Metastable and interface dynamics for the hyperbolic Jin-Xin system in one space dimension

Marta Strani,
Sapienza Università di Roma,
Dipartimento di Matematica

14th International Conference on Hyperbolic Problems:
Theory, Numerics, Applications
Università di Padova, June 25-29 2012

Outline

1. Slow motion of internal shock layers for the Jin-Xin system in one space dimension
   - Overview of the problem
   - Spectral analysis
   - Main results

The main problem

We describe slow motion for the Jin-Xin system, with Dirichlet boundary conditions in the bounded interval $I = (-\ell, \ell)$, that is

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + a^2 \partial_x u = \dfrac{1}{\varepsilon}(f(u) - v) \\ u(\pm\ell, t) = u_\pm & t \geq 0 \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \equiv f(u_0(x)) & x \in I \end{cases} \tag{1}$$

for some $\varepsilon, \ell, a > 0$, $u_\pm \in \mathbb{R}$ and flux function f that satisfies

$$f''(s) \geq c_0 > 0, \quad f'(u_+) < 0 < f'(u_-), \quad f(u_+) = f(u_-)$$

In vector form

$$\partial_t Z = \mathcal{F}^\varepsilon[Z], \quad Z\big|_{t=0} = Z_0$$

where

$$\mathcal{F}^\varepsilon[Z] := \begin{pmatrix} \mathcal{P}_1^\varepsilon[Z] \\ \mathcal{P}_2^\varepsilon[Z] \end{pmatrix} = \begin{pmatrix} -\partial_x v \\ -a^2 \partial_x u + \dfrac{1}{\varepsilon}(f(u) - v) \end{pmatrix}$$

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Metastable dynamics

- First time scale where solutions are close to some non stationary state.
- Exponentially long time convergence to the asymptotic limit.
- Presence of a first small eigenvalue of the linearized operator.

Allen-Cahn:

- Carr, Pego, Comm. Pure Appl. Math. 1989
- Fusco, Hale, J. Dyn. Diff. Eq. 1989

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The relaxation limit

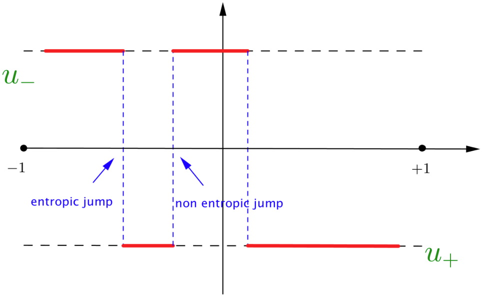
In the relaxation limit ($\varepsilon \to 0^+$), system (1) can be approximated to leading order by

$$\partial_t u + \partial_x f(u) = 0, \quad v = f(u) \tag{2}$$

The first equation is a quasi-linear equation of hyperbolic type, whose standard setting is given by the *entropy formulation*, hence possessing discontinuous solutions with speed of propagation s given by the Rankine–Hugoniot relation

$$s[\![u]\!] = [\![f(u)]\!]$$

together with appropriate entropy conditions.
Concerning the stationary solutions



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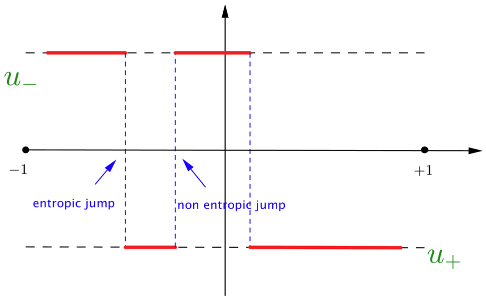

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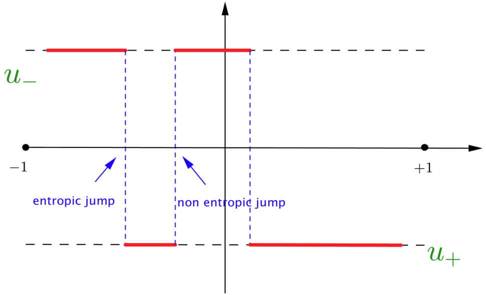

