Stability of supersonic flow onto a wedge with the attached weak shock under the fulfillment of the weak Lopatinsky condition

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Introduction

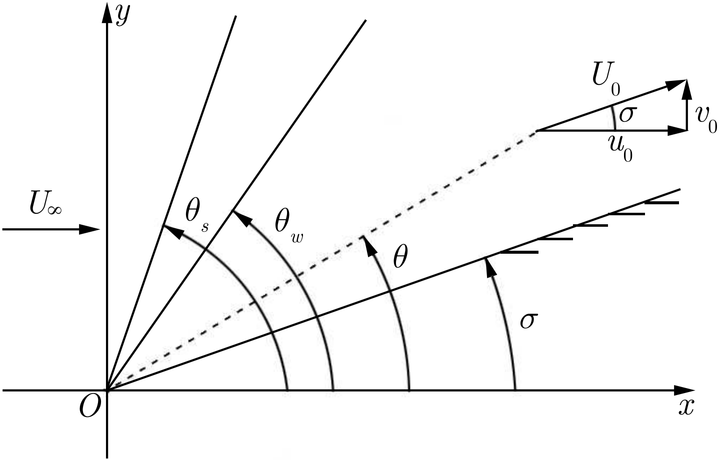

Fig. 1.

Introduction

As is well known, theoretically the classical problem of a supersonic stationary inviscid nonheatconducting gas flow onto a planar infinite wedge when the gas is in the thermodynamic equilibrium has two solutions [1-3]. One of these solutions corresponds to the case of a weak shock, when the flow behind the shock is generically supersonic, i.e., $u_0^2 + v_0^2 > c_0^2$, and the another one corresponds to the case of a strong shock, when the flow behind the shock is subsonic, $u_0^2 + v_0^2 < c_0^2$ (here u_0 and v_0 are components of the velocity field, and c_0 is the sound speed).

Introduction

1. R. Courant, K.O. Friedrichs, Supersonic flow and shock waves, Interscience Publishers, New York,1948.
2. L.V. Ovsyannikov, Lectures on Fundamentals of Gas Dynamics, Institute of computer investigations, Moscow-Izhevsk, 2003, 336 p.
3. G.G. Chernyj, Gas Dynamics, Nauka,Moscow, 1988, 424 p.

Introduction

Paradoxically, if one does not harshly control the process [4,5,10], then the solution with a weak shock is realized in physical experiments and numerical simulations. Up to now, in spite of numerous qualitative studies (see, e.g., [6-9]), there was no rigorous explanation of this phenomenon. It should be noted that it will be absolutely unclear which of two possible solutions is realized in any concrete case until a strict result is obtained. Moreover, as is noted in [10], the appearance of hybrid "solutions" is also possible.

4. M.D. Salas, B.D. Morgan, Stability of shock waves attached to wedges and cones, AIAA J. 21 (12) (1983), 1611-1617.
5. A.N. Lubimov, V.V. Rusanov, Gas Flow Around Pointed Bodies, Nauka, Moscow, 1970.

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6. A.I. Rylov, On regimes of flowing around peaked bodies of finite thickness for arbitrary supersonic sounds of incoming flow, Prikl. Mat. Mech. 55 (1) (1991), 95-99.
7. A.A. Nikolsky, On plane turbulent gas flow, Theoretical Study in Mechanics of Gas and Liquid: Proc. Central Aerohydrodyn. Inst. (2122) (1981), 74-85.
8. B.M. Bulach, Nonlinear Conic Gas Flows, Nauka, Moscow, 1970, 344 p.
9. B.L. Rozhdestvensky, Revision of the theory on flowing around a wedge by a inviscid supersonic gas flow, Math. Model. 1 (8) (1989), 99-102.
10. V. Elling, T.-P. Liu, Exact Solution to Supersonic Flow onto a Solid Wedge Hyperbolic Problems: Theory, Numerics, Applications, (Proceedings of the Eleventh International Conference on Hyperbolic Problems), Lyon, July 17-21, (2006), 101-112.

