# A high-order unstaggered constrained transport method for the 3D ideal magnetohydrodynamic equations based on the method of lines

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(joint work with C. Helzel & J.A. Rossmanith)

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Research unit 1048:
"Instabilities, turbulence and transport in cosmic magnetic fields"

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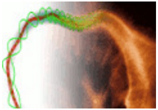
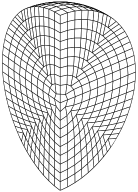
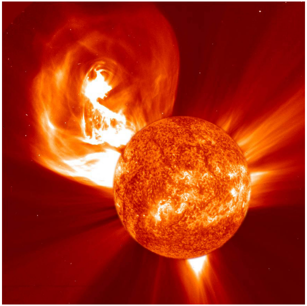


# Contents

# Motivation



Ball mapping
taken
from [Calhoun et al., 2008]



Observation of
Coronal mass ejection (CME)
taken from [SOHO,2002]

# The ideal MHD equations in conservation form

[Brackbill & Barnes, 1980]

$$\begin{bmatrix} \rho \\ \rho\mathbf{u} \\ \mathcal{E} \\ \mathbf{B} \end{bmatrix}_t + \nabla \cdot \begin{bmatrix} \rho\mathbf{u} \\ \rho\mathbf{u}\mathbf{u} + \left(p + \frac{1}{2}|\mathbf{B}|^2\right)\mathbf{Id} - \mathbf{B}\mathbf{B} \\ \mathbf{u}\left(\mathcal{E} + p + \frac{1}{2}|\mathbf{B}|^2\right) - \mathbf{B}(\mathbf{u}\cdot\mathbf{B}) \\ \mathbf{u}\mathbf{B} - \mathbf{B}\mathbf{u} \end{bmatrix} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

- the thermal pressure is related via the ideal gas law:

$$p = (\gamma - 1)(\mathcal{E} - \frac{1}{2}||\mathbf{B}||^2 - \frac{1}{2}\rho||\mathbf{u}||^2)$$

# Problems due to insufficient control of $\nabla \cdot \mathbf{B}$

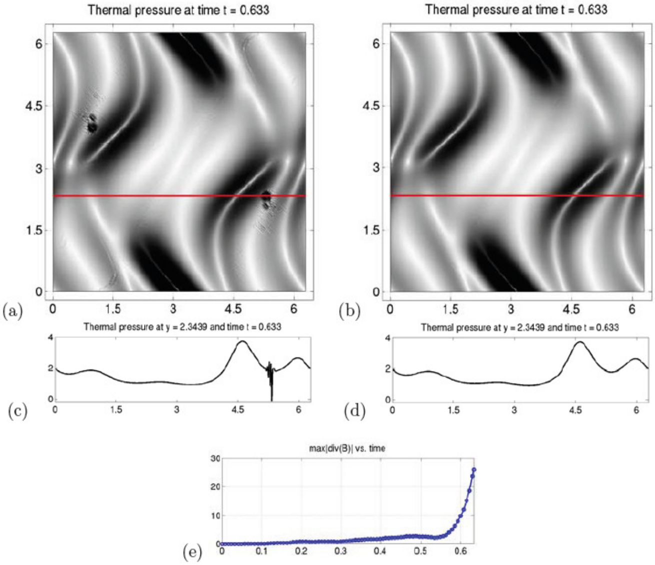



Figure: taken from [Rossmanith, 2006].

# Approaches to control $\nabla \cdot \mathbf{B}$

Numerical methods for ideal MHD must in general satisfy (or at least control) some discrete version of the divergence free condition on the magnetic field.