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Stationary solutions for $\varepsilon = 0$

Stationary solutions

We have a one-parameter family of stationary solutions

$$U_{\rm hyp}(x;\xi) = u_-\chi_{(-\ell,\xi)}(x) + u_+\chi_{(\xi,\ell)}(x)$$
$$V_{\rm hyp}(x;\xi) = f(u_-)\chi_{(-\ell,\xi)}(x) + f(u_+)\chi_{(\xi,\ell)}(x)$$

The dynamics determined by initial-value problem for (2) is very simple.

Hypotheses: If $f(u)$ is convex and such that $f(u_-) = f(u_+)$, where $u_+ \leq u_-$

every entropy solution converges *in finite time* to an element of the family $\{U_{\rm hyp}(\cdot;\xi), V_{\rm hyp}(\cdot;\xi)\}$.

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Metastable dynamics for $\varepsilon > 0$

Stationary solution

For $\varepsilon > 0$, the presence of the Laplace operator has the effect of a drastic reduction of the number of stationary solutions: in this case there exists a unique stationary solution that is asymptotically stable.

Such solution, denoted here by $(\bar{U}^{\varepsilon}_{rel}(x), \bar{V}^{\varepsilon}_{rel}(x))$, converges in the limit $\varepsilon \to 0^+$ to a specific element of the family $\{U_{\mathrm{hyp}}(\cdot;\xi), V_{\mathrm{hyp}}(\cdot;\xi)\}$.

Question

What happens to the dynamics generated by an initial datum localized that still presents a sharp transition from u^- to u^+, but it is localized far from the equilibrium solution?

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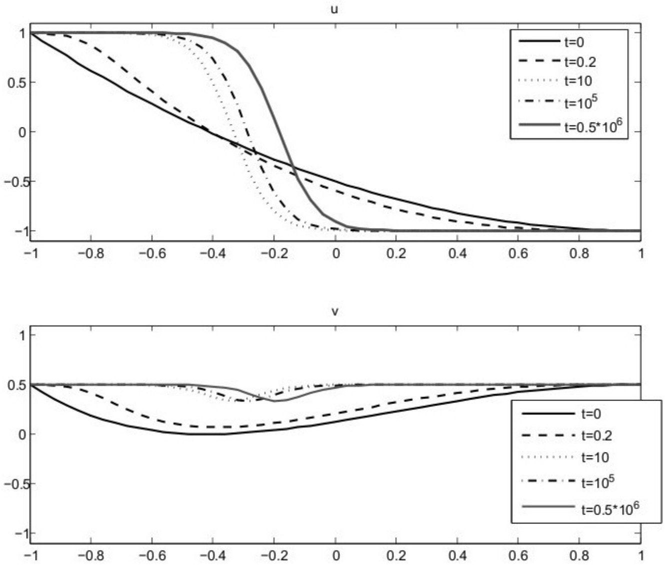

Figure: Profiles of (u, v), solutions to (1), with $f(u) = u^2/2$, a $=1$ $\varepsilon = 0.04$ and $u_\pm = \mp 1$. The initial data is given by the couple $(u_0(x), f(u_0(x)))$, with $u_0(x)$ a decreasing function connecting u_+ and u_-.

The strategy

To describe the dynamics generated by an initial datum localized far from the position of the steady state, our strategy is:

to build up a one parameter family of approximate stationary solutions $\mathbf{W}^{\varepsilon}(x;\xi) = \{U^{\varepsilon}(x;\xi), V^{\varepsilon}(x;\xi)\}$, parametrized by ξ that represent the location of the internal shock, such that $\mathcal{F}^{\varepsilon}[\mathbf{W}^{\varepsilon}] \to 0$ as $\varepsilon \to 0$.

