L^1 Error Estimates for Difference Approximations of Degenerate Convection-Diffusion Equations

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The Equation

- We consider the strongly degenerate convection-diffusion problem:

$$\begin{cases} \partial_t u + \partial_x f(u) = \partial_x^2 A(u), & (x,t) \in \Pi_T, \\ u(x,0) = u^0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

where $\Pi_T = \mathbb{R} \times (0,T)$. $A \in C^1(\mathbb{R}), A(0) = 0, A' \geq 0$. $f \in Lip_{\text{loc}}(\mathbb{R})$.

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  where $\Pi_T = \mathbb{R} \times (0,T)$. $A \in C^1(\mathbb{R}), A(0) = 0, A' \geq 0$. $f \in Lip_{\text{loc}}(\mathbb{R})$.
- J. Carrillo [1]: For $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, there exists a unique solution $u \in C((0,T); L^1(\mathbb{R}^d))$, $u \in L^\infty(\Pi_T)$ of (1) such that $\partial_x A(u) \in L^2(\Pi_T)$ and for all convex functions $S : \mathbb{R} \to \mathbb{R}$ with $q_S' = f'S'$ and $r_S' = A'S'$,

$$\partial_t S(u) + \partial_x q_S(u) - \partial_x^2 r_S(u) \leq 0$$

  in the weak sense on $[0,T) \times \mathbb{R}$.

Viscous Approximation

$$\begin{cases} u_t^\eta + f(u^\eta)_x = A(u^\eta)_{xx} + \eta u_{xx}, & (x,t) \in \Pi_T, \\ u^\eta(x,0) = u^0(x), & x \in \mathbb{R}. \end{cases}$$

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- Concerning the corresponding *boundary value problem* R. Eymard, T. Gallouët, R. Herbin [3]:

$$\|u_\eta(t,\cdot) - u(t,\cdot)\|_{L^1} \leq C\,\eta^{1/5}.$$

Finite Difference Scheme

Let $x_j = j\Delta x, I_j = [x_{j-1/2}, x_{j+1/2})$. We consider the semidiscrete finite difference scheme

$$\begin{cases} \frac{d}{dt}u_j(t) + D_-F(u_j, u_{j+1}) = D_-D_+A(u_j), \\ u_j(0) = \frac{1}{\Delta x}\int_{I_j} u^0(x)\,dx, \end{cases} \tag{2}$$

where $F \in C^1(\mathbb{R}^2)$ monotone and consistent with f.

$$D_+\sigma_j = \frac{\sigma_{j+1} - \sigma_j}{\Delta x}, \qquad D_-\sigma_j = \frac{\sigma_j - \sigma_{j-1}}{\Delta x}.$$

Result

Let

$$u_{\Delta x}(x,t) = u_j(t) \quad \text{for } x \in I_j = [x_{j-1/2}, x_{j+1/2}). \qquad (3)$$

Assumptions on the initial data u^0:

(i) $u^0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$.

(ii) $A(u^0)_x \in BV(\mathbb{R})$.

Theorem

If u^0 satisfies (i) *and* (ii) *above, then for each $b > 0$ there exist a constant C_b such that for all $\Delta x \in (0,b), t \in (0,T)$,*

$$\|u_{\Delta x}(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbb{R})} \leq \left\|u^0_{\Delta x} - u^0\right\|_{L^1(\mathbb{R})} + C_b \Delta x^{\frac{1}{3}},$$

where the constant C_b depends on A, f, u^0, and T, but not on Δx.

Earlier Results

- N.N. Kuznetsov for $A \equiv 0$ [5]

$$\|u_{\Delta x}(\cdot, t) - u(\cdot, t)\|_{L^1(\mathbb{R})} \leq C\sqrt{\Delta x}.$$

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- K. H. Karlsen, U. Koley and N.H. Risebro [4]

$$\int_{-L+Mt}^{L-Mt} |u(t,x) - u_{\Delta x}(t,x)|\, dx \leq C\Delta x^{1/11}, \qquad (t \leq T), \tag{4}$$

where $M > \max_{|u|<|u_0|} |f'(u)|$ and $L > MT$.

- Following Carillo [1]. Let u, v be entropy solutions of (1) with initial data u^0, v^0. If we can prove

$$\iiiint_{\Pi_T^2} |u(x,t)-v(y,s)|\psi'(t)\omega_r(x-y)\rho_r(t-s)\,dX \geq 0$$

then

$$\int_{\mathbb{R}} |u(x,\tau) - v(x,\tau)|\,dx \leq \int_{\mathbb{R}} |u^0(x) - v^0(x)|\,dx.$$

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- Let

$$\varphi(x,t,y,s) = \psi(t)\omega_r(x-y)\rho_r(t-s).$$

Then

$$\begin{aligned} \varphi_t + \varphi_s &= \psi'\omega_r\rho_r, \\ \varphi_x + \varphi_y &= 0, \\ \varphi_{xx} + 2\varphi_{xy} + \varphi_{yy} &= 0. \end{aligned}$$