Introduction

R. Courant and K.O. Friedrichs [1] proposed to choose solutions according to their stability property, i.e., to study their (asymptotic) Lyapunov's stability/instability. Indeed, in numerical simulations (usually performed by stabilization method) or in a physical experiment, which culls "bad" solutions, this property plays an important role.

Exactly R. Courant and K.O. Friedrichs supposed that the solution corresponding to a strong shock is unstable whereas the solution corresponding to a weak shock is stable by Lyapunov (for $t \to \infty$) against small perturbations of the steady gas flow. That is, actually the question in hand is whether solutions of the corresponding linearized problem are stable or unstable (for various values of parameters).

Introduction

In the case when small perturbations depend only on one "space" variable (angular coordinate) the Courant-Friedrichs hypothesis was fully justified. Though, because of the complexity of coefficients of the linearized problem, for arbitrary upstream Mach numbers M_∞ and an arbitrary angular coordinate this was done only numerically [11-13].

11. A.M. Blokhin, E.N. Romensky, Stability of limit stationary solution in problem on flowing around a circular cone, Proc. Siberian Branch Acad. Sci. USSR 13 (3) (1978), 87-97.
12. A.M. Blokhin, E.N. Romensky, The influence of the properties of the limit steady solution to its stabilization, Proc. Siberian Branch Acad. Sci. USSR 3 (1) (1980), 44-50.
13. V.V. Rusanov, A.A. Sharakshane, Study of linearized nonstationary model of flowing around an infinite wedge, preprint No. 13, Keldysh Inst. of Appl. M., AS USSR, Moscow, 1980.

Introduction

For the essentially more complicated 2D case a certain progress is made for the situation when the main solution corresponds to a strong shock. Firstly, in [14] the well-posedness of the linearized initial boundary value problem has been proved at least for the case of small angles at the wedge's vertex. Secondly, in [15-17] an implicit generalized solution of the linearized problem has been found for compactly supported initial data and under the fulfillment of an additional integral condition at the wedge's vertex (again the angle at the wedge's vertex is assumed small enough).

14. A.M. Blokhin, Energy Integrals and their Applications to Problem of Gas Dynamics, Nauka, Novosibirsk, 1986, 240 p.
15. A.M. Blokhin, D.L. Tkachev, L.O. Baldan, Study of the stability in the problem on flowing around a wedge. The case of strong wave, J. Math. Anal. Appl. 319 (2006), 248-277.

Introduction

For the first time one has managed to realize that the boundary singularity influences on the character of the solution itself. The point is that even for compactly supported initial data there appears a wave at the wedge's vertex that destroys the solution. We can avoid its appearance if we impose an additional integral condition on the initial data. However, this integral condition has purely theoretical character. In practice, it enables one to approach discretely the chosen steady solution. Note that these results were obtained thanks to the technique developed in [18] and described in the monograph [19].

Introduction

16. A.M. Blokhin, D.L. Tkachev, Yu.Yu. Pashinin, Stability condition for strong shock waves in the problem of flow around an infinite plane wedge, Nonlinear Analysis: Hybrid Systems, 2 (2008), 1-17.
17. A.M. Blokhin, D.L. Tkachev, Yu.Yu. Pashinin, The Strong Shock Wave in the Problem on Flow Around Infinite Plane Wedge, Hyperbolic Problems: Theory, Numerics, Applications, (Proceedings of the Eleventh International Conference on Hyperbolic Probles), Lyon, July 17-21, (2006), 1037-1044.
18. D.L. Tkachev, Mixed problem for the wave equation in a quadrant, Sib. J. Diff. Eq. 1 (3) (1998), 269-283.
19. A.M. Blokhin, D.L. Tkachev, Mixed Problems for the Wave Equation in Coordinate Domain, Nova Science Publishers Inc., New York, 1998, 133 p.

Introduction

The case of a weak shock requires an approach which is essentially different from that for a strong shock. The point is that after the application of the Laplace transform with respect to the time there appears a hyperbolic problem which needs a modification of the research technique.

