The onset of instability in quasi-linear systems

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Instability $\equiv$ instantaneous, relative amplification in low norms of solutions issued from initial perturbations that are arbitrarily small in high norms.

Instability of reference solution

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$$\forall\, T > 0, \quad \forall\, m \in \mathbb{R}, \quad \forall\, \mathcal{U}_m \text{ n'hood of 0 in } H^m, \quad \forall\, \alpha \in (0, \alpha_0], \quad \alpha_0 \leq 1$$

$$\sup_{0 \leq t \leq T} \sup_{(u - u_s)(0) \in \mathcal{U}_m} \frac{|(u - u_s)(t)|_{H^s}}{|(u - u_s)(0)|^{\alpha}_{H^m}} = \infty, \qquad \forall\, s > 1 + d/2.$$

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- u_s unstable $\equiv$ lack of $H^m \to H^s$ Hölder continuity for flow at $u_s(0)$.
- Weaker than lack of continuity of flow.

  Compare with symmetric systems: $A_j \equiv A_j^*$, $s > 1 + d/2$, for which flow is:

  continuous: $B_{H^s}(0, R) \to C^0([0, T(R)], H^s(\mathbb{R}^d))$,

  Lipschitz: $B_{H^{s+1}}(0, R) \to C^0([0, T(R)], H^s(\mathbb{R}^d))$.

Point of view

Instability criteria in terms of growth properties of the micro-local flow.

Micro-local flow $S(\varepsilon, \tau; t, x, \xi)$ defined as solution to family of ODEs:

$$\partial_t S + A(u_s(\varepsilon t, x), i\xi) S = 0, \qquad S(\tau; \tau) \equiv \mathrm{Id},$$

where

$$(t, x, \xi) \to A(u_s(t, x), \xi) = \sum_{1 \leq j \leq d} \xi_j A_j(u_s(t, x)) \in \mathbb{R}^{n \times n}.$$

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- Symbol of a differential operator $\equiv$ freeze coefficients then Fourier, i.e. change ∂_x into $i\xi$ while keeping dependence in t, x.

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- stability: Rouché's Theorem vs Weyl-type theorems
- BUT exact description of S typically not available
- AND does growth for S imply growth of small initial perturbations ?

Assumption: growth for the micro-local flow

Define advected micro-local flow $\tilde{S}$ by $\partial_t \tilde{S} + \tilde{A}(\varepsilon t, x, i\xi)\tilde{S} = 0$, where
$\tilde{A}(t, x, \xi) = \Big(Q^{-1}(A(u_s) - \mu)Q\Big)(t, x_\star(t, x, \xi), \xi_\star(t, x, \xi))$, with bicharacteristics:
$\partial_t(x_\star, \xi_\star) = (-\partial_\xi \mu, \partial_x \mu)(t, x_\star, \xi_\star)$, $(x_\star, \xi_\star)(0, x, \xi) = (x, \xi)$.

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ex. $\partial_t + \partial_x + it\partial_x$, factor out transport part.

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Assumption

The advected micro-local flow $\tilde{S}$ *satisfies the bounds*

$$\exp\left(\gamma(x,\xi)\varepsilon^\ell t^{\ell+1}\right) \lesssim |\tilde{S}|, \qquad |\partial_x^\alpha \partial_\xi^\beta \tilde{S}| \lesssim \exp\left((\gamma + c)(x,\xi)\varepsilon^\ell t^{\ell+1}\right),$$

locally around (x_0, ξ_0), *where* γ *and* c *are continuous at* (x_0,ξ_0), *with* $\gamma(x_0,\xi_0) > 0$ *and* $c(0, x_0, \xi_0) = 0$.

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ISSUES:

- Verification of such an assumption ?
- Consequence for the system of PDEs ?

Result

$\partial_t u + A(u, \partial_x) u = 0$, $A(u, \partial_x) = \sum_{1 \le j \le d} A_j(u) \partial_{x_j}$.

Reference solution: $u_s(t, x)$, local, smooth.

Symbol: $A(u_s(t, x), \xi)$.

