Almost sure existence of global weak solutions for the supercritical Navier-Stokes equations

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Goal of the talk:

We show that after suitable data randomization there exists a large set of super-critical periodic initial data, in $H^{-\alpha}(\mathbb{T}^d)$ for some $\alpha(d) > 0$, for both 2d and 3d Navier-Stokes equations for which global energy bounds are proved.

We then obtain almost sure super-critical global weak solutions.

We also show that in 2d these global weak solutions are unique.

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The Navier-Stokes Equations

Consider a viscous, homogenous, incompressible fluid with velocity $\vec{u}$ on $\Omega = \mathbb{R}^d$ or $\mathbb{T}^d$, d=2, 3 and which is not subject to any external force. Then the initial value problem for the Navier-Stokes equations is given by

(NSEp)
$$\begin{cases} \vec{u}_t + \vec{v} \cdot \nabla \vec{u} = -\nabla p + \nu \Delta \vec{u}; \quad x \in \Omega \ t > 0 \\ \nabla \cdot \vec{u} = 0 \\ \vec{u}(x,0) = \vec{u}_0(x), \end{cases}$$

where $0 < \nu =$inverse Reynols number (non-dim. viscosity);
$\vec{u} : \mathbb{R}^+ \times \Omega \to \mathbb{R}^d$, $p = p(x,t) \in \mathbb{R}$ and $\vec{u}_0 : \Omega \to \mathbb{R}^d$ is divergence free.

- For smooth solutions it is well known that the pressure term p can be eliminated via Leray-Hopf projections and view (NSEp) as an evolution equation of $\vec{u}$ alone[1],
- the mean of $\vec{u}$ is easily seen to be an invariant of the flow (conservation of momentum) so can reduce to the case of mean zero $\vec{u}_0$.

[1] Although understanding the pressure term might be important.

Then the incompressible Navier-Stokes equations (NSEp) (assume $\nu = 1$) can be expressed as

(NSE)
$$\begin{cases} \vec{u}_t = \Delta\vec{u} - \mathbb{P}\nabla\cdot(\vec{u}\otimes\vec{u}); \quad x \in \Omega, \quad t > 0 \\ \nabla\cdot\vec{u} = 0 \\ \vec{u}(x,0) = \vec{u}_0(x), \end{cases}$$

where $\mathbb{P}$ is the Leray-Hopf projection operator into divergence free vector fields given via

$$\mathbb{P}\vec{h} = \vec{h} - \nabla\frac{1}{\Delta}(\nabla\cdot\vec{h}) = (I + \vec{R}\otimes\vec{R})\vec{h}$$

($\vec{R}$ = Riesz transforms vector) and $\vec{u}_0$ is mean zero and divergence free.

By Duhamel's formula we have

(NSEi)
$$\vec{u}(t) = e^{t\Delta}\vec{u}_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(\vec{u}\otimes\vec{u})\,ds$$

- In fact, under suitable general conditions on $\vec{u}$ the three formulations (NSEp), (NSE) and (NSEi) can be shown to be equivalent (weak solutions, mild solutions, integral solutions. Work by Leray, Browder, Kato, Lemarie, Furioli, Lemarie and Terraneo, and others.)

- Recall if the velocity vector field $\vec{u}(x,t)$ solves the Navier-Stokes equations in $\mathbb{R}^d$ or $\mathbb{T}^d$ then $\vec{u}_\lambda(x,t)$ with

$$\vec{u}_\lambda(x,t) = \lambda\vec{u}(\lambda x, \lambda^2 t),$$

is also a solution to the system (NSE) for the initial data

$$\vec{u}_{0\,\lambda} = \lambda\vec{u}_0(\lambda x) \quad .$$

In particular,

$$\|\vec{u}_{0\,\lambda}\|_{\dot{H}^{s_c}} = \|\vec{u}_0\|_{\dot{H}^{s_c}}, \qquad s_c = \frac{d}{2} - 1.$$

The spaces which are invariant under such a scaling are called critical spaces for Navier-Stokes. Examples:

$$\dot{H}^{\frac{d}{2}-1} \hookrightarrow L^d \hookrightarrow \dot{B}_{p,\infty}^{-1+\frac{d}{p}} \hookrightarrow BMO^{-1} \qquad (1 < p < \infty).$$

$v \in BMO^{-1}$ iff $\exists\ f^i \in BMO$ such that $v = \sum \partial_i f^i$ (Koch-Tataru)

- Classical solutions to the (NSE) satisfy the decay of energy which can be expressed as:

$$\|u(x,t)\|_{L^2}^2 + \int_0^t \|\nabla u(x,\tau)\|_{L^2}^2 \, d\tau = \|u(x,0)\|_{L^2}^2.$$

- **When $d = 2$:** the energy $\|u(x,t)\|_{L^2}$, which is globally controlled, is exactly the scaling invariant $\dot{H}^{s_c} = L^2$-norm. In this case the equations are said to be *critical*. Classical global solutions have been known to exist; see Ladyzhenskaya (1969).
- **When $d = 3$:** the global well-posedness/regularity problem of (NSE) is a long standing open question!
  - The energy $\|u(x,t)\|_{L^2}$ is at the super-critical level with respect to the scaling invariant $\dot{H}^{\frac{1}{2}}$-norm, and hence the Navier-Stokes equations are said to be *super-critical*
  - The lack of a known bound for the $\dot{H}^{\frac{1}{2}}$ contributes in keeping the large data global well-posedness question for the initial value problem (NSE) still open.

Some Background

One way of studying the initial value problem (NSE) is via weak solutions introduced by Leray (1933-34).