Introduction

We note that in [20] an a priori estimate guaranteeing the exponential in time decay of the solution of the linearized initial boundary value problem (this solution converges to the steady solution with a weak shock) was obtained by the dissipative integrals technique provided that

$$M_1(\theta) = \frac{u_0 cos\theta + v_0 sin\theta}{c_0} > 1, \quad \sigma \leq \theta \leq \theta_s,$$

where σ is the angle at the wedge's vertex, θ_s is the angular coordinate of the adjoint weak shock.

20. A.M. Blokhin, Well-posedness of linear mixed problem on supersonic flowing around a wedge, Siberian Math. J. 29 (5) (1988), 48-57.

This estimate was deduced under rather restrictive assumptions on a class of the generalized solution. In the work [21] which ideas and results are crucially used in the present paper it was proved that the problem has no growing normal modes.

21. A.M. Blokhin, A.D. Birkin, Study of stability of stationary rigimes of supersonic flowing around an infinite wedge, Appl. Math. Techn. Phys. 36 (2) (1995), 181-195.

Introduction

The Courant-Friedrichs hypothesis on the linear level for the case of weak shock was justified in [22,23] (it was assumed that the strong Lopatinsky condition holds on the shock wave and the initial data are compactly supported with supports separated from the coordinate axes).

22. A.M. Blokhin, D.L. Tkachev. Stability of a supersonic flow about a wedge with weak shock wave. Sbornik: Mathematics, 2009. Vol. 200:2, 157-184.

23. D.L. Tkachev, A.M. Blokhin. Courant - Frie drichs hypothesis and its justification at the linear level. Hyperbolic Problems: Theory, Numerics and Applications. Proceedings of Symposia in Applied Mathematics, Maryland, USA, 2009. Amer Mathematical Society.Vol. 67. Pp. 959-967.

The author's results on the justification of the Courant-Friedrichs hypothesis on the linear level are collected in the monograph [24].
24. A.M. Blokhin, D.L. Tkachev. Justification of the Courant-Friedrichs hypothesis in the problem on flow onto a wedge. Novosibirsk, Tamara Rozhkovskaya, 2011, 140 p. (in Russian)
Among recent works in which some particular results were obtained we mention the paper [25]
25. Elling V., Liu T. - P. Supersonic flow onto a solid wedge. Comm. Pure Appl. Math. 2008, 51 (10), p. 1347-1448,
in which the validity of the Courant-Friedrichs hypothesis is studied for the original nonlinear problem for the case of weak shock. Strong assumptions like the flow potentiality and self-similarity enabled one to reduce the analysis to the study of some nonlinear second order equation.

Introduction

Unlike [22], [23], in my talk I consider a more general case when the weak (but not strong) Lopatinsky condition holds on the shock front. It turns out that the Courant-Friedrichs hypothesis is still valid in this case in spite of a number of special feature. These features are connected with the appearance of additional perturbations which are plane waves.

Statement of main and reduced problems.

The linearized problem of supersonic stationary inviscid gas flow onto a planar infinite wedge is formulated as follows.

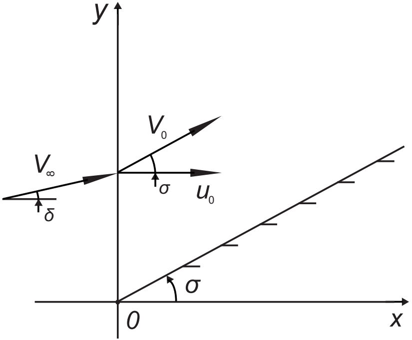

Fig. 2.

Statement of main and reduced problems.