Advected micro-local flow: $\partial_t \tilde{S} + \tilde{A}(\varepsilon t, x, i\xi)\tilde{S} = 0$, with advected symbol $\tilde{A} = Q(A - \mu)Q^{-1}(x_\star, \xi_\star)$, where $(x_\star, \xi_\star)$ bicharacteristics of μ.

Growth of advected micro-local flow: locally near (x_0, ξ_0) :

$$\exp\left(\gamma(x, \xi)\varepsilon^\ell t^{\ell+1}\right) \lesssim |\tilde{S}|,$$

$$|\partial_x^\alpha \partial_\xi^\beta \tilde{S}| \lesssim \exp\left((\gamma + c)(x, \xi)\varepsilon^\ell t^{\ell+1}\right),$$

Theorem (Lerner, Nguyen, T)

Growth of the advected micro-local flow implies instability of reference solution u_s.

Trivial examples: defect and loss of hyperbolicity

Defect of hyperbolicity ($\ell = 0$ in main Assumption):

$$(\partial_t + i\partial_x)u = 0, \quad u(0,x) = \varepsilon^M e^{ix\xi_0/\varepsilon}\theta(x), \quad \text{with } \hat{\theta}(0) \neq 0.$$

Fourier:

$$(\partial_t - \xi)\hat{u} = 0, \quad \hat{u}(0,\xi) = \varepsilon^M \hat{\theta}(\xi - \xi_0/\varepsilon),$$

so that $\hat{u}(\varepsilon t, \xi_0/\varepsilon) = \varepsilon^M e^{t\xi_0}\hat{\theta}(0) \geq O(1)$ in time $O(|\log \varepsilon|)$.

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Loss of hyperbolicity ($\ell = 1$ in main Assumption):

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Hyperbolicity as necessary condition for stability

Symbol: $A(u_s(t,x),\xi) = \sum_{1\leq j\leq d} \xi_j A_j(u_s(t,x))$.

Theorem (Defect of hyperbolicity: Lax 1957, . . . , Métivier 2005)

If the symbol is analytic and for some (x_0,ξ_0), the spectrum of $A(u_s(0,x_0),\xi_0)$ is not included in $\mathbb{R}$, then u_s is unstable.

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Exponential growth effective only if persistence of estimates in v is shown over time $O(|\log\varepsilon|)$, i.e. long-time C-K result.

Example: compressible gas dynamics

Van der Waals pressure law:

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x p(u) = 0, \end{cases} \qquad \text{with } p'(u) < 0, \text{ for some } u \in \mathbb{R}.$$

Symbol is $\xi \begin{pmatrix} 0 & 1 \\ p'(u) & 0 \end{pmatrix}$, with eigenvalues $\lambda_\pm = \pm\xi(p'(u))^{1/2}$.

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Defect of hyperbolicity: $p'(u(0, x_0)) < 0$ for some $x_0 \in \mathbb{R}$.

Loss of hyperbolicity: $p'(u(0)) \geq 0$, $p'(u(0, x_0) = 0$, and

$$\begin{aligned} \lambda_+(t, x_0, 1) &= \Big(0 + tp''(0, x_0)\partial_t u(0, x_0) + O(t^2)\Big)^{1/2} \\ &= \Big(-tp''(0, x_0)\partial_x v(0, x_0) + O(t^2)\Big)^{1/2}. \end{aligned}$$

If $p''(u(0, x_0))\partial_x v(0, x_0) > 0$, then e-values branch out of real axis at $(0, x_0, \xi_0)$ like $\sqrt{t}$.

