ENTROPY FORMULATION FOR FORWARD-BACKWARD PARABOLIC EQUATION

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Collaboration

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Introduction

Forward-backward parabolic equation

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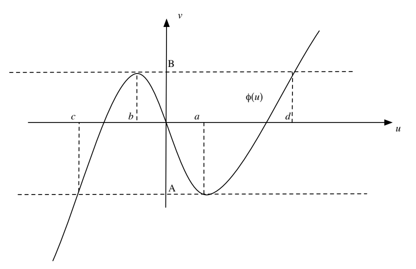
Example 1

Model of phase separation

$$\phi'(u) > 0 \text{ if } u \in (-\infty, b) \cup (a, \infty),\ \phi'(u) < 0 \text{ if } u \in (b, a);$$



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Model of image processing (1D), Perona–Malik equation

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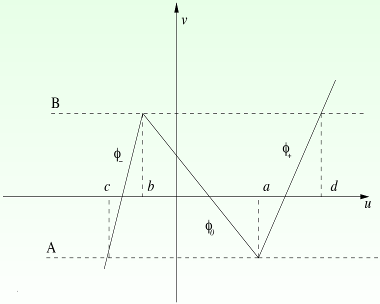
Hollig (Trans. Amer. Math. Soc. 83) ϕ piecewise linear, there is an infinite number of solutions of the Neumann boundary problem.



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Problem is ill-posed since some relevant physical terms are neglected

Phase transition, Cahn–Hilliard equation

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Model of population dynamic, Padron

$$u_t = (\phi(u) + \epsilon u_t)_{xx}.$$

Phase transition

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In the following $\delta = 0$, ϕ is of cubic type.

Novick Cohen-Pego (*Trans. Amer. Math. Soc.* 1991) study the viscosity problem

$$\begin{cases} u_t = \Delta v & \text{in} \quad \Omega \times (0,T] =: Q_T \\ \frac{\partial v}{\partial \nu} = 0 & \text{in} \quad \partial\Omega \times (0,T] \\ u = u_0 & \text{in} \quad \Omega \times \{0\}, \end{cases} \qquad (2)$$

where

$$v := \phi(u) + \epsilon u_t \qquad (\epsilon > 0)\,, \qquad (3)$$

is the *chemical potential*, $\Omega \subseteq \mathbb{R}^n$ is bounded , $\partial\Omega$ regular, $T > 0$.

Partial differential equation in (2) can be rewritten

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Moreover $v = (I - \epsilon\Delta)^{-1}\phi(u)$.
Using the standard theory of ODE in the Banach spaces we have

Theorem

(Novick Cohen-Pego) Given $u_0 \in L^\infty(\Omega)$, $\epsilon > 0$ *there exists a unique solution* (u_ϵ, v_ϵ) *defined in* $(0, T_\epsilon)$, $u_\epsilon \in C^1([0, T_\epsilon), L^\infty(\Omega))$.

A priori estimates

For every $g \in C^1(\mathbb{R})$ such that $g' \geq 0$

$$G(u) = \int_0^u g(\phi(s))\, ds + c.$$

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Then

$$[G(u_\epsilon)]_t = \mathrm{div}\left[g(v_\epsilon)\nabla v_\epsilon\right] - g'(v_\epsilon)|\nabla v_\epsilon|^2 +$$

$$-\frac{1}{\epsilon}\left[g(\phi(u_\epsilon)) - g(v_\epsilon)\right]\left(\phi(u_\epsilon) - v_\epsilon\right) .$$

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Integrating in Ω and using boundary condition we have

$$\frac{d}{dt}\int_\Omega G(u_\epsilon(x,t))\, dx \leq 0$$

that gives a priori estimates in L^∞. Moreover choosing $g(u) \equiv u$ we have

$$\iint_{Q_T}\left\{|\nabla v^\epsilon|^2 + \epsilon|\partial_t u^\epsilon|^2\right\} dxdt \leq C_2\, .$$

Entropy formulation

In analogy with conservation laws we characterize an entropy solution of problem

$$\begin{cases} u_t = \Delta\phi(u) & \text{in} \quad \Omega \times (0,T] = Q_T \\ \frac{\partial \phi(u)}{\partial \nu} = 0 & \text{in} \quad \partial\Omega \times (0,T] \\ u = u_0 & \text{in} \quad \Omega \times \{0\}, \end{cases} \tag{4}$$

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For every $\epsilon > 0$ and $g \in C^1(\mathbb{R})$, $g' \geq 0$ we have

$$\iint_{Q_T} \left\{ G(u^\epsilon)\psi_t - g(v^\epsilon)\nabla v^\epsilon \cdot \nabla\psi - g'(v^\epsilon)|\nabla v^\epsilon|^2\psi \right\} \geq 0 \tag{5}$$

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for every $\psi \in C_0^\infty(Q_T)$, $\psi \geq 0$.
The idea is to pass in the limit in (5) to characterize an entropy solution of (4).