In the domain $t, x > 0$, $y > x \cdot tg\sigma$ we seek for a solution of the acoustics system

$$AU_t + BU_x + C_\sigma U_y = 0 \qquad (1.1)$$

satisfying the following boundary conditions on the shock (at $x = 0$) and on the wedge (at $y = x \cdot tg\sigma$):

$$u_1 + du_3 = 0, \quad u_3 + u_4 = 0, \quad u_2 = \frac{\lambda}{\mu} F_y, \qquad (1.2)$$

$$F_t + F_y tg\sigma = \mu u_3;$$

$$u_2 = u_1 \cdot tg\sigma, \qquad (1.3)$$

and the initial data for $t = 0$:

$$U(0, x, y) = U_0(x, y), \quad F(0, y) = F_0(y). \qquad (1.4)$$

Statement of main and reduced problems.

Here $U(t,x,y) = (u_1, u_2, u_3, u_4)^T$; u_1, u_2, u_3, u_4 are small perturbations of the velocity, the pressure, and the entropy respectively, $x = F(t,y)$ is a small disturbance of the shock front with

$$F(t,0) = F_0(0) = 0, \qquad (1.5)$$

Statement of main and reduced problems.

the components of $U_0(x,y)$ are compactly supported functions, i.e., $supp\ u_{0i} \subset R^2_+ = \{(x,y)|x,y>0\}$, $i=1,2,3,4$. The matrices A, B, and C_σ read

$$A = diag(M^2, M^2, 1, 1), B = \begin{pmatrix} M^2 & 0 & 1 & 0 \\ 0 & M^2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

Statement of main and reduced problems.

$C_\sigma = C + tg\sigma \cdot A$; $M = \frac{u_0}{c_0}$, $M < 1$ is the downstream Mach number (u_0, v_0 are components of the velocity field for the steady solution, c_0 is the sound speed in the gas at the rest), and the d, λ, and μ are some physical constants.

Statement of main and reduced problems.

If the solution of the initial boundary value problem (1.1)–(1.4) is continuous up to the boundary $x = 0,\ y = x \cdot tg\sigma$, taking into account (1.5), it follows from the boundary conditions (1.2), (1.3) that the compatibility condition
$(\lambda + d \cdot tg^2\sigma)u_3(t, 0, 0) = 0, \quad t \geq 0$, should be satisfied on the edge $t \geq 0,\ x = y = 0$. That is, if $D_1 = \lambda + d \cdot tg^2\sigma \neq 0$, then

$$U(t, 0, 0) = 0, \quad t \geq 0. \tag{1.6}$$

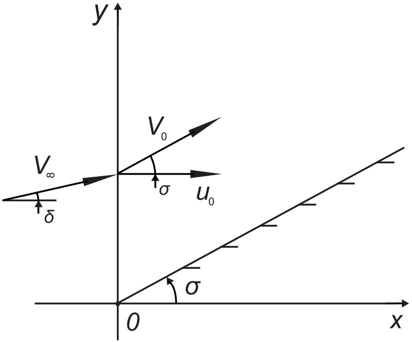

Fig. 2.

Remark

The initial boundary value problem (1.1)–(1.4) was formulated for the case when the main solution corresponds to the gas flow onto the wedge with a shock wave directed along the axis Oy (see Fig. 2).

We consider the case of a weak shock, i.e., we assume that

$$M_0 = \sqrt{\frac{u_0^2 + v_0^2}{c_0^2}} = \frac{M}{cos\sigma} > 1. \qquad (1.7)$$

Statement of main and reduced problems.

Using known relations on an oblique shock wave [26], inequality (1.7) can be rewritten as

$$\left(tg^2\delta - \frac{\gamma - 1}{\gamma + 1}\right)M_N^4 - \frac{3-\gamma}{1+\gamma}M_N^2 + \frac{2}{\gamma + 1} > 0. \qquad (1.8)$$

Here $M_N = M_\infty \cdot cos\delta$, $M_\infty = U_\infty / c_\infty$ is the upstream Mach number, $\gamma > 1$ is the adiabat index.