Example: compressible gas dynamics

Van der Waals pressure law:

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x p(u) = 0, \end{cases} \quad \text{with } p'(u) < 0, \text{ for some } u; \quad \text{symbol: } \xi \begin{pmatrix} 0 & 1 \\ p'(u) & 0 \end{pmatrix}.$$

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Prepared data so as to have loss of hyperbolicity:

$$p'(u(t, x_0)) = \underbrace{p'(u(0, x_0))}_{=0} - t\, \underbrace{p''(u(0, x_0))\partial_x v(0, x_0)}_{>0} + O(t^2) = -t + t^2.$$

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Micro-local flow:

$$\partial_t S + \xi \begin{pmatrix} 0 & 1 \\ p'(u) & 0 \end{pmatrix} S = \partial_t S + \xi \begin{pmatrix} 0 & 1 \\ -\varepsilon t & 0 \end{pmatrix} S = 0.$$

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Column (y_1, y_2) of S solves $y_1' + i\xi y_2 = 0$, $y_2' + i\xi\varepsilon t y_1 = 0$, implying

$$y_1'' = \xi^2 \varepsilon t y_1 \qquad \text{Airy equation}$$

Hence $y_1 \sim e^{(2\xi/3)\varepsilon^{1/2} t^{3/2}}$: main Assumption satisfied with $\ell = 1/2$.

Example

Van der Waals example generalizes into:

Proposition (Lerner, Nguyen, T)

Assume that the characteristic polynomial

$$P(t, x, \xi, \lambda) = \det\big(A(u_s(t, x), \xi) - \lambda\big)$$

satisfies at $\omega_0 = (0, x_0, \xi_0, \lambda_0)$, *with* $\lambda_0 \in \mathbb{R}$,

$$P(\omega_0) = 0, \quad \partial_\lambda P(\omega_0) = 0, \quad (\partial_t P \partial_\lambda^2 P)(\omega_0) > 0.$$

Then advected micro-local flow grows like Airy function, i.e. main assumption is satisfied with $\ell = 1/2$.

Growth condition for micro-local flow implied by conditions bearing only on data.

Assumption: growth for the micro-local flow

Advected micro-local flow $\tilde{S}$ by $\partial_t \tilde{S} + \tilde{A}(\varepsilon t, x, i\xi)\tilde{S} = 0$, where $\tilde{A}(t,x,\xi) = \Big(Q^{-1}(A(u_s) - \mu)Q\Big)(t, x_\star(t,x,\xi), \xi_\star(t,x,\xi))$, with bicharacteristics: $\partial_t(x_\star, \xi_\star) = (-\partial_\xi \mu, \partial_x \mu)(t, x_\star, \xi_\star)$, $(x_\star, \xi_\star)(0,x,\xi) = (x,\xi)$.

Assumption

The advected micro-local flow $\tilde{S}$ satisfies the bounds

$$\exp\left(\gamma(x,\xi)\varepsilon^\ell t^{\ell+1}\right) \lesssim |\tilde{S}|,$$
$$|\partial_x^\alpha \partial_\xi^\beta \tilde{S}| \lesssim \exp\left((\gamma + c)(x,\xi)\varepsilon^\ell t^{\ell+1}\right),$$

locally around (x_0, ξ_0), where γ and c are continuous at (x_0,ξ_0), with $\gamma(x_0,\xi_0) > 0$ and $c(0, x_0, \xi_0) = 0$.

ISSUES:

- Verification of such an assumption ?
- Consequence for the system of PDEs ?

Key Lemma in proof

Lemma

Consider $\partial_t u + \mathrm{op}(M)u = f \in L^\infty([0,T], H^s)$, $u(0,x) = u_0(x) \in H^s$. *Assume that symbol* M *compactly supported in* ξ, *uniformly in* t, x. *Then the unique solution* $u \in C^0([0,T], H^s)$ *is*

$$u(t) = \Sigma(0;t)\tilde{u}_0 + \int_0^t \Sigma(t';t)\tilde{f}(\tau)\,d\tau,$$

where

- $\tilde{u}_0$ *is almost* u_0 *and* $\tilde{f}$ *is almost* f.

  For some R_1, R_2 *with nice bounds,* $\tilde{u}_0 = (\mathrm{Id} + \varepsilon R_1)u_0$, $\tilde{f} = (\mathrm{Id} + \varepsilon R_2)f$.