Plotnikov's results

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Using previous a priori estimate we deduce that there exist two subsequence $\{u^{\epsilon_n}\}$, $\{v^{\epsilon_n}\}$ and a couple (u, v) $u \in L^\infty(Q_T)$, $v \in L^\infty(Q_T) \cap L^2((0,T); H^1(\Omega))$ such that for every $T > 0$:

$$u^{\epsilon_n} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(Q_T)\,,$$

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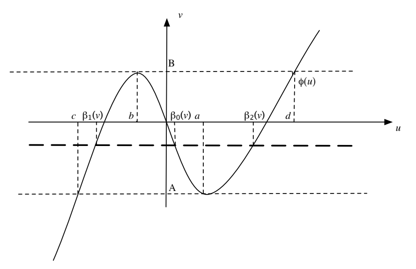
Unfortunately this is not enough to pass to the limit in (5). Plotnikov gives a characterization of the Young measure $\nu(x,t)$ associate to the sequence $\{u^{\epsilon_n}\}$ proving that this is given by superposition of Dirac measures,

$$\nu_{(x,t)}(\tau) = \sum_{i=0}^{2} \lambda_i(x,t)\delta(\tau - \beta_i(v(x,t))) \text{ a.e in } Q_T$$

where $\beta_i(v)$, $i = 0, 1, 2$ are the three branches of the graph $v = \phi(u)$



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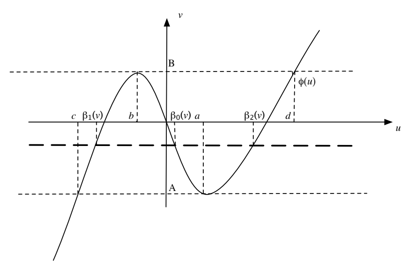

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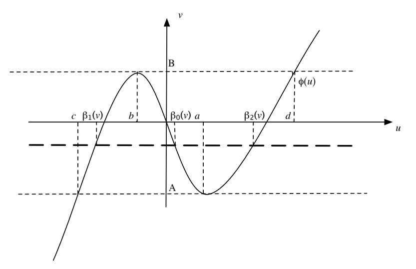

Moreover, $0 \leq \lambda_i \leq 1$ e $\sum_{i=0}^{2} \lambda_i(x,t) = 1$.
It could be proved that a subsequence of $\{v^{\epsilon_n}\}$ converges strongly to the function v and (u, v) satisfies equation $u_t = \Delta v$ in the weak sense,

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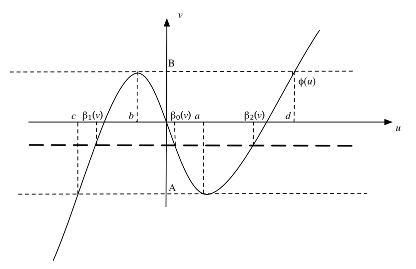

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It could be proved that a subsequence of $\{v^{\epsilon_n}\}$ converges strongly to the function v and (u, v) satisfies equation $u_t = \Delta v$ in the weak sense, meanwhile u satisfies $u_t = \Delta(\phi(u))$ in the sense of measured valued solutions.

Letting $\epsilon_n \to 0^+$ in the viscous entropy inequality

$$\iint_{Q_T} \left\{ G(u^{\epsilon_n})\psi_t - g(v^{\epsilon_n})\nabla v^{\epsilon_n} \cdot \nabla\psi - g'(v^{\epsilon_n})|\nabla v^{\epsilon_n}|^2\psi \right\} dxdt +$$

$$\int_\Omega G(u_0(x))\psi(x,0)dx \geq 0$$

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we have

$$\iint_{Q_T} \Big\{ \overline{G}(u)\psi_t - g(v)\nabla v \cdot \nabla\psi - g'(v)|\nabla v^2|\psi \Big\}\, dxdt +$$

$$\int_\Omega G(u_0(x))\psi(x,0)\, dx \geq 0 \tag{6}$$

where $\overline{G}(u) = \sum_{i=0}^{2} \lambda_i G(\beta_i(v))$.

