-dimensional restriction of L to π_{j-1}

$$V_{\pi_{j-1}} := \{v \in \pi_{j-1} | \ L_{\pi_{j-1}}(\cdot) \text{ is not twice differentiable in } v\}.$$

(HYP(j-1)) We suppose $V_{\pi_{j-1}}$ is contained in a finite union of $(j-2)$-dimensional planes $\Pi_{\pi_{j-1}}$, $\forall \pi_{j-1} \in \Pi_{\pi_j}$.

Theorem (Bianchini and T. [1])

Under the above assumptions (HYP(0)),(HYP(n)),...,(HYP(2)), the Radon measure $\mathrm{div} d(t, \cdot)$ *has Cantor part on* Ω_t *only for a countable number of* t*'s in* $[0, T]$*.*

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