The vanishing viscosity limit for Navier-Stokes equations in bounded domains with slip boundary conditions

Stefano Spirito
joint work with Luigi Carlo Berselli

26 giugno 2012

Vanishing viscosity

We consider the Navier-Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^3$ and study the behavior of solutions in terms of the viscosity $\nu > 0$

$$\begin{aligned} \partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu - \nu \Delta u^\nu + \nabla p^\nu &= 0, \\ \nabla \cdot u^\nu &= 0, \end{aligned} \qquad \text{(NS)}$$

with initial data

$$u^\nu(0, x) = u_0(x)$$

and some type of boundary conditions on $\Gamma = \partial\Omega$.

Vanishing viscosity

formally, when

$$\nu \to 0$$

we have the convergence $u^\nu \to u^E$, where u^E is the solution to the Euler equations in $\Omega \times (0, T)$

$$\begin{aligned} \partial_t u^E + (u^E \cdot \nabla)\, u^E + \nabla p^E &= 0, \\ \nabla \cdot u^E &= 0, \end{aligned} \qquad \text{(E)}$$

with the following boundary conditions and initial data

$$u^E \cdot n = 0, \quad u^E(0, x) = u_0(x).$$

Convergence taken into account

In particular the following convergence can be considered.

- Convergence and explicit rate of convergence of the inviscid limit in energy norm $L^\infty(0, T^E; L^2(\Omega))$ towards the unique local smooth solutions of Euler equations.

Convergence taken into account

In particular the following convergence can be considered.

- Convergence and explicit rate of convergence of the inviscid limit in energy norm $L^\infty(0, T^E; L^2(\Omega))$ towards the unique local smooth solutions of Euler equations.
- Local in time convergence of the inviscid limit in higher norm, say $C(0, T; H^3(\Omega))$ or $C(0, T; W^{2,p}(\Omega))$ with $p > 3$.

Whole Space

- In the case when there are no physical boundaries the previous problems can be positively solved.

Whole Space

- In the case when there are no physical boundaries the previous problems can be positively solved.
- In particular in the higher norms convergence the key observation is that the Navier-Stokes equations and the Euler equations can be solved in the same interval of time independent on the viscosity.

Whole Space

- In the case when there are no physical boundaries the previous problems can be positively solved.
- In particular in the higher norms convergence the key observation is that the Navier-Stokes equations and the Euler equations can be solved in the same interval of time independent on the viscosity.

$$\partial_t \|u^\nu\|_{H^3}^2 \leq C\|u^\nu\|_{H^3}^3 \qquad \partial_t \|u^E\|_{H^3}^2 \leq C\|u^E\|_{H^3}^3$$

Whole Space

Swann (1971), Kato (1971)

$$u^\nu \overset{*}{\rightharpoonup} u^E \qquad \text{in } L^\infty(0, T^*; H^3).$$

Ebin and Marsden (1970), (using differential geometry)

$$u^\nu \to u^E \qquad \text{in } L^\infty(0, T^*; H^s),$$

for $s > \frac{13}{2}$.
Kato (1975), Beirão da Veiga (1994)

$$u^\nu \to u^E \qquad \text{in } L^\infty(0, T^*; H^s),$$

for $s > 5/2$.
Masmoudi (2007), Beirão da Veiga (2010): shorter proof of the previous result.

Bounded domains

- When we consider bounded domains the situation is different

Bounded domains

- When we consider bounded domains the situation is different
- The obstruction to the convergence even in the energy norm is given by the presence of boundary layers.

Bounded domains

- When we consider bounded domains the situation is different
- The obstruction to the convergence even in the energy norm is given by the presence of boundary layers.
- A rigorous study of the boundary layers created in the inviscid limit of Navier-Stokes equations is very difficult to perform.