- Leray (1934) and Hopf (1951) showed existence of a global weak solution of the Navier-Stokes equations corresponding to initial data in $L^2(\mathbb{R}^d)$.
- Lemarié extended this construction and obtained existence of uniformly locally square integrable weak solutions.

- Questions addressing uniqueness and regularity of these solutions when $d = 3$ have **not** been answered yet. **But important contributions** in understanding partial regularity and conditional uniqueness of weak solutions **by many**; **see e.g.**
  - Caffarelli-Kohn-Nirenberg (82'); Struwe (88'-07'); Lin (98'); P.L. Lions-Masmoudi (98'), Seregin-Sverak (02') Escauriaza-Seregin-Šverak (03'); Vasseur (07'), Kenig-G. Koch (11'), and many others.

Another approach is to construct solutions to the corresponding integral equation ('mild' solutions) pioneered by Kato and Fujita (1961).

- Mild solutions to the Navier-Stokes equations for $d \geq 3$ has been studied locally in time and globally for small initial data in various sub-critical or critical spaces. **Many references**, **see e.g.**
  - T. Kato (84'), Giga-Miyakawa (89'), Taylor (92'), Planchon (96'), Cannone (97'), H.Koch-Tataru (01'), Gallagher-Planchon (02'), Gallagher-Iftimie-Planchon(05'), Germain-Pavlovic-Staffilani(07'), Kenig-G. Koch (09'), others.

Periodic Navier-Stokes Below L^2

We consider the periodic Navier-Stokes problem (NSE)

$$\text{(NSE)} \qquad \begin{cases} \vec{u}_t = \Delta \vec{u} - \mathbb{P}\nabla \cdot (\vec{u} \otimes \vec{u}); \quad x \in \mathbb{T}^d \quad t > 0 \\ \nabla \cdot \vec{u} = 0 \\ \vec{u}(x,0) = \vec{u}_0(x), \end{cases}$$

where $d = 2, 3$ and $\vec{u}_0$ is divergence free and mean zero and $\mathbb{P}$ is the Leray projection into divergence free vector fields.

- We address the question of long time existence of weak solutions for super-critical randomized initial data both in $d = 2$, 3.
- For $d = 2$ we address uniqueness as well.

- **Periodic setting:** similar supercritical randomized well-posededness results were obtained for the 2D cubic NLS by Bourgain (96') and for the 3D cubic NLW by Burq and Tzvetkov (11').

- This approach was applied in the context of the Navier-Stokes to obtain:
  - **Local in time** solutions to the corresponding integral equation for randomized initial data in $L^2(\mathbb{T}^3)$ by Zhang and Fang (2011) and by Deng and Cui (2011). Also **global in time** solutions to the corresponding integral equation for randomized **small initial data**.
  - Deng and Cui (2011) obtained local in time solutions to the corresponding integral equation for randomized initial data in $H^s(\mathbb{T}^d)$, for $d = 2, 3$ with $-1 < s < 0$.

- We are concern with existence of **global in time weak solutions**(NSE) for randomized initial data (without any smallness assumption) in negative Sobolev spaces $H^{-\alpha}(\mathbb{T}^d)$, $d = 2, 3$, for some $\alpha = \alpha(d) > 0$.

Main Ideas

- We start with a divergence free and mean zero initial data $\vec{f} \in (H^{-\alpha}(\mathbb{T}^d))^d$, $d = 2, 3$ and suitably randomize it to obtain $\vec{f}^\omega$ preserving the divergence free condition.
- **Key point:** although the initial data is in $H^{-\alpha}$ for some $\alpha > 0$, the randomized data and its heat flow have almost surely improved L^p bounds. These bounds yield improved nonlinear estimates arising in the analysis of the difference equation for $\vec{w}$ almost surely.

  This induced 'smoothing' phenomena -akin to the role of Kintchine inequalities in Littlewood-Paley theory- stems from classical results of Rademacher, Kolmogorov, Paley and Zygmund proving that random series on the torus enjoy better L^p bounds than deterministic ones.

For example, consider *Rademacher Series*

$$f(\tau) := \sum_{n=0}^{\infty} a_n \, r_n(\tau) \qquad \tau \in [0,1), \quad a_n \in \mathbb{C}$$

$$r_n(\tau) := \operatorname{sign} \sin(2^{n+1}\pi\,\tau), \qquad n \geq 0$$

$$r_{k,j}(\tau) := r_k(\tau) r_j(\tau), \qquad 0 \leq k < j < \infty \quad \text{is o.n. over}(0,1)$$

- If $a_n \in \ell^2$ the sum $f(\tau)$ converges a.e.

Classical Theorem (cf. Zygmund Vol I)

If $a_n \in \ell^2$ then the sum $f(\tau)$ belongs to $L^p([0,1))$ for all p. More precisely,

$$\left(\int_0^1 |f|^p \, d\tau \right)^{1/p} \approx_p \| a_n \|_{\ell^2}$$

- In particular, $p > 2$!

- These ideas were already exploited in Bourgain's work on NLS, KdV, mKdV, Zakharov system.
  - Almost surely global well-posedness on the statistical ensemble via the existence and invariance of the Gibbs measure (after Lebowitz, Rose and Speer's and Zhidkov's works).
  - Well-posedness' failure might come from certain 'exceptional' initial data set. The virtue of the Gibbs measure-or weighted Wiener measure- is that it does not see that exceptional set.
- The starting point of this method is a good local theory on the statistical ensemble (support of the measure) of randomized data of the form

$$\phi = \phi^\omega(x) = \sum \frac{g_n(\omega)}{|n|^\alpha} e^{i\langle x,n\rangle},$$

  where $\{g_n(\omega)\}_n$ are independent standard (complex/real) Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$ (morally '$\widehat{\phi}_n = \frac{g_n}{|n|^\alpha}$ ') and αdepends on the equation and dimension.
- If $S(t)\phi^\omega$ is the linear evolution of the problem at hand and u the solution, one shows that $w = u - S(t)\phi^\omega$ almost surely in ω is smoother than the linear part.