Entropy solution

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Definition

Given $u_0 \in L^\infty(\Omega)$ *an entropy solution of problem forward–backward (4) is given by the functions* $\lambda_i \in L^\infty(Q_T)$, $i = 0, 1, 2$, $u \in L^\infty(Q_T)$, $v \in L^\infty(Q_T) \cap L^2((0, T), H^1(\Omega))$. *Such that*
(i) $\sum_{i=0}^{2} \lambda_i = 1$, $\lambda_i \geq 0$, $u = \sum_{i=0}^{2} \lambda_i \beta_i(v)$;
(ii) $u_t = \Delta v$ *in the weak sense;*
(iii) u *and* v *satisfy (6) for every* $g \in C^1(\mathbb{R})$, $g' \geq 0$ *(entropy condition).*

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(iii) u and v satisfy (6) for every $g \in C^1(\mathbb{R})$, $g' \geq 0$ (entropy condition).

Problems: Existence in a stronger sense? Uniqueness? Study of the evolution of the different phases.

Two phase entropy solution

Case $n = 1$**.** Let $\Omega = (-L, L)$, $u_0 \leq b$ in $(-L, 0)$, $u_0 \geq a$ in $(0, L)$, initial data in the two stable phases.

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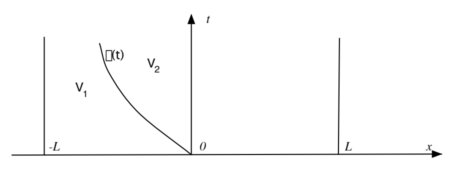
We search a solution in which the two stable phases are separated by an interface ξ, $\xi(0) = 0$,
$V_1 := \{(x, t) \in Q_T \mid -L \leq x < \xi(t),\ t \in (0, T)\}$, $V_2 := Q_T \setminus \overline{V}_1$



An entropy solution is a triple of functions (ξ, u, v) such that :
(a) $\xi \in C^1((0,T))$, $\xi(0) = 0$, $u \in L^\infty(Q_T)$,
$v \in L^\infty(Q_T) \cap L^2((0,T)\,;\, H^1((-L,L))$;
(b) u, v satisfy

$$u = \beta_i(v) \text{ in } V_i \quad (i = 1,2) \quad (v = \phi(u));$$

(c) $u_t = v_{xx}$ in the weak sense, entropy condition, boundary and initial condition.

Determinate conditions for the interface.

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Theorem

(Evans–Portilheiro Math. Models Methods Appl. Sci. (2004))
Let u, v, ξ a two phase entropy "piecewise regular" solution then
(i) Rankine-Hugoniot condition*:*

$$\xi' = -\frac{[v_x]}{[u]} \quad a.e \text{ } on \ \gamma := \{(\xi(t), t) \ : \ t \in (0, T)\}.$$

(ii) entropy condition*:*

$$\xi' \, [G(u)] \geq -g(v)[v_x] \quad a.e. \ \ su \ \gamma \, .$$

where $[h] := h(\xi(t)+, t) - h(\xi(t)-, t)$.

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$$\begin{cases} (a)\ \xi'(t) > 0 \implies \phi(u(\xi(t), t)) = A\,; \\ (b)\ \xi'(t) < 0 \implies \phi(u(\xi(t), t)) = B. \end{cases}$$

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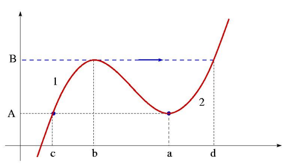
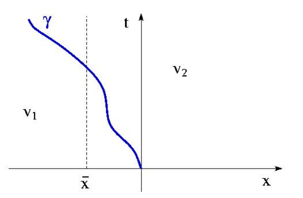
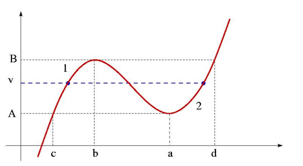
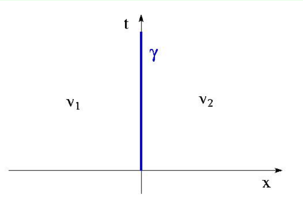
Condition for the phase change. We can pass from phase 1 to phase 2 only if $v = B$





If $v \in (A, B)$ phase does not change



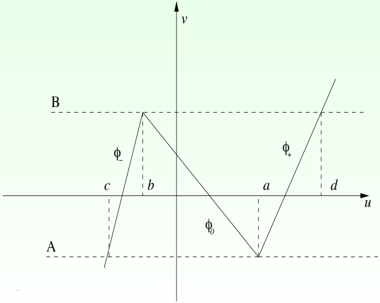


Piecewise linear case

$$\phi(u) = \begin{cases} \phi_-(u) & \text{if} \quad u \leq b \\ \phi_0(u) & \text{if} \quad b < u < a \\ \phi_+(u) & \text{if} \quad u \geq a\,, \end{cases}$$

where

$$\phi_\pm(u) := \alpha_\pm\, u + \beta_\pm\,, \qquad \phi_0(u) := \frac{A(u-b) - B(u-a)}{a-b}\,.$$



Theorem

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i) $\alpha_- u_0(0-) + \beta_- = \alpha_+ u_0(0+) + \beta_+ \in (A, B)$;

ii) $\alpha_- u_0'(0-) \neq \alpha_+ u_0'(0+)$.