Bounded domains

- When we consider bounded domains the situation is different
- The obstruction to the convergence even in the energy norm is given by the presence of boundary layers.
- A rigorous study of the boundary layers created in the inviscid limit of Navier-Stokes equations is very difficult to perform.
- It is difficult the convergence in higher norms since the existence time of strong solutions of the Navier-Stokes equations depends on the viscosity

Bounded Domains: Dirichlet

In the case of a bounded domain Ω with Dirichlet boundary conditions we have only partial result:
Kato (1984)

$$u^\nu \to u^E \quad \text{in} \quad L^2(\Omega), \text{ uniformly in } t \in [0,T], \quad \text{as} \quad \nu \to 0,$$

if and only if

$$\nu \int_0^T \|\nabla u^\nu(\tau)\|^2_{L^2(\Omega^\nu)}\, d\tau \to 0, \quad \text{as} \quad \nu \to 0,$$

where u^E is the solution to the Euler equation and Ω^ν is a boundary strip of width ν. See also Kelliher (2005) for an extension of this result.

Bounded Domains: Navier

If we consider the following Navier boundary condition,

$$[\mathcal{D}(u^\nu)\, n + \beta\, u^\nu]_{\text{tan}} = 0 \qquad u^\nu \cdot n = 0$$

where $\mathcal{D}(u^\nu) = \frac{1}{2}\left[\nabla u^\nu + (\nabla u^\nu)^T\right]$ is the deformation tensor, $\beta \geq 0$ is a constant (the friction coefficient) the situation is simpler.

- 2D case: Bardos (1987), Yudovich (1966),Lopes Filho, Nussenzveig Lopes, and Planas (2005)
- 3D case Iftimie and Planas (2006)

Vorticity Based Boundary Conditions

We study the problem of Navier-Stokes equations with the following vorticity-based slip boundary conditions

$$\begin{cases} u^\nu \cdot n = 0 & \text{on } \Gamma \times]0, T], \\ \omega^\nu \times n = 0 & \text{on } \Gamma \times]0, T], \end{cases} \tag{BC}$$

- n external unit normal to the boundary Γ.
- ω vorticity field, $\omega = \operatorname{curl} u$.

Remarks

- Boundary conditions (BC) coincides with the general Navier boundary conditions in domains with flat boundary and in 2 dimensions (see Xiao-Xin (2007)).

Remarks

- Boundary conditions (BC) coincides with the general Navier boundary conditions in domains with flat boundary and in 2 dimensions (see Xiao-Xin (2007)).
- They differ with the Navier boundary conditions in a lower order term (see Xiao-Xin (2007)).

Previuos results

- Xiao and Xin (2007): Existence of weak solutions, L^2 theory, Convergence of the vanishing viscosity in higher norms (Flat case)

Previuos results

- Xiao and Xin (2007): Existence of weak solutions, L^2 theory, Convergence of the vanishing viscosity in higher norms (Flat case)
- Beirão da Veiga - Berselli (2009): Existence and regularity of strong solutions, geometric conditions for the global regularity.

Previuos results

- Xiao and Xin (2007): Existence of weak solutions, L^2 theory, Convergence of the vanishing viscosity in higher norms (Flat case)
- Beirão da Veiga - Berselli (2009): Existence and regularity of strong solutions, geometric conditions for the global regularity.
- Beirão da Veiga-Crispo (2010): Convergence of the vanishing viscosity in the flat case, L^p theory.

Convergence in higher norms

Let Ω a general bounded regular domains, we consider local smooth solutions of Navier-Stokes and Euler equations constructed in the space

$$C([0,T);W^{2,p}(\Omega)) \qquad p>3$$

Given $u_0 \in W^{2,p}(\Omega)$

- There exist a time $T^\nu = T^\nu(u_0,\nu)$ such that there exists a unique local smooth solution of Navier-Stokes equations in the above space
- There exists a time $T^E = T^E(u_0)$ such that there exists a unique local smooth solution of the Euler equations in the above space

Convergence in higher norms

Theorem (Berselli, S.)