Some known methods to control $\nabla \cdot \mathbf{B}$ on the discrete level:

- projection methods, e.g. [Tóth, 2000]
- 8-wave-formulation, [Powell, 1994]
- divergence cleaning, [Dedner et al.,2002]
- flux-distribution methods, [Torrilhon,2003],[Mishra & Tadmor, 2012]
- contrained transport (CT) methods, e.g. [Evans and Hawley, 1988, Rossmanith, 2006]

# The idea of CT in 3D

Consider the induction equation

$$\mathbf{B}_t + \nabla \times (\mathbf{B} \times \mathbf{u}) = 0$$

and assume that $\mathbf{u}$ is a given vector valued function.
Set $\mathbf{B} = \nabla \times \mathbf{A}$ to obtain

$$\nabla \times (\mathbf{A}_t + (\nabla \times \mathbf{A}) \times \mathbf{u}) = 0$$

$$\Rightarrow \mathbf{A}_t + (\nabla \times \mathbf{A}) \times \mathbf{u} = -\nabla\psi$$

$\psi$ is an arbitrary scalar function taken to be $\psi = 0$ (Weyl gauge) in the following.

Different choices of $\psi$ represent different **gauge conditions**.
See e.g. [C.Helzel, J.A.Rossmanith & B.Taetz, 2011] for discussions on different choices.

# The evolution of the magnetic potential

$$\mathbf{A}_t + N_1(\mathbf{u})\,\mathbf{A}_x + N_2(\mathbf{u})\,\mathbf{A}_y + N_3(\mathbf{u})\,\mathbf{A}_z = 0,$$

with

$$N_1 = \begin{bmatrix} 0 & -u^2 & -u^3 \\ 0 & u^1 & 0 \\ 0 & 0 & u^1 \end{bmatrix},\ N_2 = \begin{bmatrix} u^2 & 0 & 0 \\ -u^1 & 0 & -u^3 \\ 0 & 0 & u^2 \end{bmatrix},\ N_3 = \begin{bmatrix} u^3 & 0 & 0 \\ 0 & u^3 & 0 \\ -u^1 & -u^2 & 0 \end{bmatrix}.$$

This system is **weakly hyperbolic**, which means that we do not have a full set of linearly independent eigenvectors in all directions.

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# Weak hyperbolicity of the system matrix

- The system matrix in an arbitrary direction $\mathbf{n} \in S^2$ is

$$n^1 N_1 + n^2 N_2 + n^3 N_3 = \begin{bmatrix} n^2u^2 + n^3u^3 & -n^1u^2 & -n^1u^3 \\ -n^2u^1 & n^1u^1 + n^3u^3 & -n^2u^3 \\ -n^3u^1 & -n^3u^2 & n^1u^1 + n^2u^2 \end{bmatrix}$$

- the eigenvalues are

$$\lambda = \{0, \mathbf{n}\cdot\mathbf{u}, \mathbf{n}\cdot\mathbf{u}\};$$

- the eigenvectors read

$$R = \begin{bmatrix} n^1 & n^2u^3 - n^3u^2 & u^1(\mathbf{u}\cdot\mathbf{n}) - n^1\|\mathbf{u}\|^2 \\ n^2 & n^3u^1 - n^1u^3 & u^2(\mathbf{u}\cdot\mathbf{n}) - n^2\|\mathbf{u}\|^2 \\ n^3 & n^1u^2 - n^2u^1 & u^3(\mathbf{u}\cdot\mathbf{n}) - n^3\|\mathbf{u}\|^2 \end{bmatrix};$$

- the determinant of $R$ can be written as

$$\det(R) = -\|\mathbf{u}\|^3 \cos(\alpha)\sin^2(\alpha),$$

  $\alpha$ is the angle between $\mathbf{n}$ and $\mathbf{u}$.

  $\rightarrow$ Thus we do not have a full set of linearly independent eigenvectors in all directions $\mathbf{n}$.