26. Si-I. Bai, Introduction to the theory of compressible fluid, Moscow: Publishing house of foreign literature, 1962.

At the same time, with the explicit formulas for the coefficients d and λ for a polytropic gas the condition $D_1 \neq 0$ becomes

$$(tg^2\delta - \frac{\gamma - 1}{\gamma + 1})M_N^4 + \left(tg^2\delta - \frac{3-\gamma}{1+\gamma}\right)M_N^2 \qquad (1.9)$$

$$+\frac{2}{\gamma + 1} \neq 0.$$

Statement of main and reduced problems.

Assume that the solution of problem (1.1)–(1.4) is not only continuous but also have the second-order derivatives that are continuous up to the boundary. Using cross differentiation, problem (1.1)–(1.4) under the condition (1.6) can be reduced to the following problem for the component u_3.

Statement of main and reduced problems.

In the domain $t, x > 0,\ y > x \cdot tg\sigma$ we seek for a smooth solution of the wave equation

$$\Big\{M^2 L_1^2 - L_2^2 - \big(\partial_y\big)^2\Big\} u_3 = 0, \qquad (1.10)$$

satisfying the following boundary conditions on the shock wave (at $x = 0$) and on the wedge (at $y = x \cdot tg\sigma$):

$$\Big\{m L_1^2 + n L_2^2 - \frac{\beta}{M^2} L_1 L_2\Big\} u_3 = 0; \qquad (1.11)$$

$$\{cos\sigma \cdot \partial_y - sin\sigma \cdot \partial_x\} u_3 = 0; \qquad (1.12)$$

$$u_3(t, 0, 0) = 0; \qquad (1.13)$$

and the initial data for $t = 0$

$$u_3|_{t=0} = u_0(x, y),\ \ (u_3)_t|_{t=0} = u_1(x, y) \qquad (1.14)$$

Statement of main and reduced problems.

(the derivative $(u_3)_t|_{t=0}$ is found from the third equation of system (1.1)). The following notations were used above:

$L_1 = \frac{1}{\beta} \cdot l_1, \quad l_1 = \partial_t + tg\sigma \cdot \partial_y, \quad L_2 = \beta \cdot \partial_x - \frac{M^2}{\beta} \cdot l_1;$

$\beta^2 = 1 - M^2, \quad n = -\frac{\lambda}{\beta}, \quad m = \beta d + \frac{\lambda M^2}{\beta}.$

Statement of main and reduced problems.

The opposite is true as well, i.e., any smooth solution u_3 of problem (1.10)–(1.14) uniquely defines a smooth solution $U(t,x,y)$, $F(t,y)$ of problem (1.1)–(1.5), (1.6). Thus, these two problems are equivalent.

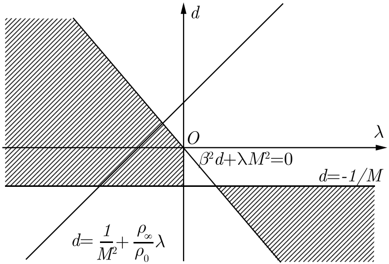

Fig. 3.

Statement of main and reduced problems.

We will assume that the weak Lopatinski condition [23] is satisfied on the boundary
$x = 0$ for problem (1.10)–(1.14), i.e.,

$$\begin{cases} d > -\frac{1}{M},\ d < -\frac{\lambda M^2}{\beta^2}, \\ \lambda < 0 \end{cases} \qquad \begin{cases} d < -\frac{1}{M}, \\ d > -\frac{\lambda M^2}{\beta^2}. \end{cases} \quad (1.15)$$

(see Fig. 3 for the corresponding shaded domain of admissible parameters λ and d).

23. R. Sakamoto, Hyperbolic boundary value problems, Cambridge, 1978, 210 p.

Note that on the boundary $y = x \cdot tg\sigma$ the uniform Lopatinski condition is satisfied.

Statement of main and reduced problems.