- The invariance of the measure, just like the usual conserved quantities, is used to control the growth in time of those solutions in its support and extend the local in time solutions to global ones almost surely.

- Some recent works (after Bourgain's):
  - Tzvetkov for subquintic radial nonlinear wave equation on the disc.
  - Burq-Tzvetkov for subcubic and subquartic (radial if via measure) nonlinear wave equations on 3D ball.
  - T. Oh in his thesis for the periodic KdV-type coupled systems. Then for white noise for the KdV equation, and Schrödinger-Benjamin-Ono system.
  - Nahmod- Oh - Rey Bellet- S.– and by Nahmod - Rey Bellet - Sheffield - S.– for the derivative NLS equation on $\mathbb{T}$. Need to understand how gaussian measures and their supports transform under gauge transformations (periodic setting).
  - Colliander-Oh for the cubic NLS below $L^2(\mathbb{T})$ (no measure).

Back to Navier-Stokes: Main Steps

- We start with an initial data $\vec{f} \in H^{-\alpha}$, $\alpha > 0$, hence supercritical. Assume $\{\vec{a}_n\}$ are the Fourier coefficients of $\vec{f}$.
- Randomizing $\vec{f}$ means that we replace $\{\vec{a}_n\}$ by $\{l_n(\omega)\vec{a}_n\}$, where $\{l_n(\omega)\}$ are independent random variables, and we take its Fourier series $\vec{f}^\omega$ as the new randomized initial data.
- We seek a solution to the initial value problem (NSE) in the form $\vec{u} = e^{t\Delta}\vec{f}^\omega + \vec{w}$ and identify the difference equation that $\vec{w}$ should satisfy.
- The heat flow of the randomized data gives almost surely improved L^p bounds. These bounds yield improved nonlinear estimates arising in the analysis of the difference equation for $\vec{w}$ almost surely.
- We revisit the proof of equivalence between the initial value problem for the difference equation and the integral formulation of it in our context (similar to Lemarie and Furioli, Lemarie and Terraneo).

- We prove a priori energy estimates for $\vec{w}$. The integral equation formulation is used near time zero and the other one away from zero.
- A construction of a global weak solution to the difference equation via a Galerkin type method is thus possible.
- We prove uniqueness of weak solutions when $d = 2$. Our proof is done 'from scratch' for the difference equation (in spirit of Ladyzhenskaya-Prodi-Serrin condition).
- Put all ingredients together to conclude.

Remark

We should immediately notice that although in our paper we use improved properties for $e^{t\Delta}\vec{f}^{\omega}$, one can show that already $\vec{f}^{\omega}$ belongs to certain **critical** *Besov spaces for wich Gallagher-Planchon already proved in 2d global well-posedness. On the other hand while their proof is based on a combination of the high-low argument of Bourgain and the H. Kock-Tataru small BMO^{-1} data result, ours is much more self contained and gives more precise energy estimate. Moreover our existence result extends to 3d, as mentioned.*

Randomization Setup

Lemma [Large deviation bound] Burq-Tzevtkov 08'

Let $(l_r(\omega))_{r=1}^{\infty}$ be a sequence of real, 0 mean, independent random variables on a probability space (Ω, A, p) with associated sequence of distributions $(\mu_r)_{r=1}^{\infty}$. Assume that μ_r satisfy the property

$$(\dagger) \qquad \exists c > 0 \, : \, \forall \gamma \in \mathbb{R}, \forall r \geq 1, \, \left| \int_{-\infty}^{\infty} e^{\gamma x} \, d\mu_r(x) \right| \leq e^{c\gamma^2}.$$

Then there exists $\alpha > 0$ such that for every $\lambda > 0$, every sequence $(c_r)_{r=1}^{\infty} \in \ell^2$ of real numbers,

$$p\left(\omega \, : \, \left|\sum_{r=1}^{\infty} c_r l_r(\omega)\right| > \lambda\right) \leq 2e^{-\frac{\alpha\lambda^2}{\sum_r c_r^2}}.$$

As a consequence there exists $C > 0$ such that for every $q \geq 2$ and every $(c_r)_{r=1}^{\infty} \in \ell^2$,

$$\left\| \sum_{r=1}^{\infty} c_r l_r(\omega) \right\|_{L^q(\Omega)} \leq C\sqrt{q} \left(\sum_{r=1}^{\infty} c_r^2 \right)^{\frac{1}{2}}.$$

- Burq and Tzvetkov showed that the standard real Gaussian as well as standard Bernoulli variables satisfy the assumption (†)

Definition [Diagonal randomization]

Let $(l_n(\omega))_{n\in\mathbb{Z}^d}$ be a sequence of of real, independent, random variables on a probability space (Ω, A, p) For $\vec{f} \in (H^s(\mathbb{T}^d))^d$, let (a_n^i), $i = 1, 2, \ldots, d$, be its Fourier coefficients. We introduce the map from (Ω, A) to $(H^s(\mathbb{T}^d))^d$ equipped with the Borel sigma algebra, defined by

$$\text{(DR)} \qquad \omega \longrightarrow \vec{f}^\omega, \qquad \vec{f}^\omega(x) = \left(\sum_{n\in\mathbb{Z}^d} l_n(\omega) a_n^1 e_n(x), \ldots, \sum_{n\in\mathbb{Z}^d} l_n(\omega) a_n^d e_n(x) \right),$$

where $e_n(x) = e^{in\cdot x}$ and call such a map randomization.