- $A' > 0 \implies \text{sign}(u-v) = \text{sign}(A(u) - A(v))$.

$$[u_t + f(u)_x = A(u)_{xx}]\,\text{sign}(u-v)\varphi$$

Gives

$$\int_{\Pi_T^2} |u-v|\varphi_t + \text{sign}(u-v)(f(u)-f(v))\varphi_x \, dX$$

$$+ \int_{\Pi_T^2} |A(u)-A(v)|\varphi_{xx} \, dX$$

$$= \int_{\Pi_T^2} \text{sign}'(A(u)-A(v))(A(u)_x)^2 \varphi \, dX.$$

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$$= \int_{\Pi_T^2} \text{sign}'(A(u)-A(v))(A(u)_x)^2 \varphi \, dX.$$

- $$\int_{\Pi_T^2} |A(u)-A(v)|\varphi_{xy} \, dX =$$
$$- \int_{\Pi_T^2} \text{sign}'(A(u)-A(v)) A(u)_x A(v)_y \varphi \, dX. \quad (5)$$

Adding up the equations

$$\int_{\Pi_T^2} |u-v|(\varphi_t+\varphi_s) + \mathrm{sign}(u-v)(f(u)-f(v))(\varphi_x+\varphi_y)\,dX$$
$$+ \int_{\Pi_T^2} |A(u)-A(v)|(\varphi_{xx}+2\varphi_{xy}+\varphi_{yy})\,dX$$
$$= \int_{\Pi_T^2} \mathrm{sign}'(A(u)-A(v))[A(u)_x-A(v)_y]^2\varphi\,dX.$$

So

$$\int_{\Pi_T^2} |u-v|(\varphi_t+\varphi_s)\,dX \geq 0.$$

- For any sequence $\{a_j\}_{j\in\mathbb{Z}}$ we associate the piecewise constant function

$$\overline{a}_j(x) = a_j \quad \text{for } x \in I_j. \tag{6}$$

where $I_j = [x_{j-1/2}, x_{j+1/2})$.

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- Let the approximation of the sign function be given by

$$\text{sign}_\varepsilon(\sigma) = \begin{cases} \sin(\frac{\pi\sigma}{2\varepsilon}) & \text{for } |\sigma| < \varepsilon, \\ \operatorname{sign}(\sigma) & \text{otherwise,} \end{cases}$$

where $\varepsilon > 0$. Then we define

$$|u|_\varepsilon = \int_0^u \text{sign}_\varepsilon(z)\, dz.$$

Lemma

Let $\{u_j\}_{j\in\mathbb{Z}}$ be some sequence in $\mathbb{R}$ and $A:\mathbb{R}\to\mathbb{R}$ a strictly increasing continuously differentiable function. For any $u\in\mathbb{R}$ there exist sequences $\{\tau_j\}_{j\in\mathbb{Z}}, \{\theta_j\}_{j\in\mathbb{Z}}$ such that for each $j\in\mathbb{Z}$ both τ_j and θ_j are in $\operatorname{int}(u_j,u_{j+1})$ and

$$\begin{aligned} D_+\mathrm{sign}_\varepsilon(A(u_j)-A(u)) &= \mathrm{sign}'_\varepsilon(A(\tau_j)-A(u))D_+A(u_j),\\ D_+|A(u_j)-A(u)|_\varepsilon &= \mathrm{sign}_\varepsilon(A(\theta_j)-A(u))D_+A(u_j). \end{aligned}$$

If u is a differentiable function of y then for each $j\in\mathbb{Z}$

$$\mathrm{sign}'_\varepsilon(A(\tau_j)-A(u))A(u)_y = -(\mathrm{sign}_\varepsilon(A(\theta_j)-A(u)))_y.$$

Both $\{\tau_j\}_{j\in\mathbb{Z}}$ and $\{\theta_j\}_{j\in\mathbb{Z}}$ depend on u and ε.

$$
\begin{aligned}
\int_{\Pi_T^2} |u-\overline{u}_j| (\varphi_s+\varphi_t) + \mathrm{sign}\,(u-\overline{u}_j)\,(f(u)-f(\overline{u}_j))\,\varphi_y\,dX \\
- \int_{\Pi_T^2} \mathrm{sign}\,(\overline{u}_j - u)\, D_- F(\overline{u}_j, \overline{u}_{j+1})\varphi\,dX \\
= - \int_{\Pi_T^2} |A(u)-A(\overline{u}_j)| \left(D_- D_+\varphi + 2D_+\varphi_y + \varphi_{yy}\right) dX \\
+ \lim_{\varepsilon\downarrow 0} \int_{\Pi_T^2} \mathsf{sign}'_\varepsilon(A(\overline{\tau}_j) - A(u))(A(u)_y - D_+A(\overline{u}_j))^2 \varphi^{\Delta x}\,dX \\
+ \lim_{\varepsilon\downarrow 0} \int_{\Pi_T^2} \Big[\mathsf{sign}'_\varepsilon(A(\overline{u}_j) - A(u))\varphi \\
- \mathsf{sign}'_\varepsilon(A(\overline{\tau}_j) - A(u))\varphi^{\Delta x}\Big](A(u)_y)^2\,dX \\
+ \lim_{\varepsilon\downarrow 0} \int_{\Pi_T^2} \Big[\mathsf{sign}_\varepsilon(A(\overline{u}_j) - A(u)) \\
- \mathsf{sign}_\varepsilon(A(\overline{\theta}_j) - A(u))\Big] D_+A(\overline{u}_j) D_+\varphi\,dX.
\end{aligned}
$$