- *Let Ω be a smooth bounded and simply connected domain*
- *Assume that $u_0^\nu = u_0^E =: u_0 \in W^{2,p}(\Omega)$, with $p > 3$, is div.-free and satisfies the boundary conditions (BC) and $\omega_0 = 0$ on Γ*
- *Let u^E be the unique solution of the Euler eq. in $C([0, T_E]; W^{2,p}(\Omega))$ starting from u_0 at time $t = 0$.*

THEN

Convergence in higher norms

Theorem (Berselli, S.)

- *Then, there exists* $T_0 \in [0, T_E]$, *independent on* ν, $\nu_0 > 0$ *such that for all* $0 < \nu < \nu_0$

$$u^\nu \in C([0, T_0]; W^{1,p}(\Omega)) \cap L^\infty([0, T_0]; W^{2,p}(\Omega)),$$

Convergence in higher norms

Theorem (Berselli, S.)

- *Then, there exists* $T_0 \in [0, T_E]$, *independent on* ν, $\nu_0 > 0$ *such that for all* $0 < \nu < \nu_0$

$$u^\nu \in C([0, T_0]; W^{1,p}(\Omega)) \cap L^\infty([0, T_0]; W^{2,p}(\Omega)),$$

- *Moreover, in the same interval we have*

$$u^\nu \rightharpoonup u^E \text{ weakly* in } L^\infty(0, T_0; W^{2,p}(\Omega)),$$
$$u_t^\nu \rightharpoonup u_t^E \text{ weakly* in } L^\infty(0, T_0; L^p(\Omega)),$$
$$\|u^\nu(t) - u^E(t)\|_{s,p} \leq C\nu^{\frac{2-s}{2}} \qquad \forall\, t \in [0, T_0],\ \forall\, s \in [1, 2[,$$

Some remark

The following problem are open

- The vanishing viscosity problem up to three derivative, i.e. in the norm $C_t(H^3_x)$, certain surface integral arising in the integrations by parts are not under control.
- The strong convergence in the norm $C_t(W^{2,p}_x)$, for this problem our method doesn't work.

Initial datum

The condition on the vorticity of the initial datum is related to the fact that in general one cannot impose additional boundary condition to the smooth Euler flow. Indeed, let us suppose that we have the convergence of the vanishing viscosity limit in a norm such that the boundary terms make sense, then the following equations is valid on every point on the boundary

$$\partial_t(\omega^E \times n) + \mathrm{curl}\,(u^E \times \omega^E) \times n = 0$$

Since the convergence of the vanishing viscosity limit holds in a quite strong norm, then the Euler flow should satisfy the extra condition $\omega^E \times n = 0$ on $\partial\Omega \times (0, T^E)$

Initial datum

THEN

we should have that

$$\text{curl}\,(u^E \times \omega^E) \times n = 0 \text{ on } \partial\Omega \times (0, T^E) \qquad \text{(P)}$$

- For general smooth vector fields satisfying the boundary conditions (BC) the condition (P) is generically false (Beirão da Veiga Crispo (2011))
- It follows that if the initial datum is such that (P)$\neq 0$ the convergence of the vanishing viscosity limit in higher norm cannot be exist at least for small time.

Initial datum

- If $\omega_0 = 0$ on Γ we have that there exists T^* such that for any $t \in [0, T^*)$

$$\omega^E(x,t) = 0 \qquad (x,t) \in \partial\Omega \times [0, T^*).$$

Then, under this hypothesis we have that (P)$= 0$.

Initial datum

- If $\omega_0 = 0$ on Γ we have that there exists T^* such that for any $t \in [0, T^*)$

$$\omega^E(x,t) = 0 \qquad (x,t) \in \partial\Omega \times [0, T^*).$$

  Then, under this hypothesis we have that (P)$= 0$.
- It is worth noting that this condition down not trivialize the problem, i.e. if for the initial datum for the Navier-Stokes we have $\omega_0^\nu \neq 0$ on Γ then u^ν is not such that ω^ν vanishes on the boundary.