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# An operator splitting approach (dimensional splitting)

**Sub-problem 1:**

$$\begin{aligned} \mathbf{A}^1_t - u^2\mathbf{A}^2_x - u^3\mathbf{A}^3_x &= 0, \\ \mathbf{A}^2_t + u^1\mathbf{A}^2_x &= 0, \\ \mathbf{A}^3_t + u^1\mathbf{A}^3_x &= 0, \end{aligned}$$

**Sub-problem 2:**

$$\begin{aligned} \mathbf{A}^1_t + u^2\mathbf{A}^1_y &= 0, \\ \mathbf{A}^2_t - u^1\mathbf{A}^1_y - u^3\mathbf{A}^3_y &= 0, \\ \mathbf{A}^3_t + u^2\mathbf{A}^3_y &= 0, \end{aligned}$$

**Sub-problem 3:**

$$\begin{aligned} \mathbf{A}^1_t + u^3\mathbf{A}^1_z &= 0, \\ \mathbf{A}^2_t + u^3\mathbf{A}^2_z &= 0, \\ \mathbf{A}^3_t - u^1\mathbf{A}^1_z - u^2\mathbf{A}^2_z &= 0. \end{aligned}$$

- developed for Cartesian grids to $2^{nd}$ order of accuracy using Strang splitting.

# A high-order unsplit spatial discretization for weakly hyperbolic systems

Consider the integral form of a weakly hyperbolic system:

$$\mathbf{q}_t + N(x)\mathbf{q}_x = 0.$$

We write the semi-discrete form for the cell-averages $Q_i(t)$ as:

$$\partial_t Q_i(t) = -\frac{1}{\Delta x}(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2} + \mathcal{A}\Delta Q_i)$$

with

$$\begin{aligned}\mathcal{A}\Delta Q_i &:= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \tilde{N}_i(x)\,\tilde{\mathbf{q}}_{i,x}\,dx,\\ \mathcal{A}^-\Delta Q_{i-\frac{1}{2}} + \mathcal{A}^+\Delta Q_{i-\frac{1}{2}} &= \lim_{\varepsilon\to 0}\int_{x_{i-\frac{1}{2}}-\varepsilon}^{x_{i-\frac{1}{2}}+\epsilon} \tilde{N}(x)\left(Q^{\varepsilon}_{i-\frac{1}{2}}(t,x)\right)_x\,dx.\end{aligned}$$

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# Definition of fluctuations

Use regularization $Q^{\varepsilon}_{i-\frac{1}{2}}(t,x)$ with a straight-line path

$$\Psi_{i-\frac{1}{2}} = Q^-_{i-\frac{1}{2}} + l\left(Q^+_{i-\frac{1}{2}} - Q^-_{i-\frac{1}{2}}\right), \quad 0 \le l \le 1$$

to derive

$$\mathcal{A}^-\Delta Q_{i-\frac{1}{2}} + \mathcal{A}^+\Delta Q_{i-\frac{1}{2}} = \underbrace{N\big|_{\Psi_{i-\frac{1}{2}}}}_{\frac{1}{2}(N^-_{i-1/2}+N^+_{i-1/2})} (Q^+_{i-\frac{1}{2}} - Q^-_{i-\frac{1}{2}}).$$

Use generalized Rusanov-flux similar to [Castro et al., 2010]:

$$\mathcal{A}^-\Delta Q_{i-1/2} := \frac{1}{2}\left[N\big|_{\Psi_{i-1/2}} - \underbrace{\alpha_{i-1/2}}_{\ge|\lambda^p_{i-1/2}|\ \forall p} \mathbf{Id}\right]\left(Q^+_{i-\frac{1}{2}} - Q^-_{i-\frac{1}{2}}\right),$$