Remarks

- The map (DR) is measurable and $\vec{f}^{\omega} \in L^2(\Omega; (H^s(\mathbb{T}^d))^d)$, is an $(H^s(\mathbb{T}^d))^d$-valued random variable.
- The diagonal randomization defined in (DR) commutes with the Leray projection $\mathbb{P}$.
- No H^s regularization $\|\vec{f}^{\omega}\|_{H^s} \sim \|\vec{f}\|_{H^s}$ (Burq-Tzvetkov).
- **But randomization gives improved L^p estimates (almost surely).**

Main Results

Theorem [Existence and Uniqueness in 2D]

Fix $T > 0$, $0 < \alpha < \frac{1}{2}$ and let $\vec{f} \in (H^{-\alpha}(\mathbb{T}^2))^2$, $\nabla \cdot \vec{f} = 0$ and of mean zero. Then there exists a set $\Sigma \subset \Omega$ of probability 1 such that for any $\omega \in \Sigma$ the initial value problem (NSE) with datum $\vec{f}^\omega$ has a unique global weak solution $\vec{u}$ of the form

$$\vec{u} \, = \, \vec{u}_{\vec{f}^\omega} \, + \, \vec{w}$$

where $\vec{u}_{\vec{f}^\omega} = e^{t\Delta}\vec{f}^\omega$ and $\vec{w} \in L^\infty([0,T];(L^2(\mathbb{T}^2))^2) \cap L^2([0,T];(\dot{H}^1(\mathbb{T}^2))^2)$.

Theorem [Existence in 3D]

Fix $T > 0$, $0 < \alpha < \frac{1}{3}$ and let $\vec{f} \in (H^{-\alpha}(\mathbb{T}^3))^3$, $\nabla \cdot \vec{f} = 0$, and of mean zero. Then there exists a set $\Sigma \subset \Omega$ of probability 1 such that for any $\omega \in \Sigma$ the initial value problem (NSE) with datum $\vec{f}^\omega$ has a global weak solution $\vec{u}$ of the form

$$\vec{u} \, = \, \vec{u}_{\vec{f}^\omega} \, + \, \vec{w},$$

where $\vec{u}_{\vec{f}^\omega} = e^{t\Delta}\vec{f}^\omega$ and $\vec{w} \in L^\infty([0,T];(L^2(\mathbb{T}^3))^3) \cap L^2([0,T];(\dot{H}^1(\mathbb{T}^3))^3)$.

Free Evolution of the Randomized Data

Deterministic estimates.

For $0 < \alpha < 1$, $k \geq 0$ integer and $\vec{u}_{\vec{f}^\omega} = e^{t\Delta}\vec{f}^\omega$, $\vec{f}^\omega \in (H^{-\alpha}(\mathbb{T}^d))^d$, we have:

$$\|\nabla^k \vec{u}_{\vec{f}^\omega}(\cdot, t)\|_{L^2_x} \lesssim (1 + t^{-\frac{\alpha+k}{2}})\, \|\vec{f}\|_{H^{-\alpha}}.$$

$$\|\nabla^k \vec{u}_{\vec{f}^\omega}\|_{L^\infty_x} \lesssim \left(\max\{t^{-1}, t^{-(k+\alpha+\frac{d}{2})}\}\right)^{\frac{1}{2}} \|\vec{f}\|_{H^{-\alpha}}.$$

Probabilistic estimates.

Let $T > 0$ and $\alpha \geq 0$. Let $r \geq p \geq q \geq 2$, $\sigma \geq 0$ and $\gamma \in \mathbb{R}$ be such that $(\sigma + \alpha - 2\gamma)q < 2$. Then there exists $C_T > 0$ such that for every $\vec{f} \in (H^{-\alpha}(\mathbb{T}^d))^d$

$$\|t^\gamma (-\Delta)^{\frac{\sigma}{2}} e^{t\Delta}\vec{f}^\omega\|_{L^r(\Omega; L^q([0,T]; L^p_x)} \leq C_T\, \|\vec{f}\|_{H^{-\alpha}},$$

where C_T may depend also on p, q, r, σ, γ and α.

Probabilistic estimates (cont.)

Moreover, if we set

$$E_{\lambda,T,\vec{f},\sigma,p} = \{\omega \in \Omega \,:\, \|t^{\gamma}(-\Delta)^{\frac{\sigma}{2}} e^{t\Delta}\vec{f}^{\omega}\|_{L^q([0,T];L^p_x)} \geq \lambda\},$$

then there exists $c_1, c_2 > 0$ such that for every $\lambda > 0$ and for every $\vec{f} \in (H^{-\alpha}(\mathbb{T}^d))^d$

$$P(E_{\lambda,T,\vec{f},\sigma,p}) \leq c_1 \exp\left[-c_2 \frac{\lambda^2}{C_T\|\vec{f}\|^2_{H^{-\alpha}}}\right].$$

Difference Equation. Equivalent Formulations

Let

$$H = \text{the closure of } \{\vec{f} \in (C^\infty(\mathbb{T}^d))^d \mid \nabla \cdot \vec{f} = 0\} \text{ in } (L^2(\mathbb{T}^d))^d,$$
$$V = \text{the closure of } \{\vec{f} \in (C^\infty(\mathbb{T}^d))^d \mid \nabla \cdot \vec{f} = 0\} \text{ in } (\dot{H}^1(\mathbb{T}^d))^d,$$
$$V' = \text{ the dual of } V.$$

and recall

$$\vec{u} - \vec{u}_{\vec{f}^\omega} =: \vec{w},$$

We consider two formulations of the initial value problem for the difference equation that $\vec{w}$ solves and re-prove in our context an equivalence lemma, which is similar to the version for the Navier-Stokes equations themselves (Lemarie, Furioli-Lemarie, Terraneo).