- $(U^{\varepsilon}(\cdot;\bar{\xi}), V^{\varepsilon}(\cdot,\bar{\xi})) := (\bar{U}^{\varepsilon}_{rel}, \bar{V}^{\varepsilon}_{rel}), \quad \exists\, \bar{\xi} \in I$

to describe the dynamics of the system in a neighborhood of the family, by linearizing the equation around an element $\mathbf{W}^{\varepsilon}(x;\xi)$, in order to obtain a coupled system for the shock layer location $\xi(t)$ and the perturbation Y.

$$Z(x,t) = Y(x,t) + \mathbf{W}^{\varepsilon}(x;\xi(t))$$

to determine spectral properties of the linearized operator at such states.

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Example

In the case of Burgers flux, i.e. $f(s) = \frac{1}{2}s^2$ and $u_\pm := \mp u_*$, for some $u_* > 0$, an approximate solution $U^\varepsilon(x;\xi)$ is obtained by matching two different steady states satisfying, respectively, the left and the right boundary condition together with the request $U^\varepsilon(\xi) = 0$; in formulas,

$$U^\varepsilon(x;\xi) = \begin{cases} \kappa_- \tanh\left(\kappa_-(\xi - x)/2\varepsilon\right) & \text{in} \quad (-\ell, \xi) \\ \kappa_+ \tanh\left(\kappa_+(\xi - x)/2\varepsilon\right) & \text{in} \quad (\xi, \ell), \end{cases}$$

where $\kappa_\pm = \kappa_\pm(u_*)$. Moreover, by the condition $v = \frac{c^2}{2}$, we have

$$V^\varepsilon(x;\xi) = \begin{cases} \kappa_-^2/2 & \text{in } (-\ell, \xi) \\ \kappa_+^2/2 & \text{in } (\xi, \ell) \end{cases}$$

By direct substitution, we obtain the identity

$$\mathcal{P}_1^\varepsilon[\mathbf{W}^\varepsilon(\cdot;\xi)] = [\![\partial_x U^\varepsilon]\!]_{x=\xi}\,\delta_{x=\xi}, \qquad \mathcal{P}_2^\varepsilon[\mathbf{W}^\varepsilon(\cdot;\xi)] = 0$$

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The equations

By linearizing around and element $\mathbf{W}^\varepsilon$ of the family of approximate steady states, we obtain the following coupled system

$$\begin{cases} \dfrac{d\xi}{dt} = \theta^\varepsilon(\xi)\big(1 + \langle \partial_\xi \psi_1^\varepsilon, Y\rangle\big), \\ \partial_t Y = H^\varepsilon(\xi) + (\mathcal{L}_\xi^\varepsilon + \mathcal{M}_\xi^\varepsilon) Y \end{cases} \tag{3}$$

to be complemented with initial conditions

$$\xi(0) = \xi_0 \in (-\ell, \ell) \qquad \text{and} \qquad Y(x,0) = Y_0(x) \in L^2(I;\mathbb{R}^2). \tag{4}$$

Here $\mathcal{L}_\xi^\varepsilon$ is the operator arising from the linearization, and $\mathcal{M}_\xi^\varepsilon$ is a linear bounded operator.

Leading order term in the equation for $\xi(t)$

$$\theta^\varepsilon(\xi) := \langle \psi_1^\varepsilon, \mathcal{F}^\varepsilon[\mathbf{W}^\varepsilon]\rangle, \qquad \theta^\varepsilon(\xi) \to 0 \text{ as } \varepsilon \to 0$$

When $f(u) = u^2/2$ and $\mathbf{W}^\varepsilon = (U^\varepsilon, V^\varepsilon)$, we have