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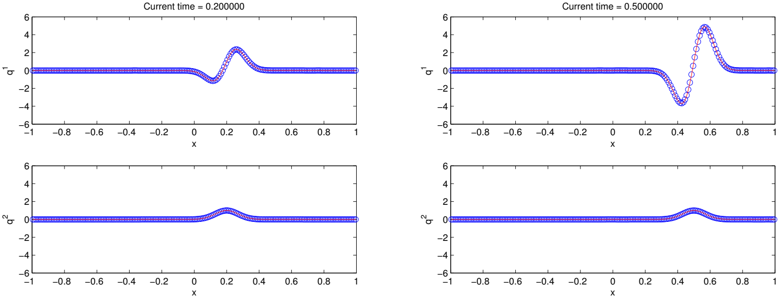
# A numerical example

Consider the weakly hyperbolic system:

$$\begin{bmatrix} q^1 \\ q^2 \end{bmatrix}_t + \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{N} \begin{bmatrix} q^1 \\ q^2 \end{bmatrix}_x = 0$$

with initial data:

$$q^1(x,0) = q^2(x,0) = e^{-(10x)^2}.$$



# High-order 3D extension for mapped grids

The semi-discrete form reads:

$$\partial_t Q_{i,j,k}(t) = \frac{-1}{|C_{i,j,k}|}(\underbrace{\mathcal{A}\Delta Q_{i,j,k}}_{\text{inner integral}} + \sum_{n=1}^{3}[(|A|(\breve{\mathcal{A}}^{+}\Delta Q))_{\mathbf{I}_n} + (|A|(\breve{\mathcal{A}}^{-}\Delta Q))_{\mathbf{I}_{n+\mathbf{e}_n}}]).$$

E.g. on the x-lower face with index $I_1 = (i - 1/2, j, k)$, we have:

$$(|A|\breve{\mathcal{A}}^{\pm}\Delta Q)_{\mathbf{I}_1} = \int_0^1\int_0^1 [\breve{\mathcal{A}}^{\pm}\Delta Q(\underbrace{\mathbf{X}(0,\eta,\zeta)}_{\text{local trilinear map}})\underbrace{\sqrt{a(\eta,\zeta)}]_{\mathbf{I}_1}\, d\eta d\zeta}_{\text{area element}}$$

On a Gaussian point $I_1^{l,m}$ with corresponding $\mathbf{n}, Q^{+}, Q^{-}$ the fluctuations are:

$$\breve{\mathcal{A}}^{+}\Delta Q^{l,m}_{i-1/2,j,k} = \frac{1}{2}(N|_{\Psi}(\mathbf{n})^{l,m} + \alpha(Q^{+},Q^{-})\mathbf{Id})(Q^{+} - Q^{-})$$

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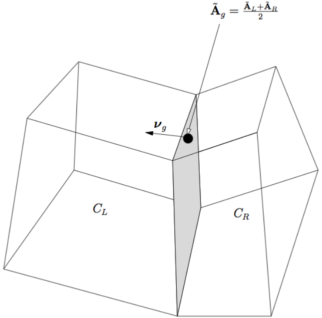
# The discretization of $\nabla \times \mathbf{A}$ to high-order and on mapped grids

$$
\begin{aligned}
\frac{1}{|C_{i,j,k}|} \iiint_{C_{i,j,k}} \mathbf{B}\, dC &= \frac{1}{|C_{i,j,k}|} \iiint_{C_{i,j,k}} \nabla \times \mathbf{A}\, dC \\
&= \frac{1}{|C_{i,j,k}|} \iint_{\partial C_{i,j,k}} \boldsymbol{\nu} \times \mathbf{A}\, dA.
\end{aligned}
$$

- Conservative computation of $\mathbf{B}$:



# Temporal discretization of the CT method using a SSP Runge Kutta method

**Stage 1.**

$$Q_{\text{MHD}}^{(1*)} = Q_{\text{MHD}}^{n} + \Delta t\, \mathcal{L}_1\left(Q_{\text{MHD}}^{n}\right),$$

$$Q_{\mathbf{A}}^{(1)} = Q_{\mathbf{A}}^{n} + \Delta t\, \mathcal{L}_2\left(Q_{\mathbf{A}}^{n}, Q_{\text{MHD}}^{n}\right),$$

Replace $\mathbf{B}^{(1*)}$ with $\nabla \times Q_{\mathbf{A}}^{(1)} \to Q_{\text{MHD}}^{(1)}$.