The Equivalence Lemma

Let $T > 0$. Assume that $\nabla \cdot \vec{g} = 0$, $\|\vec{g}(x,t)\|_{L^2} \lesssim (1 + \frac{1}{t^{\frac{\alpha}{2}}})$ and

$$\begin{cases} \|\vec{g}\|_{L^4([0,T],L^4_x)} \le C, & \text{if } d = 2 \\ \|\vec{g}\|_{L^6([0,T],L^6_x)} \le C, & \text{if } d = 3, \end{cases}$$

for some $C > 0$. Then the following statements are equivalent.

(DE) $\vec{w}$ is a weak solution to the initial value problem

$$\begin{cases} \partial_t \vec{w} = \Delta \vec{w} - \mathbb{P}\nabla(\vec{w} \otimes \vec{w}) + c_1[\mathbb{P}\nabla(\vec{w} \otimes \vec{g}) + \mathbb{P}\nabla(\vec{g} \otimes \vec{w})] + c_2 \mathbb{P}\nabla(\vec{g} \otimes \vec{g}) \\ \nabla \cdot \vec{w} = 0, \\ \vec{w}(x,0) = 0. \end{cases}$$

(IE) The function $\vec{w} \in L^\infty((0,T); H) \cap L^2((0,T), V)$, solves

$$\vec{w}(t) = -\int_0^t e^{(t-s)\Delta} \nabla \vec{F}(x,s)\, ds, \qquad \text{where}$$

$$\vec{F}(x,s) = -\mathbb{P}(\vec{w} \otimes \vec{w}) + c_1[\mathbb{P}(\vec{w} \otimes \vec{g}) + \mathbb{P}(\vec{g} \otimes \vec{w})] + c_2 \mathbb{P}(\vec{g} \otimes \vec{g}).$$

Energy Estimates for the Difference Equation

$$E(\vec{w})(t) = \|\vec{w}(t)\|_{L^2}^2 + c\int_0^t \int_{\mathbb{T}^d} |\nabla \otimes \vec{w}|^2 \, dx \, ds$$

Theorem (Energy Estimates).

Let $T > 0$, $\lambda > 0$, $\gamma < 0$, and $\alpha > 0$ be given. Let $\vec{g}$ be s.t. $\nabla \cdot \vec{g} = 0$ and

$$\|\vec{g}(x,t)\|_{L^2} \lesssim (1 + \frac{1}{t^{\frac{\alpha}{2}}}), \quad \|\nabla^k \vec{g}(x,t)\|_{L^\infty} \lesssim \left(\max\{t^{-1}, t^{-(k+\alpha+\frac{d}{2})}\}\right)^{\frac{1}{2}} \; k = 0, 1;$$

$$\begin{cases} \|t^\gamma \vec{g}\|_{L^4([0,T];L^4_x)} \le \lambda, & \text{if } d = 2 \\ \|t^\gamma \vec{g}\|_{L^6([0,T];L^6_x)} \le \lambda, & \text{if } d = 3. \end{cases}$$

Let $\vec{w} \in L^\infty((0,T);H) \cap L^2((0,T);V)$ be a solution to (DE). Then,

$$E(\vec{w})(t) \lesssim C(T, \lambda, \alpha), \quad \text{for all } t \in [0, T].$$

$$\|\frac{d}{dt}\vec{w}\|_{L^p_t H^{-1}_x} \le C(T, \lambda, \alpha),$$

where $p = 2$, if $d = 2$ and $p = \frac{4}{3}$, if $d = 3$.

- We rely on the equivalence lemma and use the integral equation formulation for $\vec{w}$ near time zero and the other one away from zero.
- These *a priori* estimates for $\vec{w}$ are then used in conjunction with Galerkin approximations to construct weak solutions.
- Related work by T. Tao (07').

Sketch of proof: consider two cases: t near zero and t away from zero.

Case **t** near zero: By (IE):

$$\vec{w}(t) = -\int_0^t e^{(t-s)\Delta} \nabla \vec{F}(x,s)\, ds,$$

We use a continuity argument: assume $0 \leq \tau \leq \delta^*$, where δ^* will be determined later. Then for $\tau \in [0, \delta^*]$ we have:

$$\|\vec{w}(t)\|_{L^2_x} \lesssim \|\vec{F}\|_{L^2_{t\in[0,\tau]}L^2_x}, \qquad \text{for all } t \in [0,\tau].$$

Also by applying the maximal regularity we obtain:

$$\|\vec{w}\|_{L^2_{t\in[0,\tau]}H^1_x} \lesssim \|\vec{F}\|_{L^2_{t\in[0,\tau]}L^2_x}.$$

Hence it suffices to analyze $\|\vec{F}\|_{L^2_{t\in[0,\tau]}L^2_x}$.