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Spectral analysis

$$\mathcal{L}^{\varepsilon}_{\xi} Y := \begin{pmatrix} -\partial_x v \\ -a^2\partial_x u + \frac{1}{\varepsilon}(f'(U^{\varepsilon})u - v) \end{pmatrix} \Rightarrow \begin{cases} \lambda\varphi = -\partial_x \psi \\ \lambda\psi = -a^2\partial_x\varphi + \frac{1}{\varepsilon}(f'(U^{\varepsilon})\varphi - \psi) \end{cases}$$

By differentiating the second equation with respect to x, we obtain

$$\varepsilon a^2 \partial_{xx}\varphi - \partial_x(f'(U^{\varepsilon})\varphi) = \lambda(1 + \varepsilon\lambda)\varphi$$

To study the eigenvalue problem for the differential linear diffusion-transport operator

$$\mathcal{L}^{\varepsilon, vsc}\varphi := \varepsilon\partial_{xx}\varphi - \partial_x(f'(U^{\varepsilon}(x;\xi))\varphi)$$

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- Kreiss, Kreiss, Appl. Numer. Math. 1986
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Spectral analysis

$$\mathcal{L}^{\varepsilon}_{\xi} Y := \begin{pmatrix} -\partial_x v \\ -a^2\partial_x u + \frac{1}{\varepsilon}(f'(U^{\varepsilon})u - v) \end{pmatrix} \Rightarrow \begin{cases} \lambda\varphi = -\partial_x\psi \\ \lambda\psi = -a^2\partial_x\varphi + \frac{1}{\varepsilon}(f'(U^{\varepsilon})\varphi - \psi) \end{cases}$$

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Theorem (Spectral analysis)

Under opportune hypotheses on the family of functions $f'(U^{\varepsilon}(x;\xi))$ the spectrum of the linearized operator $\mathcal{L}^{\varepsilon}_{\xi}$ can be decomposed as follow

1. $\lambda_1^{JX} \in \mathbb{R}$ *and* $\quad -e^{-C'/\varepsilon} \leq \lambda_1^{JX} < 0$
2. *All the remaining eigenvalues* λ_n^{JX} *are such that*

$$Re[\lambda_n^{JX}] \leq -C/\varepsilon$$

First real small eigenvalue

$f(u) = u^2/2$

$$|\lambda_{1,+}^{JX}(\xi)| \sim \frac{\frac{u_*^2}{\varepsilon}\left[e^{-u_*\varepsilon^{-1}(\ell-\xi)} + e^{-u_*\varepsilon^{-1}(\ell+\xi)}\right]}{1+\sqrt{1-2u_*^2\left[e^{-u_*\varepsilon^{-1}(\ell-\xi)} + e^{-u_*\varepsilon^{-1}(\ell+\xi)}\right]}}$$

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Hypotheses

H1. The family $\{\mathbf{W}^\varepsilon(\cdot,\xi)\}$ is such that there exist two families of smooth positive functions $\Omega_1^\varepsilon = \Omega_1^\varepsilon(\xi)$ and $\Omega_2^\varepsilon = \Omega_2^\varepsilon(\xi)$, uniformly convergent to zero as $\varepsilon \to 0$, such that

$$|\langle \psi(\cdot), \mathcal{P}_1^\varepsilon[\mathbf{W}^\varepsilon(\cdot,\xi)]\rangle| \le \Omega_1^\varepsilon(\xi)|\psi|_{L^\infty} \quad \forall \psi \in C(I)$$
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H2. $\Omega_1^\varepsilon(\xi) + \Omega_2^\varepsilon(\xi) \le C|\lambda_{1,+}^{JX}(\xi)|$, for all $\xi \in (-\ell, \ell)$.