**Stage 2.**

$$Q_{\text{MHD}}^{(2*)} = \frac{3}{4} Q_{\text{MHD}}^{n} + \frac{1}{4} Q_{\text{MHD}}^{(1)} + \frac{1}{4} \Delta t\, \mathcal{L}_1(Q_{\text{MHD}}^{(1)}),$$

$$Q_{\mathbf{A}}^{(2)} = \frac{3}{4} Q_{\mathbf{A}}^{n} + \frac{1}{4} Q_{\mathbf{A}}^{(1)} + \frac{1}{4} \Delta t\, \mathcal{L}_2(Q_{\mathbf{A}}^{(1)}, Q_{\text{MHD}}^{(1)}),$$

Replace $\mathbf{B}^{(2*)}$ with $\nabla \times Q_{\mathbf{A}}^{(2)} \to Q_{\text{MHD}}^{(2)}$.

**Stage 3.**

$$Q_{\text{MHD}}^{(*)} = \frac{1}{3} Q_{\text{MHD}}^{n} + \frac{2}{3} Q_{\text{MHD}}^{(2)} + \frac{2}{3} \Delta t\, \mathcal{L}_1(Q_{\text{MHD}}^{(2)}),$$

$$Q_{\mathbf{A}}^{n+1} = \frac{1}{3} Q_{\mathbf{A}}^{n} + \frac{2}{3} Q_{\mathbf{A}}^{(2)} + \frac{2}{3} \Delta t\, \mathcal{L}_2(Q_{\mathbf{A}}^{(2)}, Q_{\text{MHD}}^{(2)}),$$

Replace $\mathbf{B}^{(*)}$ with $\nabla \times Q_{\mathbf{A}}^{n+1} \to Q_{\text{MHD}}^{n+1}$.

# Temporal discretization of the CT method using a SSP Runge Kutta method

**Stage 1.**

$$Q_{\mathrm{MHD}}^{(1*)} = Q_{\mathrm{MHD}}^{n} + \Delta t\, \mathcal{L}_1\left(Q_{\mathrm{MHD}}^{n}\right),$$

$$Q_{\mathbf{A}}^{(1)} = Q_{\mathbf{A}}^{n} + \Delta t\, \mathcal{L}_2\left(Q_{\mathbf{A}}^{n}, Q_{\mathrm{MHD}}^{n}\right),$$

Replace $\mathbf{B}^{(1*)}$ with $\nabla \times Q_{\mathbf{A}}^{(1)} \to Q_{\mathrm{MHD}}^{(1)}$.

**Stage 2.**

$$Q_{\mathrm{MHD}}^{(2*)} = \frac{3}{4} Q_{\mathrm{MHD}}^{n} + \frac{1}{4} Q_{\mathrm{MHD}}^{(1)} + \frac{1}{4} \Delta t\, \mathcal{L}_1(Q_{\mathrm{MHD}}^{(1)}),$$

$$Q_{\mathbf{A}}^{(2)} = \frac{3}{4} Q_{\mathbf{A}}^{n} + \frac{1}{4} Q_{\mathbf{A}}^{(1)} + \frac{1}{4} \Delta t\, \mathcal{L}_2(Q_{\mathbf{A}}^{(1)}, Q_{\mathrm{MHD}}^{(1)}),$$

Replace $\mathbf{B}^{(2*)}$ with $\nabla \times Q_{\mathbf{A}}^{(2)} \to Q_{\mathrm{MHD}}^{(2)}$.