$$[u_y + f(u)_y = A(u)_{yy}]\,\mathsf{sign}_\varepsilon(A(u) - A(u_j))\varphi$$

$$\Longrightarrow$$

$$\begin{aligned}
\int_{\Pi_T} \mathsf{sign}'_\varepsilon(A(u) - A(u_j))\,(A(u)_y)^2\,\varphi\,dyds \\
= \int_{\Pi_T} [\mathsf{sign}_\varepsilon(A(u) - A(u_j))A(u)_y]_y\,\varphi\,dyds \\
- \int_{\Pi_T} \mathsf{sign}_\varepsilon(A(u) - A(u_j))u_s\varphi\,dyds \\
- \int_{\Pi_T} \mathsf{sign}_\varepsilon(A(u) - A(u_j))f(u)_y\varphi\,dyds.
\end{aligned}$$

$$\left[u_y + f(u)_y = A(u)_{yy}\right] \mathrm{sign}_\varepsilon(A(u) - A(\theta_j))\varphi$$

$$\Longrightarrow$$

$$\begin{aligned}
&\int_{\Pi_T} \mathrm{sign}'_\varepsilon(A(u) - A(\tau_j))\left(A(u)_y\right)^2 \varphi\, dyds \\
&\quad = \int_{\Pi_T} \left[\mathrm{sign}_\varepsilon(A(u) - A(\theta_j))A(u)_y\right]_y \varphi\, dyds \\
&\qquad - \int_{\Pi_T} \mathrm{sign}_\varepsilon(A(u) - A(\theta_j))u_s\varphi\, dyds \\
&\qquad\quad - \int_{\Pi_T} \mathrm{sign}_\varepsilon(A(u) - A(\theta_j))f(u)_y\varphi\, dyds.
\end{aligned}$$

$$
\begin{aligned}
&\int_{\Pi_T} \Big(\mathrm{sign}'_\varepsilon(A(u) - A(u_j)) - \mathrm{sign}'_\varepsilon(A(u) - A(\tau_j)) \Big) \left(A(u)_y\right)^2 \varphi \, dyds \\
&= \int_{\Pi_T} \Big[\Big(\mathrm{sign}_\varepsilon(A(u) - A(u_j)) \\
&\qquad\qquad - \mathrm{sign}_\varepsilon(A(u) - A(\theta_j)) \Big) A(u)_y \Big]_y \varphi \, dyds \\
&\quad - \int_{\Pi_T} \Big(\mathrm{sign}_\varepsilon(A(u) - A(u_j)) \\
&\qquad\qquad - \mathrm{sign}_\varepsilon(A(u) - A(\theta_j)) \Big) u_s \varphi \, dyds \\
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&\qquad\qquad - \mathrm{sign}_\varepsilon(A(u) - A(\theta_j)) \Big) f(u)_y \varphi \, dyds.
\end{aligned}
$$

- $$\lim_{\varepsilon \downarrow 0} [\mathsf{sign}_\varepsilon(A(u) - A(u_j)) - \mathsf{sign}_\varepsilon(A(u) - A(\theta_j)] = H_j'(u).$$

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- Then $|H_j'(u)|$ is bounded by 2 and has support in $\mathrm{int}(u_j, u_{j+1})$.

So for instance if we let

$$P_j(u) = \int_0^u H_j'(z)A'(z)\,dz,$$

then

$$\begin{aligned}
&\Big|\int_{\Pi_T} \underbrace{H_j'(u)A'(u)}_{P_j'(u)} u_y \varphi_y \, dyds\Big| \\
&= \Big|\int_{\Pi_T} P_j(u)\varphi_{yy}\, dyds\Big| \\
&\leq \frac{C}{r^2}|u_{j+1} - u_j| = C\frac{\Delta x}{r^2}|D_+ u_j|.
\end{aligned}$$

Taking all terms into account we obtain the following inequality

$$\int_{\Pi_T^2} |u - v|(\varphi_t + \varphi_s)\, dX \geq -\Gamma\Big(r, \frac{\Delta x}{r}\Big)\frac{\Delta x}{r^2}.$$

Independent of η. It follows

$$\|\overline{u}_j(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|\overline{u}_j^0 - u^0\|_{L^1(\mathbb{R})} + Cr + \Gamma\Big(r, \frac{\Delta x}{r}\Big)\frac{\Delta x}{r^2}.$$

Picking $\Delta x = r^3$ proves the theorem.

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