For example, if $f(u) = u^2/2$ and $\mathbf{W}^\varepsilon = (U^\varepsilon, V^\varepsilon)$

$$\boldsymbol{\Omega}^\varepsilon(\xi) = (\Omega_1^\varepsilon(\xi), \Omega_2^\varepsilon(\xi)) \sim \left(\frac{2\,{u_*}^2}{\varepsilon}(e^{-u_*(\ell+\xi)/\varepsilon} - e^{-u_*(\ell-\xi)/\varepsilon}), 0\right)$$

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Slow motion for the shock layer

The perturbation Y

- The operator $\mathcal{L}^{\varepsilon}_{\xi(t)} + \mathcal{M}^{\varepsilon}_{\xi(t)}$ depends on time $\to$ Theory of Stable family of generators.
- $|Y|_{L^2} \leq C(|\mathbf{\Omega}^{\varepsilon}|_{L^\infty} + e^{-\mu^{\varepsilon} t}|Y_0|_{L^2}), \quad \mu^{\varepsilon} = \sup_{\xi} \lambda_1^{JX}(\xi) - C|\mathbf{\Omega}^{\varepsilon}|_{L^\infty} > 0$

Theorem (Slow motion of the shock layer)

Let hypotheses **H1-2** *be satisfied. Assume also*

$$s\,\theta^{\varepsilon}(s) < 0 \quad \textit{for any } s \in I,\ s \neq 0 \qquad \textit{and} \qquad {\theta^{\varepsilon}}'(\bar{\xi}) < 0.$$

Then for ε and $|Y_0|_{L^2}$ sufficiently small, than the solution $\xi(t)$ converges to $\bar{\xi}$ as $t \to +\infty$.
More precisely, the dynamics of $\xi(t)$ is described by

$$\frac{d\xi}{dt} = \theta^{\varepsilon}(\xi)\ , \qquad \theta^{\varepsilon}(\xi) = \langle \psi_1^{\varepsilon}, \mathcal{F}[\mathbf{W}^{\varepsilon}] \rangle$$

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this estimate shows the exponentially slow motion of the shock layer for small ε.

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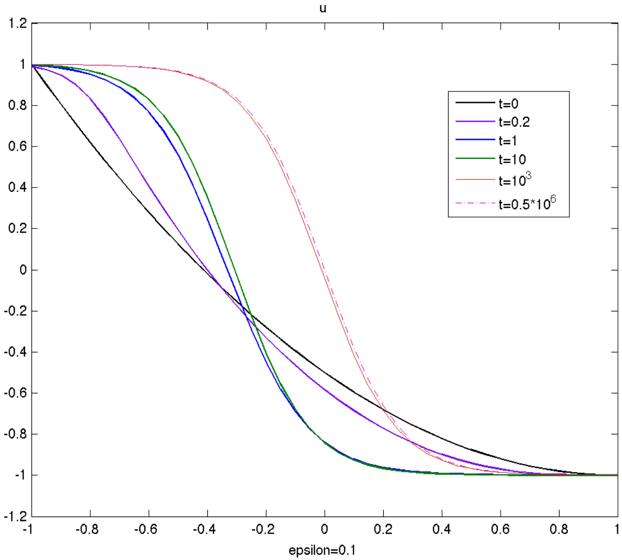
Slow motion for the shock layer-Numerical computations

This is a table containing the values of the different locations of the shock layer obtained numerically for different values of the parameter ε.

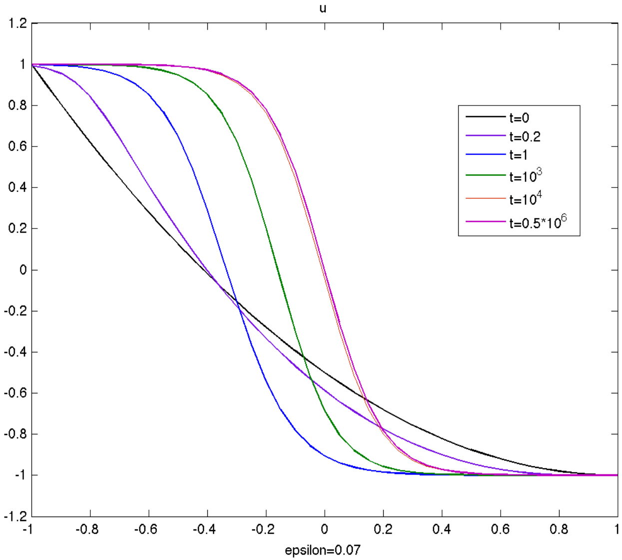
The numerical location of the shock layer $\xi(t)$ for different values of the parameter ε