**Stage 3.**

$$Q_{\mathrm{MHD}}^{(*)} = \frac{1}{3} Q_{\mathrm{MHD}}^{n} + \frac{2}{3} Q_{\mathrm{MHD}}^{(2)} + \frac{2}{3} \Delta t\, \mathcal{L}_1(Q_{\mathrm{MHD}}^{(2)}),$$

$$Q_{\mathbf{A}}^{n+1} = \frac{1}{3} Q_{\mathbf{A}}^{n} + \frac{2}{3} Q_{\mathbf{A}}^{(2)} + \frac{2}{3} \Delta t\, \mathcal{L}_2(Q_{\mathbf{A}}^{(2)}, Q_{\mathrm{MHD}}^{(2)}),$$

Replace $\mathbf{B}^{(*)}$ with $\nabla \times Q_{\mathbf{A}}^{n+1} \to Q_{\mathrm{MHD}}^{n+1}$.

# Temporal discretization of the CT method using a SSP Runge Kutta method

**Stage 1.**

$$Q^{(1*)}_{\mathrm{MHD}} = Q^{n}_{\mathrm{MHD}} + \Delta t\, \mathcal{L}_1\left(Q^{n}_{\mathrm{MHD}}\right),$$

$$Q^{(1)}_{\mathbf{A}} = Q^{n}_{\mathbf{A}} + \Delta t\, \mathcal{L}_2\left(Q^{n}_{\mathbf{A}}, Q^{n}_{\mathrm{MHD}}\right),$$

Replace $\mathbf{B}^{(1*)}$ with $\nabla \times Q^{(1)}_{\mathbf{A}} \to Q^{(1)}_{\mathrm{MHD}}$.

**Stage 2.**

$$Q^{(2*)}_{\mathrm{MHD}} = \frac{3}{4} Q^{n}_{\mathrm{MHD}} + \frac{1}{4} Q^{(1)}_{\mathrm{MHD}} + \frac{1}{4} \Delta t\, \mathcal{L}_1(Q^{(1)}_{\mathrm{MHD}}),$$

$$Q^{(2)}_{\mathbf{A}} = \frac{3}{4} Q^{n}_{\mathbf{A}} + \frac{1}{4} Q^{(1)}_{\mathbf{A}} + \frac{1}{4} \Delta t\, \mathcal{L}_2(Q^{(1)}_{\mathbf{A}}, Q^{(1)}_{\mathrm{MHD}}),$$

Replace $\mathbf{B}^{(2*)}$ with $\nabla \times Q^{(2)}_{\mathbf{A}} \to Q^{(2)}_{\mathrm{MHD}}$.

**Stage 3.**

$$Q^{(*)}_{\mathrm{MHD}} = \frac{1}{3} Q^{n}_{\mathrm{MHD}} + \frac{2}{3} Q^{(2)}_{\mathrm{MHD}} + \frac{2}{3} \Delta t\, \mathcal{L}_1(Q^{(2)}_{\mathrm{MHD}}),$$

$$Q^{n+1}_{\mathbf{A}} = \frac{1}{3} Q^{n}_{\mathbf{A}} + \frac{2}{3} Q^{(2)}_{\mathbf{A}} + \frac{2}{3} \Delta t\, \mathcal{L}_2(Q^{(2)}_{\mathbf{A}}, Q^{(2)}_{\mathrm{MHD}}),$$

Replace $\mathbf{B}^{(*)}$ with $\nabla \times Q^{n+1}_{\mathbf{A}} \to Q^{n+1}_{\mathrm{MHD}}$.