- *the solution operator* Σ *is almost* $\mathrm{op}(S_0)$, *where* S_0 *is the micro-local flow associated with* M :

$$\Sigma = \mathrm{op}\big(S_0 + \varepsilon(\dots)\big), \quad \partial_t S_0 + M S_0 = 0, \quad S_0(\tau;\tau) = \mathrm{Id}.$$

Remarks on representation Lemma

Pseudo-differential operators:

$$\mathrm{op}(M)v := \int e^{ix\cdot\xi} M(t,x,\varepsilon\xi)\hat{v}(\xi)\,d\xi.$$

- Calderón-Vaillancourt theorems:

$$\big|\mathrm{op}(M)v\big|_{L^2} \lesssim \sup_{\substack{|\alpha|\le d\\ |\beta|\le d}} |\partial_x^\alpha \partial_\xi^\beta M|_{L^\infty(\mathbb{R}^{2d})} |v|_{L^2}.$$

- Composition of operators:

$$\mathrm{op}(M_1)\mathrm{op}(M_2) = \mathrm{op}(M_1 M_2) + \varepsilon\,\mathrm{op}(M_1 \sharp M_2) + \varepsilon^2\big(\dots\big),$$

with $M_1 \sharp M_2 = -i\sum_{|\alpha|=1} \partial_\xi^\alpha M_1 \partial_x^\alpha M_2$.

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- Content of the Lemma (in autonomous setting: M independent of t):

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  i.e. Fourier integral operator seen as pseudo-differential operator, the symbol of which is the micro-local flow.
- More generally: dropping assumption on support of M and using above Lemma along a partition of unity in frequency space, we obtain representation of “matrix-valued Fourier integral operators” as sums of pseudo-differential operators, the symbol of which are the localized micro-local flows.

Idea of the proof of Duhamel Lemma

Goal is to solve

$$\partial_t \mathrm{op}(S) + \mathrm{op}(M)\mathrm{op}(S) = 0$$

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$$\partial_t \mathrm{op}(S_0) = \mathrm{op}(\partial_t S_0) = -\mathrm{op}(MS_0) = -\mathrm{op}(M)\mathrm{op}(S_0) + \mathrm{op}(M \sharp S_0) + \dots$$

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In semi-classical quantization:

$$\partial_t \mathrm{op}_\varepsilon(S_0) = -\mathrm{op}_\varepsilon(MS_0) = -\mathrm{op}_\varepsilon(M)\mathrm{op}_\varepsilon(S_0) + \boxed{\varepsilon}\,\mathrm{op}_\varepsilon(M\sharp S_0) + \varepsilon^2(\dots).$$

Is $\mathrm{op}_\varepsilon(M\sharp S_0)$ a small error ? Not under assumption of lack of hyperbolicity.
But (under appropriate assumptions) $M\sharp S_0$ not growing faster than S_0.

$$\partial_t \mathrm{op}_\varepsilon(S_0) = \mathrm{op}_\varepsilon(M)\mathrm{op}_\varepsilon(S_0) - \boxed{\varepsilon}\,\mathrm{op}_\varepsilon(M\sharp S_0) + \varepsilon^2(\dots).$$

Introduce corrector S_1 solving

$$\partial_t S_1 + MS_1 + M\sharp S_0 = 0, \qquad S_1(\tau;\tau) = 0.$$

Then,

$$\begin{aligned}\partial_t \mathrm{op}(S_0 + \varepsilon S_1) &= -\mathrm{op}(M(S_0 + \varepsilon S_1)) - \varepsilon \mathrm{op}(M\sharp S_0)\\ &= \mathrm{op}(M)\mathrm{op}(S_0 + \varepsilon S_1) + \varepsilon \mathrm{op}(M\sharp S_0) - \varepsilon \mathrm{op}(M\sharp S_0)\\ &\quad + \varepsilon^2\big(\mathrm{op}(M\sharp S_1) + \mathrm{op}(M\sharp_2 S_0) + \varepsilon(\dots)\big).\end{aligned}$$

Repeat (Taylor expansion) until remainder is $\sim \varepsilon^q e^{\gamma t} \ll 1$.