TIME t	$\xi(t)$, $\varepsilon = 0.10$	$\xi(t)$, $\varepsilon = 0.07$	$\xi(t)$, $\varepsilon = 0.05$	$\xi(t)$, $\varepsilon = 0.04$	$\xi(t)$, $\varepsilon = 0.02$
0.2	-0.4008	-0.4020	-0.4029	-0.4040	-0.4059
1	-0.3314	-0.3345	-0.3360	-0.3374	-0.3389
10	-0.3070	-0.3263	-0.3304	-0.3320	-0.3326
10^3	-0.0103	-0.1600	-0.2562	-0.3181	-0.3325
10^4	$\sim -10^{-12}$	-0.0084	-0.1115	-0.2531	-0.3320
$0.5 * 10^6$	$\sim -10^{-12}$	$\sim -10^{-11}$	$\sim -10^{-10}$	-0.0379	-0.3099

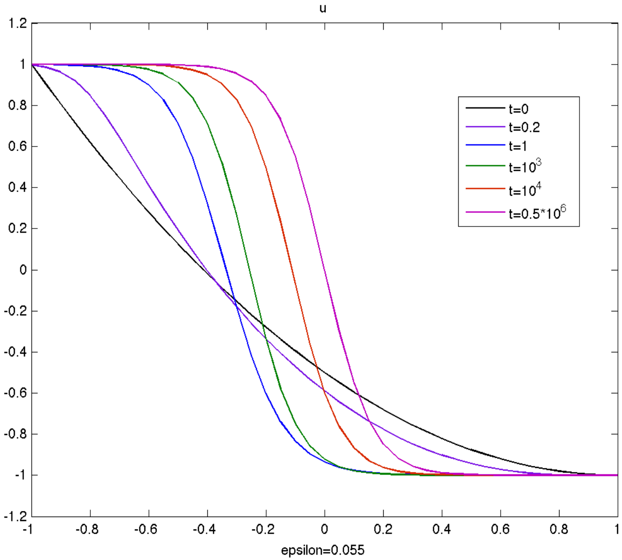
$\varepsilon = 0.10$



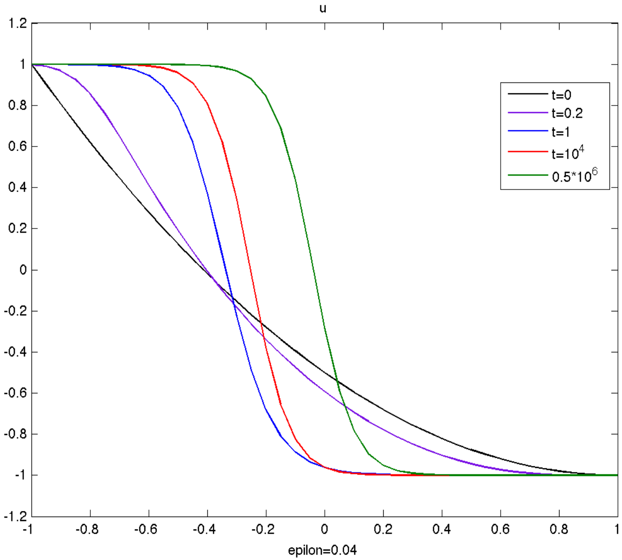
$\varepsilon = 0.07$



$\varepsilon = 0.05$



$\varepsilon = 0.04$



THANK YOU FOR YOUR ATTENTION