Initial datum

- If $\omega_0 = 0$ on Γ we have that there exists T^* such that for any $t \in [0, T^*)$

$$\omega^E(x,t) = 0 \qquad (x,t) \in \partial\Omega \times [0, T^*).$$

  Then, under this hypothesis we have that (P)$= 0$.
- It is worth noting that this condition down not trivialize the problem, i.e. if for the initial datum for the Navier-Stokes we have $\omega_0^\nu \neq 0$ on Γ then u^ν is not such that ω^ν vanishes on the boundary.
- This conditions is also optimal, in the sense that if

$$\omega^E(x,t) \neq 0 \qquad (x,t) \in \partial\Omega \times (0, T),$$

  then (P)$\neq 0$ see also Beirão da Veiga and Crispo (2012).

Idea of the proof

The proof is based on some tools

- Smoothing of the data
- Careful analysis of vorticity equations
- Precise perturbation argument

Convergence in energy norm

Theorem (Berselli, S.)

Let $\Omega \subset \mathbb{R}^3$, bounded smooth and open.Let $u_0^E \in H^3(\Omega)$ be a divergence-free vector-field s.t. $u_0^E \cdot n = 0$.
Let u^ν be a weak solution of the Navier-Stokes equations with initial datum u_0^ν and with the vorticity-based Navier's conditions and let $\{u_0^\nu\}_\nu$ converge strongly to u_0^E in $L^2(\Omega)$.
Let $u^E \in C([0,T];H^3(\Omega))$ be the unique solution of the Euler equations with initial datum u_0^E and defined in some interval $[0,T]$.
Then,

$$\sup_{t\in[0,T]} \|u^\nu(t) - u^E(t)\|^2 = \mathcal{O}(\nu^{\frac{3}{2}})$$

and

$$\int_0^T \|\nabla u^\nu(\tau) - \nabla u^E(\tau)\|^2 \, d\tau = \mathcal{O}(\nu^{\frac{1}{2}})$$

.

Convergence in energy norm

Theorem (Berselli, S.)

Moreover, if the initial datum u_0^E is such that

$$\omega_0^E(x) = 0 \qquad \forall\, x \in \Gamma.$$

Then,

$$\sup_{t\in[0,T]} \|u^\nu - u^E\|^2_{L^\infty(0,T;L^2)} = O(\nu^2),$$

$$\|\nabla u^\nu - \nabla u^E\|^2_{L^2(0,T;L^2)} = O(\nu).$$

Remarks

- A similar result has been obtained by Xiao-Xin (Chinese Ann. Math. 2011).
- Our result is different since we consider also general initial datum u_0^ν only in $L^2(\Omega)$ and we show how the boundary conditions even at level of energy norm influences the convergence
- Moreover, they consider a non standard boundary value problem for the Navier-Stokes equations

Weak solutions

$u^\nu \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$, divergence-free and tangential to the boundary, is a (Leray-Hopf) weak solution of the Navier-Stokes equations if :

$$\int_0^T \int_\Omega \big(-u^\nu \phi_t + \nu \nabla u^\nu \nabla \phi - (u^\nu \cdot \nabla)\, \phi\, u^\nu \big)\, dx d\tau$$
$$+ \nu \int_0^T \int_\Gamma u^\nu \cdot (\nabla n)^T \cdot \phi \, dS d\tau = \int_\Omega u_0^\nu \phi(0)\, dx,$$

for all $\phi \in C_0^\infty([0,T[\times\overline{\Omega})$, s.t. $\nabla \cdot \phi = 0$ in $\Omega \times [0,T[$, and $\phi \cdot n = 0$ on $\Gamma \times [0,T[$

The following energy estimate is satisfied for all $t \in [0,T]$.

$$\frac{1}{2}\int_\Omega |u^\nu(t)|^2\, dx + \nu \int_0^t \int_\Omega |\nabla u^\nu|^2\, dx d\tau +$$
$$\nu \int_0^t \int_\Gamma u^\nu \cdot (\nabla n)^T \cdot u^\nu \, dS d\tau \le \frac{1}{2}\int_\Omega |u_0^\nu|^2\, dx,$$

Sketch of the proof

- Formally, consider the equations for the differences $u := u^\nu - u^E$ and multiply by u and integrate by parts.