We have,

$$\|\vec{F}\|_{L^2_{t\in[0,\tau]}L^2_x} \lesssim \|\vec{w}\otimes\vec{w}\|_{L^2_{t\in[0,\tau]}L^2_x} + \|\vec{w}\otimes\vec{g}\|_{L^2_{t\in[0,\tau]}L^2_x} + \|\vec{g}\otimes\vec{w}\|_{L^2_{t\in[0,\tau]}L^2_x} + \|\vec{g}\otimes\vec{g}\|_{L^2_{t\in[0,\tau]}L}$$

$$\|\vec{w}\otimes\vec{w}\|_{L^2_{t\in[0,\tau]}L^2_x} = \|\vec{w}\|^2_{L^4_{t\in[0,\tau]}L^4_x} \lesssim E(\vec{w})(\tau),$$

by interpolating $L^\infty_t L^2_x$ and $L^2_t \dot{H}^1_x$ in $E(\vec{w})(t)$.
For the next two terms by Hölder's inequality we have:

$$\begin{aligned}\|\vec{w}\otimes\vec{g}\|_{L^2_{t\in[0,\tau]}L^2_x} + \|\vec{g}\otimes\vec{w}\|_{L^2_{t\in[0,\tau]}L^2_x} &\lesssim \|\vec{g}\|_{L^p_{x,t}}\|\vec{w}\|_{L^{\frac{2p}{p-2}}_{x,t}} \\ &\lesssim \|\vec{g}\|_{L^p_{x,t}}\|D^{\frac{d}{p}}\vec{w}\|_{L_t^{\frac{2p}{p-2}}L^2_x} \\ &\lesssim \|t^\gamma\vec{g}\|_{L^p_{x,t}}(\delta^*)^{-\gamma}\|D^{\frac{d}{p}}\vec{w}\|_{L_t^{\frac{2p}{d}}L^2_x}(\delta^*)^{\frac{p-2-d}{2p}},\end{aligned}$$

by Sobolev embedding and Hölder's inequality in t (recall $p \geq d+2$.).

By letting $p = 4$ when $d = 2$, and $p = 6$ when $d = 3$ it follows from the assumptions on $\vec{g}$, in conjunction with interpolation between the spaces that appear in $E(\vec{w})(t)$, that

$$\|\vec{w} \otimes \vec{g}\|_{L^2_{t\in[0,\tau]}L^2_x} + \|\vec{g} \otimes \vec{w}\|_{L^2_{t\in[0,\tau]}L^2_x} \lesssim \lambda\,(\delta^*)^{-\gamma+\beta(d)} E^{\frac{1}{2}}(\vec{w})(\tau),$$

where

$$\beta(d) = \begin{cases} 0, & \text{if } d = 2 \\ \frac{1}{12}, & \text{if } d = 3. \end{cases}$$

Finally the last term can be estimated as

$$\|\vec{g} \otimes \vec{g}\|_{L^2_{t\in[0,\tau]}L^2_x} = \|\vec{g}\|^2_{L^4_{t\in[0,\tau]}L^4_x} \leq \left(\lambda(\delta^*)^{-\gamma+\beta(d)}\right)^2,$$

with $\beta(d)$ as above.

Combining the estimates we obtain:

$$E^{\frac{1}{2}}(\vec{w})(\tau) \leq C_1 E(\vec{w})(\tau) + C_2 \lambda(\delta^*)^{-\gamma+\beta(d)} E^{\frac{1}{2}}(\vec{w})(\tau) + C_3 \left(\lambda(\delta^*)^{-\gamma+\beta(d)}\right)^2.$$

Hence if we denote $E^{\frac{1}{2}}(\vec{w})(\tau) = X$, we obtain the inequality:

$$X \leq C_1 X^2 + C_2 \lambda(\delta^*)^{-\gamma+\beta(d)} X + C_3 \left(\lambda(\delta^*)^{-\gamma+\beta(d)}\right)^2.$$

By a continuity argument X is bounded for all $\tau \in [0, \delta^*]$ provided $C_3 \left(\lambda(\delta^*)^{-\gamma+\beta(d)}\right)^2$ is small enough[2]. Hence,

$$E(\vec{w})(\tau) \leq C,$$

for all $\tau \in [0, \delta^*]$.

[2]Depending only on C_1, C_2 and C_3

Case $\mathbf{t} \in [\delta^*, T]$. By a standard energy argument for (DE) we have,

$$\frac{d}{dt}E(\vec{w})(t) = \int_{\mathbb{T}^d} 2\vec{w}(x,t)\cdot\vec{w}_t(x,t)\,dx + 2\int_{\mathbb{T}^d} |\nabla\otimes\vec{w}|^2(x,t)\,dx$$

$$= \int_{\mathbb{T}^d} 2\vec{w}(x,t)\Delta\vec{w}\,dx - 2\int_{\mathbb{T}^d} \vec{w}\cdot\mathbb{P}\nabla(\vec{w}\otimes\vec{w})\,dx + 2\int_{\mathbb{T}^d} |\nabla\otimes\vec{w}|^2(x,t)\,dx$$

$$+2\left(\int_{\mathbb{T}^d} \vec{w}\cdot\mathbb{P}\nabla(\vec{w}\otimes\vec{g})\,dx + \int_{\mathbb{T}^d} \vec{w}\cdot\mathbb{P}\nabla(\vec{g}\otimes\vec{w})\,dx\right) + 2\int_{\mathbb{T}^d} \vec{w}\cdot\mathbb{P}\nabla(\vec{g}\otimes\vec{g})dx.$$

- The expression on second line equals zero as in the case of solutions to the Navier-Stokes equations itself.
- To estimate the third line note that since $\vec{g}$ is divergence-free, $\int_{\mathbb{T}^d} \vec{w}\cdot\mathbb{P}\nabla(\vec{w}\otimes\vec{g})\,dx = \int_{\mathbb{T}^d} \vec{w}\cdot\mathbb{P}(\vec{g}\cdot\nabla)\vec{w}\,dx$ and the last expression equals zero (skew-symmetry).
- Also since $\vec{w}$ is divergence-free too:

$$\int_{\mathbb{T}^d} \vec{w}\cdot\mathbb{P}\nabla(\vec{g}\otimes\vec{w})\,dx = \int_{\mathbb{T}^d} \vec{w}\cdot\mathbb{P}(\vec{w}\cdot\nabla\vec{g})\,dx \lesssim \|\vec{w}\|_{L^2_x}^2\|\nabla\vec{g}\|_{L^\infty_x}.$$