Spectra of symbols and spectra of operators

Assume smooth family of e-values, e-vectors $(x,\xi) \to \lambda(x,\xi), \vec{e}\,(x,\xi)$ for symbol $a(x,\xi)$, in neighborhood of (x_0,ξ_0).
Then for smooth truncation $\xi \to \chi(\xi)$, $x \to \theta(x)$, $\chi \equiv 1$ in n'hood of ξ_0 and $\theta \equiv 1$ in n'hood of x_0,

$$\begin{aligned}
&\mathrm{op}(\chi a)\big(e^{ix\cdot\xi_0/\varepsilon}\vec{e}(x,\xi_0)\theta(x)\big) \\
&\qquad = e^{ix\cdot\xi_0/\varepsilon}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\chi(\xi_0+\varepsilon\xi)a(x,\xi_0+\varepsilon\xi)\,\mathcal{F}\big(\vec{e}(\cdot,\xi_0)\theta\big)(\xi)\,d\xi \\
&\qquad = e^{ix\cdot\xi_0/\varepsilon}a(x,\xi_0)\vec{e}(x,\xi_0)\theta(x) + \varepsilon O(\partial_x(\vec{e}\theta)) \\
&\qquad = \lambda(x,\xi_0)\Big(e^{ix\cdot\xi_0/\varepsilon}\vec{e}(x,\xi)\theta(x)\Big) + \varepsilon O(\partial_x(\vec{e}\theta)).
\end{aligned}$$

Hence spectrum of a appears as $H^1 \to L^2$ ε-pseudo-spectrum for semi-classical operator $\mathrm{op}_\varepsilon(\chi a)$.

Sketch of proof

Goal is to prove

$$\forall\, T > 0, \quad \forall\, m \in \mathbb{R}, \quad \forall\, \mathcal{U}_m \text{ n'hood of 0 in } H^m, \quad \forall\, \alpha \in (0,1],$$

$$\sup_{0 \le t \le T} \; \sup_{(u-u_s)(0) \in \mathcal{U}_m} \frac{|(u-u_s)(t)|_{H^s}}{|(u-u_s)(0)|^{\alpha}_{H^m}} = \infty, \qquad \forall\, s > 1 + d/2.$$

- Consider family of data $u_\varepsilon(0,x) = u_s(0,x) + \varepsilon^M \Re e\big(e^{ix\cdot\xi_0/\varepsilon}\theta(x)\big)$.
- Then $\|u_\varepsilon(0) - u_s(0)\|_{H^m} \lesssim \varepsilon^{M-m} \to 0$ as $\varepsilon \to 0$ if $M > m$.
- It suffices to prove

$$\frac{\|u_\varepsilon(t_\varepsilon) - u_s(t_\varepsilon)\|_{H^s}}{\|u_\varepsilon(0) - u_s(0)\|_{H^m}} \to \infty$$

  as $\varepsilon \to 0$ for a time sequence $t_\varepsilon \to 0$.
- Of course no a priori estimate available. Then assume that for some $C > 0$, all $0 \le t \le T$,

$$\|u_\varepsilon(t) - u_s(t)\|_{H^s} \le C\|u_\varepsilon(0) - u_s(0)\|_{H^m} = O(\varepsilon^{M-m}).$$

By contradiction, it is assumed $\|u_\varepsilon(t) - u_s(t)\|_{H^s} \le C\|u_\varepsilon(0) - u_s(0)\|_{H^m} = O(\varepsilon^{M-m})$ for solution u_ε issued from $u_s(0,x) + \varepsilon^M \Re e\big(e^{ix\cdot\xi_0/\varepsilon}\theta(x)\big)$.

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Two issues:

- Convert bounds for micro-local flow into estimates for u_ε.
- The a priori estimate is extremely weak if m is large.

  At $t = 0 : \|u_\varepsilon(0) - u_s(0)\|_{H^s} = O(\varepsilon^{M-s})$, corresponding to instantaneous loss in upper bound of $O(\varepsilon^{m-s})$.