Sketch of the proof

- Formally, consider the equations for the differences $u := u^\nu - u^E$ and multiply by u and integrate by parts.
- The previous calculation is only formal since u^ν is only a weak solution and cannot be used as test functions, however using the integral formulation and the energy inequality the following estimate can be rigorously justified.

Sketch of the proof

- Formally, consider the equations for the differences $u := u^\nu - u^E$ and multiply by u and integrate by parts.
- The previous calculation is only formal since u^ν is only a weak solution and cannot be used as test functions, however using the integral formulation and the energy inequality the following estimate can be rigorously justified.
- Energy estimate

$$\frac{\|u(t)\|^2}{2} + \nu \int_0^t \|\nabla u\|^2 d\tau \leq -\nu \int_0^t \int_\Gamma u^\nu (\nabla n)^T u \, dS d\tau$$
$$+ \int_0^t \int_\Omega (u \cdot \nabla) \, u^E u \, dx d\tau - \nu \int_0^t \int_\Omega \nabla u^E \nabla u \, dx d\tau + \frac{\|u(0)\|^2}{2}$$

Sketch of the proof

- We use the following integrations by parts formula

$$
\begin{aligned}
-\nu \int_0^t \int_\Omega \nabla u^E \nabla u \, dx d\tau = & \ \nu \int_0^t \int_\Gamma (\omega^E \times n) \, u \, dS d\tau \\
& + \nu \int_0^t \int_\Gamma u^E \cdot (\nabla n)^T \cdot u \, dS d\tau \\
& - \nu \int_0^t \int_\Omega \Delta u^E u \, dx d\tau .
\end{aligned}
$$

Sketch of the proof

We estimate the terms on the right-hand side of the energy estimate in the following way

$$\nu \left| \int_\Gamma (\omega^E \times n) u \, dS \right| \le C \nu \|u\|^{\frac{1}{2}} \|\nabla u\|^{\frac{1}{2}} \le C \, \nu^{\frac{3}{2}} + C \, \|u\|^2 + \frac{\nu}{2} \|\nabla u\|^2,$$

$$\nu \left| \int_\Gamma u \cdot (\nabla n)^T \cdot u \, dS \right| \le C \, \nu \|u\|_\Gamma^2 \le C \, \nu \|u\|^2 + \frac{\nu}{2} \|\nabla u\|^2,$$

$$\nu \left| \int_\Omega \Delta u^E u \, dx \right| \le C \left(\|u\|^2 + \nu^2 \right).$$

Sketch of the proof

Then we obtain

$$\|u(t)\|^2+\nu\int_0^t\|\nabla u(\tau)\|^2 d\tau \leq \|u(0)\|^2+C\left[\int_0^t\|u(\tau)\|^2 d\tau+\nu^2+\nu^{\frac{3}{2}}\right].$$

Sketch of the proof

Then we obtain

$$\|u(t)\|^2+\nu\int_0^t \|\nabla u(\tau)\|^2 d\tau \le \|u(0)\|^2+C\left[\int_0^t \|u(\tau)\|^2 d\tau + \nu^2 + \nu^{\frac{3}{2}}\right].$$

In particular if $\omega_0^E = 0$ on Γ we get a better rate of convergence

$$\|u(t)\|^2+\nu\int_0^t \|\nabla u(\tau)\|^2 d\tau \le \|u(0)\|^2+C\left[\int_0^t \|u(\tau)\|^2 d\tau + \nu^2\right],$$

that is the convergence stated in the Theorem.

References

- L. C. Berselli and S. Spirito.*Inviscid limits for the 3D Navier-Stokes equations with slip boundary conditions and application to the 3D Boussinesq equations*, (Preprint 2012)
- L. C. Berselli and S. Spirito *On the vanishing viscosity limit of 3D Navier-Stokes equations under slip boundary conditions in general domains* Quaderno 2011/3 del Dipartimento di Matematica Applicata U.Dini. (Accepted CMP)