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Then there exists $\tau > 0$ *such that the two phase problem has solution in* $\mathbb{R} \times (0, \tau)$.

PROOF (idea)
Local existence using two auxiliary problems: moving boundary problem and steady boundary problem.

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We can extend in time the solution until a first time τ such that $\xi'(\tau) = 0$ and $\phi(u(\xi(\tau), \tau)) = A$ or B (critical case).

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Is it possible to obtain the solution with a sequence of solutions of moving boundary problems that alternate in time with solutions of steady boundary problems?
It is necessary to study the critical case

Theorem

(T. Siam J. Mat. Anal. 2011) Let (ξ, u) be a solution of the two phase problem in Q_τ. Let $t_1 < \tau$ such that in (t_1, τ) the solution is given by the solution either of the moving boundary problem or of the steady boundary problem. Then there exists $t_2 > \tau$ such that the solution of the two phase problem can be extended in $(0, t_2)$.

Nolinear ϕ

Smarrazzo, T. (Discrete and Continuous dynamical systems A.) Rankine–Hugoniot has to be understood in a weak sense, the same for the entropy condition (theory of vector filelds with divergence–measure). We obtain the same compatibility condition for the two phase problem.

Nolinear ϕ

Smarrazzo, T. (Discrete and Continuous dynamical systems A.)
Rankine–Hugoniot has to be understood in a weak sense, the same for the entropy condition (theory of vector filelds with divergence–measure). We obtain the same compatibility condition for the two phase problem.
Uniqueness

Theorem

There exists at most one two–phase solution of problem .

Let (u^1, v^1, ξ^1) and (u^2, v^2, ξ^2) be two two–phase solutions then we have

$$\iint_{Q_T} \left\{ (u^1 - u^2)\psi_t - (v^1_x - v^2_x)\psi_x \right\} dxdt = 0 \tag{7}$$

for every test function $\psi \in H^1(Q_T) \cap C(\overline{Q}_T)$ such that $\psi(\cdot, T) \equiv 0$.

Choosing

$$\psi(x,t) := \int_T^t \{v^1(x,s) - v^2(x,s)\}\, ds, \tag{8}$$

equation reads

$$\iint_{Q_T} (u^1 - u^2)(v^1 - v^2)dxdt = \iint_{Q_T} (v_x^1 - v_x^2)\left(\int_T^t (v_x^1(x,s) - v_x^2(x,s)ds\right)dxdt. \tag{9}$$

Concerning the right-hand side of (9) we have

$$\begin{aligned} &\iint_{Q_T} (v_x^1 - v_x^2)\left(\int_T^t (v_x^1(x,s) - v_x^2(x,s)\, ds\right) dxdt \qquad (10) \\ &= \frac{1}{2}\int_{\omega_1}^{\omega_2}\int_0^T \frac{d}{dt}\left(\left[\int_T^t v_x^1(x,s) - v_x^2(x,s)\, ds\right]^2\right) dtdx \leq 0. \end{aligned}$$

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Suppose that one of the following conditions is satisfied

- there exists $\delta > 0$ such that $\phi(u_0) < B$ in $(0, \delta)$;
- there exists $\delta > 0$ such that $\phi(u_0) > A$ in $(-\delta, 0)$);

then there exists $\tau > 0$ such that u_{ϵ_n} converges strongly to the solution of the two phase problem with fixed boundary ($\xi \equiv 0$) in $(-L, L) \times (0, \tau)$.

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$$\|v_t^{\epsilon_k}\|_{L^2(Q_{\tau_{\epsilon_k}})} + \|v_x^{\epsilon_k}\|_{L^\infty(0,\tau_{\epsilon_k};L^2((-L,L))} \leq C \tag{11}$$

where τ_{ϵ_k} is the maximum time in which the two phases are separated by the stationary interface.

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Using (11) we can pass to the limit strongly and prove that there exists $\tau > 0$ such that $\tau_{\epsilon_k} \geq \tau$ for k large enough.