# Limiting with respect to the derivative

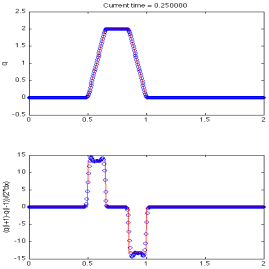
Consider the 1D scalar advection equation:

$$q_t + q_x = 0$$

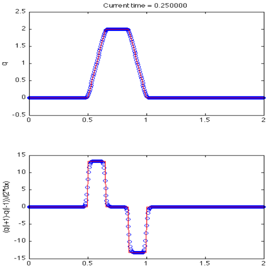
$$q_t + q_x = \epsilon(x) q_{xx}$$

- Here $\epsilon(x)$ is chosen with respect to steep gradients in the derivative. (diffusive limiter inspired by [Persson et al., 2006])

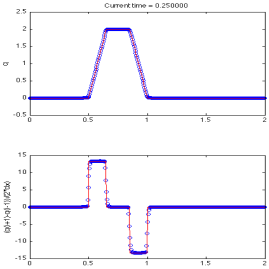
WENO limiter on solution:



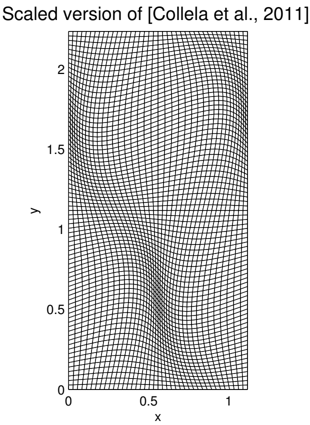

Diffusive limiter w.r.t. derivative:

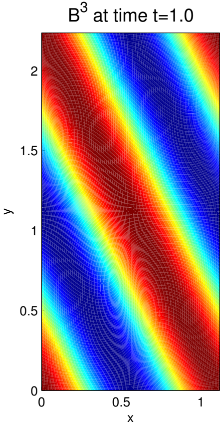



TVD limiting of [Rossmanith, 2006]:



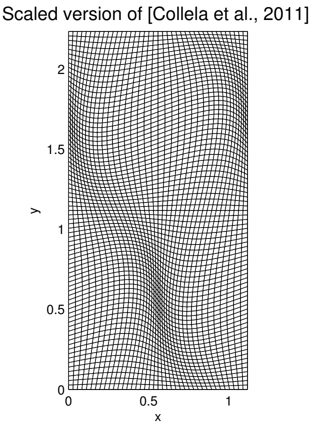
# 2.5-dimensional accuracy test on mapped grid

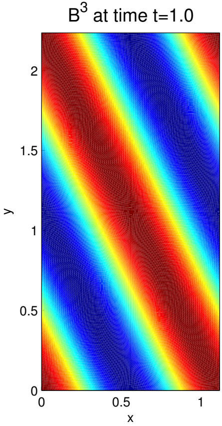



- $3^{rd}$ order of accuracy can be confirmed on mapped grid.

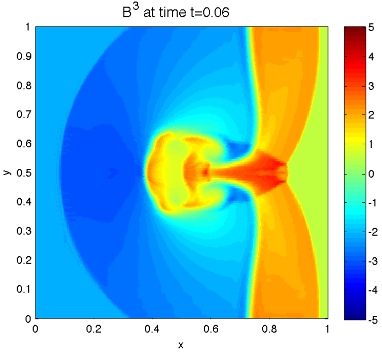
# 2.5-dimensional accuracy test on mapped grid

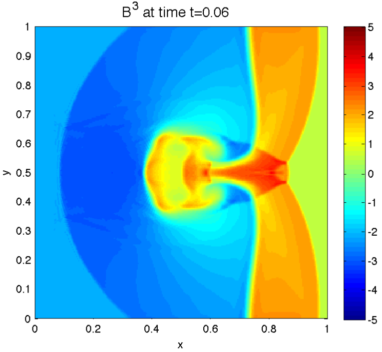


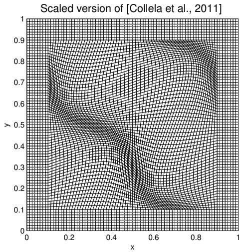



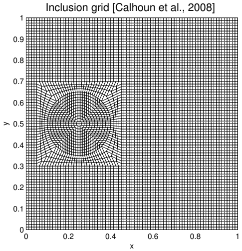

- $3^{rd}$ order of accuracy can be confirmed on mapped grid.