On the other hand by Hölder's inequality,

$$\int_{\mathbb{T}^d} \vec{w} \cdot \mathbb{P}\nabla(\vec{g} \otimes \vec{g})\, dx \leq \|\vec{w}\|_{L^2_x} \|\vec{g}\|_{L^2_x} \|\nabla \vec{g}\|_{L^\infty_x}.$$

Combining the above and using the assumptions on $\vec{g}$ we obtain:

$$\begin{aligned} \frac{d}{dt} E(\vec{w})(t) &\lesssim E(\vec{w})(t)\, \|\nabla \vec{g}\|_{L^\infty_x} + E^{\frac{1}{2}}(\vec{w})(t)\, \|\vec{g}\|_{L^2_x}\, \|\nabla \vec{g}\|_{L^\infty_x} \\ &\lesssim h(t)\, E(\vec{w})(t) + m(t)\, E^{\frac{1}{2}}(\vec{w})(t), \end{aligned}$$

whence

$$E(\vec{w})(t) \leq C(T, \delta^*, \alpha) \qquad \text{for all } t \in [\delta^*, T].$$

Construction of Weak Solutions to the Difference Equation

Write

$$\vec{f}(x,t) = \sum_{\mathbf{k}} \widehat{\vec{f}}(\mathbf{k},t)e^{i\mathbf{k}\cdot x},$$

where **k** is the discrete wavenumber:

$$\mathbf{k} = \sum_{j=1}^{d}(2\pi n_j)\mathbf{e}_j, \qquad n_j \in \mathbb{Z},$$

and $\mathbf{e}_j$ is the unit vector in the j-th direction. By P_M we denote the rectangular Fourier projection operator:

$$P_M\vec{f}(t) = \sum_{\{\mathbf{k}\,:\,|n_j|\leq M \text{ for } 1\leq j\leq d\}} \widehat{\vec{f}}(\mathbf{k},t)e^{i\mathbf{k}\cdot x}.$$

Theorem (Existence).

Let $T > 0$, $\lambda > 0$, $\gamma < 0$ and $\alpha > 0$ be given. Assume that the function $\vec{g}$ satisfies $\nabla \cdot \vec{g} = 0$ and

$$\|\vec{g}(x,t)\|_{L^2} \lesssim (1 + \frac{1}{t^{\frac{\alpha}{2}}})$$

$$\|\nabla^k P_M \vec{g}(x,t)\|_{L^\infty} \lesssim \left(\max\{t^{-1}, t^{-(k+\alpha+\frac{d}{2})}\}\right)^{\frac{1}{2}} \text{ for } k = 0, 1.$$

Furthermore, assume that we have:

$$\begin{cases} \|t^\gamma \vec{g}\|_{L^4_{x;t\in[0,T]}} \leq \lambda, & \text{if } d = 2 \\ \|t^\gamma \vec{g}\|_{L^6_{x;t\in[0,T]}} \leq \lambda, & \text{if } d = 3. \end{cases}$$

Then there exists a weak solution $\vec{w}$ for the initial value problem (DE).

Idea of the proof In the construction of weak solutions, we follow in part the approach based on Galerkin approximations of Doering and Gibbon and of Constantin and Foias. The plan is to construct a global weak solution via finding its Fourier coefficients, which, in turn, will be achieved by solving finite dimensional ODE systems for them.

Uniqueness in 2D

Theorem (Uniqueness in 2D)

Assume that $\vec{g}$ satisfies the same conditions as above Then in $d = 2$ any two weak solutions to (DE) in $L^2([0, T]; V) \cap L^\infty((0, T); H)$ coincide.

Our proof is inspired by the proof in Constantin-Foias which establishes a related uniqueness result for solutions to the Navier-Stokes equations.
Let $\vec{w}_j$ with $j = 1, 2$ be two solutions of (DE) with $\vec{g}$ as above. Let $\vec{v} = \vec{w}_1 - \vec{w}_2$; then

$$\begin{cases} \partial_t \vec{v} = \Delta \vec{v} - \mathbb{P}\nabla(\vec{w}_1 \otimes \vec{v}) - \mathbb{P}\nabla(\vec{w}_2 \otimes \vec{v}) + c_1 \left[\mathbb{P}\nabla(\vec{v} \otimes \vec{g}) + \mathbb{P}\nabla(\vec{g} \otimes \vec{v})\right] \\ \nabla \cdot \vec{v} = 0, \\ \vec{v}(x, 0) = 0. \end{cases}$$

Pair in $(L^2(\mathbb{T}^2))^2$ the first equation with $\vec{v}$ and estimate using estimates on $\vec{g}$ and $\vec{w}_j$.

Proof of Main Theorems: Gathering all the Pieces

We find solutions $\vec{u}$ to (NSE) by writing

$$\vec{u} = \vec{u}^{\omega}_{\vec{f}} + \vec{w}$$

where we recall that $\vec{u}^{\omega}_{\vec{f}}$ is the solution to the linear problem with initial datum $\vec{f}^{\omega}$ and $\vec{w}$ is a solution to (DE) with $\vec{g} = \vec{u}^{\omega}_{\vec{f}}$.

- $\vec{u}$ is a weak solution for (NSE) if and only if $\vec{w}$ is a weak solution for (DE). We also remark that uniqueness of weak solutions to (DE) is equivalent to uniqueness of weak solutions (NSE).
- The proof of the existence of weak solutions is the same for both $d = 2$ and $d = 3$ and it is a consequence existence theorem above. For the uniqueness claimed in $d = 2$ we invoke the uniqueness theorem above. Now to the details.