Transform (projection, change of reference frame, localization in (x, ξ)) equation in perturbation unknown v : $u = u_s + v$ into

$$\partial_t v + \mathrm{op}_\varepsilon^\psi(M) v = \varepsilon f(u_s, v), \qquad v(0) = \varepsilon^M \Re e \left(e^{ix\cdot\xi_0/\varepsilon} \theta(x) \right).$$

Then $|v(t)|_{L^2} \geq |\Sigma(0;t)v(0)|_{L^2} - \varepsilon \int_0^t |\Sigma(t';t) f(u_s,v)(t')|_{L^2}\, dt'$, and by assumption on micro-local flow and a priori estimate:

$$|\Sigma(0;t)v(0)|_{L^2} \geq C\varepsilon^M \exp\left(\varepsilon^\ell t^{\ell+1} \gamma(x_0, \xi_0)\right),$$

$$\int_0^t |\Sigma(t';t) f(t')|_{L^2}\, dt' \leq C \exp\left(\varepsilon^\ell t^{\ell+1} \gamma(x_0, \xi_0)\right) \varepsilon^{M-m}.$$

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Sketch of proof fails by a factor $O(\varepsilon^m)$.

Grenier: gain of $O(\varepsilon^m)$ by construction of precise WKB solution.

LNT: gain of $O(\varepsilon^m)$ by successive localization in (x, ξ).

Lower bound for action of solution operator on datum

$$\begin{aligned}
\mathrm{op}(S(0;t))\big(e^{ix\cdot\xi_0/\varepsilon}\theta(x)\big) &= \int e^{ix\cdot\xi} S(0;t,x,\varepsilon\xi)\theta(\xi-\xi_0/\varepsilon)\,d\xi \\
&= e^{ix\cdot\xi_0/\varepsilon}\int e^{ix\cdot\xi} S(0;t,x,\xi_0+\varepsilon\xi)\theta(\xi)\,d\xi \\
&= e^{ix\cdot\xi_0/\varepsilon} S(0;t,x,\xi_0)\theta(x) + \big(\sim \varepsilon\partial_\xi S\partial_x\theta\big).
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Choose θ pointing in growing direction for S.
Not necessarily eigendirection (cf VdW).

Pseudo- and para-differential calculus

Pseudo-differential calculus $\equiv$ Fourier transform with varying coefficients.

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Symbols: $(x, \xi) \to a(x, \xi) \in \mathbb{R}^{n \times n}$.

Operators (in classical quantization): $\mathrm{op}(a)u = \int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x, \xi) \hat{u}(\xi)\, d\xi$.

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A *pseudo-differential calculus* is the description of a class S^m of symbols such that:

- S^m contains differential operators of order m
- $\mathrm{op}(S^m)$ maps H^{s+m} to H^s.
- $\mathrm{op}(a_{m_1})\mathrm{op}(a_{m_2})$ has a symbol in $S^{m_1+m_2}$ given in terms of a_{m_1}, a_{m_2}.

Classes of pseudo-differential symbols

Definition

A map $a : \mathbb{R}^d_x \times \mathbb{R}^d_\xi \to \mathbb{R}^{n\times n}$ is said to belong to the class $S^m_{1,0}$ if

$$\sup_{\alpha,\beta\in\mathbb{N}^d} \sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d} \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C(1+|\xi|)^{m-|\beta|}.$$

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If $P(x,\partial_x)$ is differential, order m, with smooth, bounded coefficients, then

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Differential operators are pseudo-differential operators.

Symbols in $S^m_{1,0}$ are called classical symbols. There exists other classes $S^m_{\rho,\delta}, \ldots$

Ref: Hörmander.

Calderón-Vaillancourt theorems: $\mathrm{op}(S^0)$ operates in L^2

Theorem:

Let $a : \mathbb{R}^d_x \times \mathbb{R}^d_\xi \to \mathbb{R}^{n\times n}$ such that

$$\mathbf{M}^0_d(a) := \sup_{\substack{|\alpha|\leq d, |\beta|\leq d \\ (x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d}} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| < \infty.$$

Then

$$\mathrm{op}(a) \text{ is bounded } L^2 \to L^2,$$

with norm $\lesssim \mathbf{M}^0_d(a)$.

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Assumption is symmetric in x, ξ.