Let us make the convenient transformation of coordinates

$$\begin{cases} x' & = & x, \\ y' & = & y - tg\sigma \cdot x \end{cases}$$

(primes are dropped below). Then problem (1.10)–(1.14) is rewritten as

$$\Big\{M^2\Big(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\Big)^2-\Big(\frac{\partial}{\partial x}-tg\sigma\frac{\partial}{\partial y}\Big)^2-\Big(\frac{\partial}{\partial y}\Big)^2\Big\}u=0, \qquad (1.16)$$

$$t, x, y > 0;$$

$$\Big\{\Big(\frac{\partial}{\partial t}+tg\sigma\frac{\partial}{\partial y}\Big)\Big[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+d\Big(\frac{\partial}{\partial t}+tg\sigma\frac{\partial}{\partial y}\Big)$$

$$-\frac{1}{M^2}\Big(\frac{\partial}{\partial x}-tg\sigma\frac{\partial}{\partial y}\Big)\Big]+\lambda\Big(\frac{\partial}{\partial y}\Big)^2\Big\}u=0, \quad x=0; \qquad (1.17)$$

Statement of main and reduced problems.

$$\Big(\frac{\partial}{\partial y} - sin\sigma cos\sigma * \frac{\partial}{\partial x}\Big)u = 0, \quad y = 0; \tag{1.18}$$

$$u(t, 0, 0) = 0; \tag{1.19}$$

$$u|_{t=0} = u_0(x, y), \quad u_t|_{t=0} = u_1(x, y) \tag{1.20}$$

(we drop the subscript for the unknown u_3). By virtue of the established equivalence, it is enough to formulate main results for the solution $u(t, x, y)$ of problem (1.16)–(1.20).

Statement of main and reduced problems.

Recall that we assume the existence of second-order derivatives of the solution of problem (1.16)–(1.20) which are smooth up to the boundary. Let us additionally assume the fulfillment of the following property characterizing the behavior of the solution for large t and x: there exist parameters s_0 and p_0 such that the function $e^{-s_0 \cdot t} \cdot e^{-p_0 \cdot x} u(t, x, y)$ is bounded for $t, x \to +\infty$ for any fixed $y > 0$, i.e.,

$$u(t, x, y) = O\big(e^{s_0 t + p_0 x}\big), \quad t, x \to +\infty, \qquad (1.21)$$

$y > 0$ is fixed.

Our main results are following.

Theorem 1.
If the initial data are compactly supported, then the classical solution of problem (1.16)–(1.20) satisfying the growth condition (1.21) exists, is unique and determined by formula:

$$u(t,x,y) = \left(\frac{\partial}{\partial y''} + B_0 \frac{\partial}{\partial x''}\right) \int_0^{\left(\frac{x''+y''}{2}, \frac{B_0(x''+y'')}{2}\right)} \bar{g}(t,\xi)$$

$$*_t E\Big(t - \frac{M^2}{2\Delta}(\sqrt{\Delta}(y'' - B_0\xi) - (x'' - \xi)tg\sigma), x'' - \xi,$$

$$y'' - B_0\xi\Big)d\xi - \int_0^{(x''-y'',0)} \bar{l}(t,\xi)$$

$$*_t E\Big(t - \frac{M^2}{2\Delta}(\sqrt{\Delta}y'' - (x'' - \xi)tg\sigma), x'' - \xi, y''\Big)d\xi$$

$$+\Big(\frac{M^2}{2\sqrt{\Delta}}\frac{\partial}{\partial t} - \frac{\partial}{\partial y''}\Big)\int_0^{(x''-y'',0)} \bar{f}(t,\xi)$$

$$*_t E\Big(t - \frac{M^2}{2\Delta}(\sqrt{\Delta}y'' - (x'' - \xi)tg\sigma), x'' - \xi, y''\Big)d\xi$$

$$+\frac{M^2}{\Delta}\int\limits_{\square OQM_0P} (tg\sigma \cdot u_{0\xi} + \sqrt{\Delta}u_{0\eta})$$

$$\times E\Big(t - \frac{M^2}{2\Delta}(\sqrt{\Delta}(y'' - \eta) - (x'' - \xi)tg\sigma), x'' - \xi,$$