Let $\gamma < 0$ be such that

$$0 < \alpha < \begin{cases} \frac{1}{2} + 2\gamma, & \text{if } d = 2 \\ \frac{1}{3} + 2\gamma, & \text{if } d = 3. \end{cases}$$

By the probabilistic estimates with $\sigma = 0$, $p = q = 4$ when $d = 2$, and $p = q = 6$ when $d = 3$ we have that given $\lambda > 0$, if we define the set

$$E_\lambda := E_{\lambda,\alpha,\vec{f},\gamma,T} = \{\omega \in \Omega \ / \ \|t^\gamma \vec{u}_{\vec{f}}^\omega\|_{L^p_{[0,T],x}} > \lambda\}.$$

Then there exist $C_1, C_2 > 0$ such that

$$P(E_\lambda) \le C_1 \exp\left[-C_2\left(\frac{\lambda}{C_T\|\vec{f}\|_{H^{-\alpha}}}\right)^2\right].$$

Now, let $\lambda_j = 2^j$, $j \ge 0$ and define $E_j = E_{\lambda_j}$. Note $E_{j+1} \subset E_j$. Let

$$\Sigma := \cup E_j^c \subset \Omega.$$

Then

$$1 \ge P(\Sigma) = 1 - \lim_{j\to\infty} P(E_j) \ge 1 - \lim_{j\to\infty} \exp\left[-C_2\left(\frac{2^j}{C_T\|\vec{f}\|_{H^{-\alpha}}}\right)^2\right] = 1.$$

Final Step:

Our goal is now to show that for a fixed divergence free vector field $\vec{f} \in (H^{-\alpha}(\mathbb{T}^d))^d$ and for any $\omega \in \Sigma$, if we define $\vec{g} = \vec{u}^{\omega}_{\vec{f}}$, the initial value problem (DE) has a global weak solution. In fact given $\omega \in \Sigma$, there exists j such that $\omega \in E_j^c$. In particular we then have

$$\|t^{\gamma}\,\vec{g}\|_{L^p_{x,T}} \leq \lambda_j.$$

Hence assumptions on $\vec{g}$ in the previous theorems are satisfied. This concludes the proof.

Appendix

- Let $\mathcal{B}$ be a Banach space of functions. The space $C_{\text{weak}}((0,T),\mathcal{B})$ denotes the subspace of $L^{\infty}((0,T),\mathcal{B})$ consisting of functions which are weakly continuous, i.e. $v \in C_{\text{weak}}((0,T),\mathcal{B})$ if and only if $\phi(v(t))$ is a continuous function of t for any $\phi \in \mathcal{B}^*$.

Definition (weak solution)

Let $\vec{f} \in (H^{-\alpha}(\mathbb{T}^d))^d$, $\alpha > 0$, $\nabla \cdot \vec{f} = 0$, and of mean zero.
A weak solution to (NSE) on $[0,T]$, is a function
$\vec{u} \in L^2_{\text{loc}}((0,T);V) \cap L^{\infty}_{\text{loc}}((0,T);H) \cap C_{\text{weak}}((0,T);(H^{-\alpha}(\mathbb{T}^d))^d)$
satisfying $\dfrac{d\vec{u}}{dt} \in L^1_{\text{loc}}((0,T),V')$ and

$$\langle \frac{d\vec{u}}{dt}, \vec{v} \rangle + ((\vec{u},\vec{v})) + b(\vec{u},\vec{u},\vec{v}) = 0 \quad \text{for a.e. } t \text{ and for all } \vec{v} \in V,$$

$$\lim_{t \to 0+} \vec{u}(t) = \vec{f} \quad \text{weakly in the } (H^{-\alpha}(\mathbb{T}^d))^d \text{ topology.}$$

Given two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^d$ we use the notation

$$\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v}.$$

In $(L^2(\mathbb{T}^d))^d$ we use the inner product notation

$$(\vec{u}, \vec{v}) = \int \vec{u}(x) \cdot \vec{v}(x)\, dx.$$

In $(\dot{H}^1(\mathbb{T}^d))^d$ we use the inner product notation

$$((\vec{u}, \vec{v})) = \sum_{i=1}^{d} (D_i \vec{u}, D_i \vec{v}).$$

Finally we introduce the trilinear expression

$$b(\vec{u}, \vec{v}, \vec{w}) = \int \vec{u}_j D_j \vec{v}_i \vec{w}_i \, dx = \int \langle \vec{u} \cdot \nabla \vec{v}, \vec{w} \rangle \, dx.$$

Also we note that when $\vec{u}$ is divergence free, we have

$$b(\vec{u}, \vec{v}, \vec{w}) = \int \langle \nabla (\vec{v} \otimes \vec{u}), \vec{w} \rangle \, dx.$$

Definition

Assume that $\nabla \cdot \vec{g} = 0$. A weak solution to (DE) on $[0, T]$, is a function $\vec{w} \in L^2((0,T); V) \cap L^\infty((0,T); H)$ satisfying $\dfrac{d\vec{w}}{dt} \in L^1((0,T); V')$ and such that for almost every t and for all $\vec{v} \in V$,

$$\langle \frac{d\vec{w}}{dt}, \vec{v} \rangle + ((\vec{w}, \vec{v})) + b(\vec{w}, \vec{w}, \vec{v}) + b(\vec{w}, \vec{g}, \vec{v}) + b(\vec{g}, \vec{w}, \vec{v}) + b(\vec{g}, \vec{g}, \vec{v}) = 0$$

and

$$\lim_{t \to 0+} \vec{w}(t) = 0 \quad \text{weakly in the } H \text{ topology.}$$