# 2.5-dimensional cloud-shock interaction problem

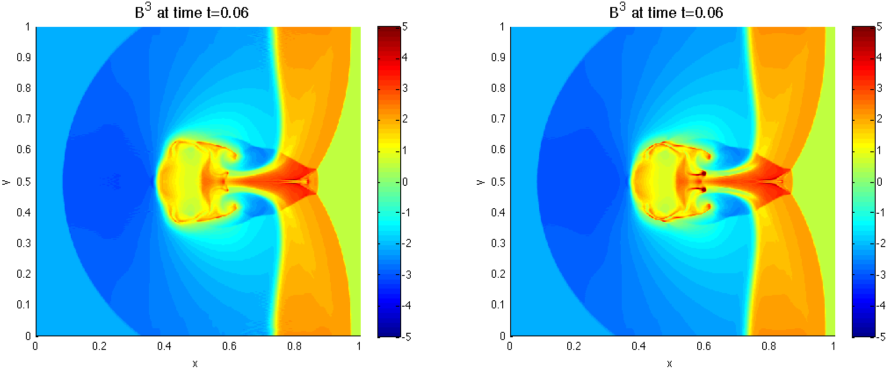


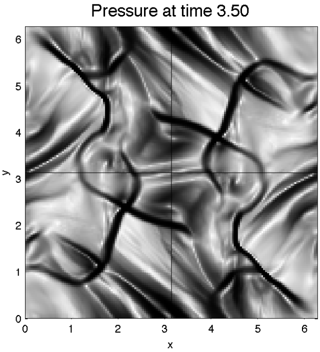



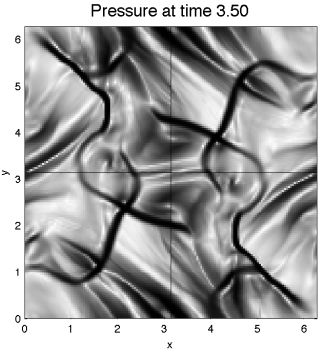
# Numerical experiments in 2.5-dimensions

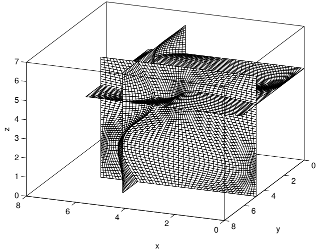



- The 2.5-dimensional cloud-shock interaction problem.
- Mesh: $512 \times 512$
- **(Left hand side)** scheme of [Rossmanith, 2006] that only uses $\mathbf{A}^3$ (scalar equation)
- **(Right hand side)** the unsplit scheme using 2-step SSP-RK time stepping and diffusive limiting on derivative on full vector potential $\mathbf{A}$.

# Numerical experiments 3D (Orzag-Tang vortex)



# Conclusions

- unsplit and unstaggered CT method
  $\rightarrow$ suitable for high-order on smooth solutions and high-resolution in the presence of shocks
- the scheme works on general (possibly non-smoothly varying) mapped grids in 2D and 3D
- scheme should be suitable for adaptive mesh refinement and parallel computing due to unstaggered and hyperbolic nature.

# Outlook

- application using 3D spherical grids of [Calhoun et al., 2008]
- extension to high-order one-step schemes (ADER schemes)
- extension to more general MHD models (two-fluid MHD)

# References

C. Helzel, J.A. Rossmanith, and B. Taetz. *An unstaggered constrained transport method for the 3D ideal magnetohydrodynamic equations*. J. Comp. Phys., 227: 9527–9553, 2011.

C. Helzel, J.A. Rossmanith, B. Taetz, *A high-order unstaggered constrained transport method for the 3d ideal magnetohydrodynamic equations based on the method of lines*, arXiv:2203.3760, [http://arxiv.org/abs/1203.3760]