$$y'' - \eta\Big) d\xi d\eta + \frac{\beta^2 M^2}{4\Delta} \frac{\partial}{\partial t} \int\limits_{\square OQM_0P} u_0(\xi, \eta)$$

$$\times E\Big(t - \frac{M^2}{2\Delta}(\sqrt{\Delta}(y'' - \eta) - (x'' - \xi)tg\sigma), x'' - \xi,$$

$$y'' - \eta\Big) d\xi d\eta + \frac{\beta^2 M^2}{4\Delta} \int\limits_{\square OQM_0P} u_1(\xi, \eta)$$

$$\times E\Big(t - \frac{M^2}{2\Delta}(\sqrt{\Delta}(y'' - \eta) - (x'' - \xi)tg\sigma), x'' - \xi,$$

$$y'' - \eta\Big) d\xi d\eta,$$

B_0, Δ - are some constants,
(where symbol " $*_t$ " serves for a designation of operation of convolution on a variable t);

$$x'' = 2(y + x\frac{tg\sigma}{\beta^2}), \quad y'' = 2\frac{\sqrt{\Delta}}{\beta^2}x,$$

the first integral is over the line $y'' = B_0 x''$, and the next two integrals are over the abscissa axis $y'' = 0$; in the last two integrals over the quadrangle $\square\, OQM_0P$ we have the following coordinates: the point $Q(x'' - y'', 0)$, the point $P(\frac{x''+y''}{2}, \frac{B_0(x''+y'')}{2})$, and the point $M_0(x'', y'')$;

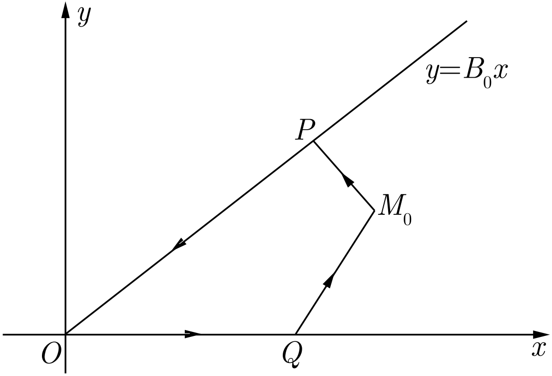

Fig. 4.

the function $E(t, x, y)$ is the fundamental solution of the operator of equation (1.16); $u_0(x, y)$ and $u_1(x, y)$ are the initial data (where the coordinates x, y are expressed through the variables x'', y''); the functions $\bar{g}(t, x'')$, $\bar{l}(t, x'')$, and $\bar{f}(t, x'')$ respectively are determined below, and in particular $\bar{f}(t, x'')$ is determined as follows:

$$\bar{f}(t,x'') = \sum_{n=0}^{N(t,x'')} L^{-1}_{p\to x'',s\to t} H_n(p,s), \qquad (1.22)$$

where $N(t,x)$ is a certain integer number, $N(t,x) \geq 0$.

Theorem 2.

The boundary value $f(t,y) = u(t,x,y)|_{x=0}$ of the solution of problem (1.16)–(1.20) on the shock wave is a superposition of a finite number of cylindric waves, i.e., the representation (1.22) takes place. If the point y belongs to a compact Y lying on the positive real semi-axis, then

$$f(t,y) \equiv 0 \text{ for } t \geq t_*\big(supp\ u_0,\ supp\ u_1,\ Y\big),\ \ y \in Y.$$

Conclusion

So,

1) we prove on the linearized level that the solution with a weak shock (under the fulfillment of only the weak Lopatinsky condition on the shock wave) is asymptotically stable (by Lyapunov).

2) Moreover, for compactly supported initial data any solution of the linearized initial boundary value problem becomes stationary for a finite time.

3) Thus, on the linearized level we completely justify the well-known Courant-Friedrichs hypothesis [R. Courant, K.O. Friedrichs, Supersonic flow and shock waves, Interscience Publishers, New York, 1948] that the solution with a strong shock is unstable whereas the solution with a weak shock is stable.

Thank you for attention