Bony's Calderón-Vaillancourt theorem

Theorem (Bony, 1981)

For some $\psi : \mathbb{R}^d_\eta \times \mathbb{R}^d_\xi \to \mathbb{R}$, *for* $a \in S^0$, *the operator*

$$\mathrm{op}^\psi(a) = \mathrm{op}\big(\mathcal{F}^{-1}\psi \star_x a\big)$$

operates in L^2, *with norm* $\lesssim M_0^0(a) = \sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d} \sup_{|\beta|\leq d} |\partial_\xi^\beta a(x,\xi)|$.

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Compare with Sobolev product inequality.

Bony's Calderón-Vaillancourt theorem

Theorem (Bony, 1981)

For some $\psi : \mathbb{R}^d_\eta \times \mathbb{R}^d_\xi \to \mathbb{R}$, *for* $a \in S^0$, *the operator*

$$\mathrm{op}^\psi(a) = \mathrm{op}\big(\mathcal{F}^{-1}\psi \star_x a\big)$$

operates in L^2, *with norm* $\lesssim M^0_0(a) = \sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d} \sup_{|\beta|\le d} |\partial^\beta_\xi a(x,\xi)|$.

Compare with Sobolev product inequality.

The remainder $\mathrm{op}(a) - \mathrm{op}^\psi(a)$ is regularizing, as in Sobolev product inequality.

Composition

Theorem

$$\mathrm{op}(a_1)\mathrm{op}(a_2) = \mathrm{op}(a_1 a_2) + \mathrm{op}(a_1 \sharp a_2) + \dots,$$

with subprincipal symbol

$$a_1 \sharp a_2 = -i \sum_{|\alpha|=1} \partial_\xi^\alpha a_1 \partial_x^\alpha a_2.$$

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In semiclassical quantization:

$$\mathrm{op}_\varepsilon(a)u = \int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x, \varepsilon\xi)\hat{u}(\xi)\, d\xi,$$

there holds

$$\mathrm{op}_\varepsilon(a_1)\mathrm{op}_\varepsilon(a_2) = \mathrm{op}_\varepsilon(a_1 a_2) + \varepsilon \mathrm{op}_\varepsilon(a_1 \sharp a_2) + \varepsilon^2 \big(\dots \big)$$

Burgers with complex forcing

$\partial_t u + u\partial_x u = i,\ u(0,x) = u_0(x) \in \mathbb{R}.$

Theorem (Lerner, Morimoto, Xu (2010))

If Burgers with complex source has a local C^2 solution, then u_0 is analytic.

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Numerical test: strong, instantaneous instability not observed.

Rather, linear growth of $\Im m\, u$.

Viscous relaxation of loss of hyperbolicity

$$\big(\partial_t + it\partial_x - \varepsilon\partial_x^2\big)u = 0, \quad u(0,x) = \varepsilon^M e^{ix\xi_0/\varepsilon}\theta(x), \quad \text{with } \hat\theta(0) \neq 0.$$

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in particular

$$\hat{u}(t,\xi_0/\varepsilon) = \varepsilon^M \hat{\theta}(0)\exp\Big(t\xi_0/\varepsilon(t-\xi_0)\Big).$$

Exponential growth after time delay $t_0 = \xi_0$.

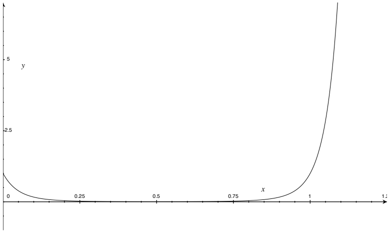

Figure: Graph of $y = \exp(10x(x-1))$.

Viscous Burgers with complex forcing

Proposition (Strani, T)

Given $a \in L^\infty(\mathbb{T})$, if ε is small enough, then the unique solution

$$u \in L^\infty([0, T(\varepsilon)], L^2(\mathbb{T})) \cap L^2([0, T(\varepsilon)], H^1(\mathbb{T}))$$

to

$$\partial_t u + u\partial_x u - \varepsilon \partial_x^2 u = i, \quad u(0, x) = a(x/\varepsilon),$$

has maximal existence time $O(1)$.

Numerical simulation of $t \to \Im m\, u(t)/t$: