Finite-time Splash and Splat Singularity for the 3-D Euler equations

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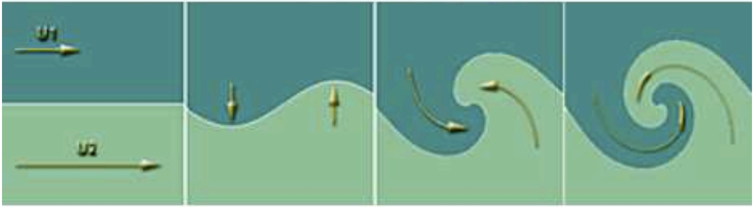
HYP2012
Padova, Italy
(In collaboration with D. Coutand)

June 28, 2012

Singularities in fluid dynamics

Surface of discontinuity self-intersects



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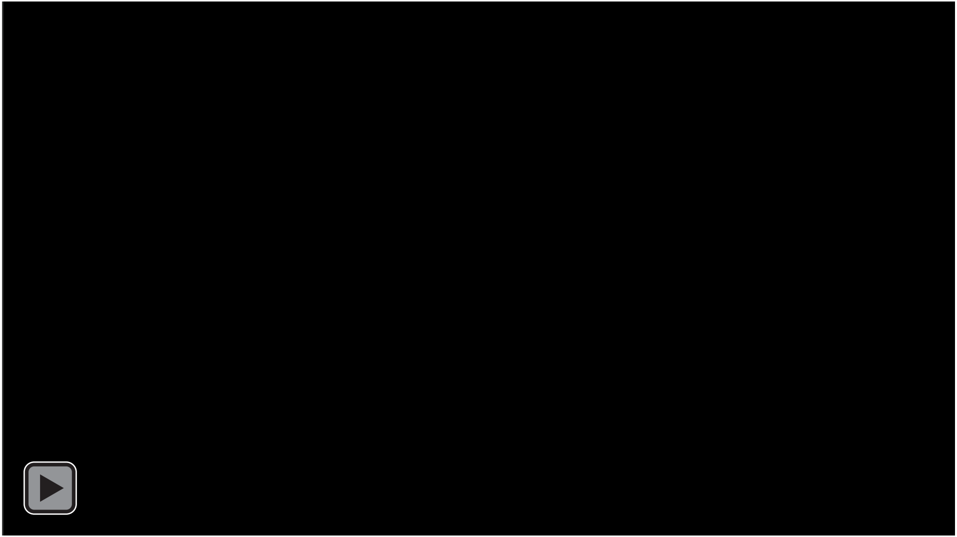
Breaking waves (Fiji Cloudbreak June 2012)



3-D Euler free-surface equations

The system of PDE

$$
\begin{aligned}
&\text{(Momentum eqn)} \quad & u_t + Du \cdot u + Dp &= 0 \quad \text{in} \quad \Omega(t) \\
& & \operatorname{div} u &= 0 \quad \text{in} \quad \Omega(t) \\
&\text{(Boundary condition)} & p &= 0 \quad \text{on} \quad \Gamma(t) \\
&\text{(Speed of free-boundary)} & \mathcal{V}(\Gamma(t)) &= u \cdot n \\
& & u &= u_0 \quad \text{on} \quad \Omega(0) \\
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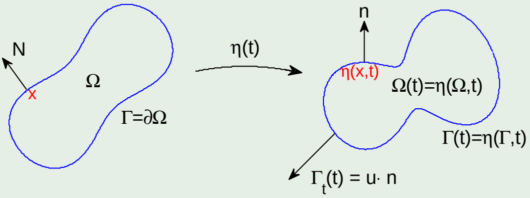
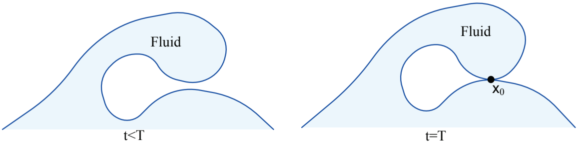
Moving Free-Boundary



The splash and splat singularities

Definition

Splash Singularity – the $\Gamma(t)$ self-intersects at a point x_0.



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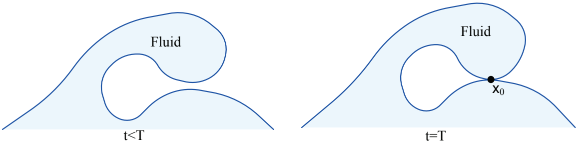

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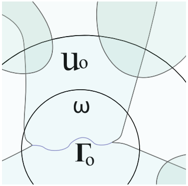
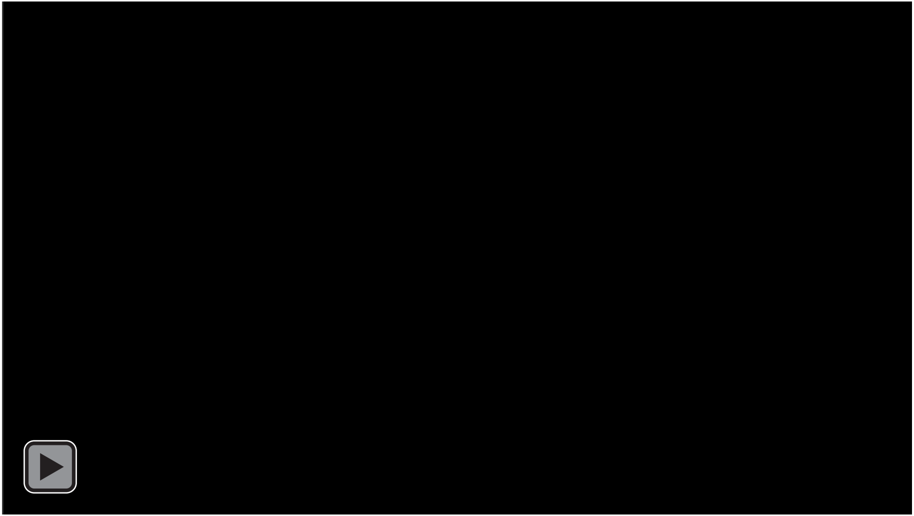


Justification of my cartoon picture: Shipsterns Bluff, Tasmania



A Brief History of the Free-Boundary Euler/WW Problem

- Water-waves equation: Assume $u = D\phi$, where ϕ is the velocity potential.
- Then curl $u = 0$ and ϕ is harmonic – problem reduces to the motion of the free-surface (complex analysis in 2-D)

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- Local well-posedness for water-waves:
  - Wu(1997,1999) (∞-depth)
  - Lannes(2005) (finite-depth, N-M)
  - Ambrose & Masmoudi (2005,2009) (limit zero-surface-tension)
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- Water-waves *long-time existence* for small data: Wu (2009,2011), Germain, Masmoudi, & Shatah (2012)

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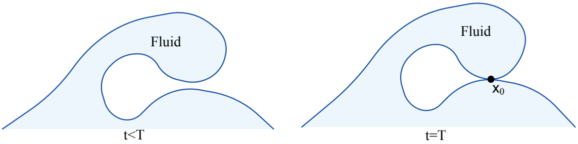

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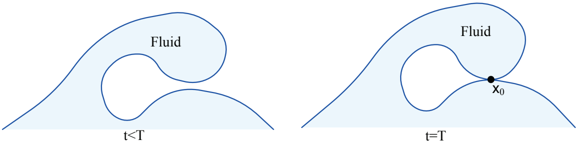

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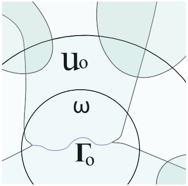

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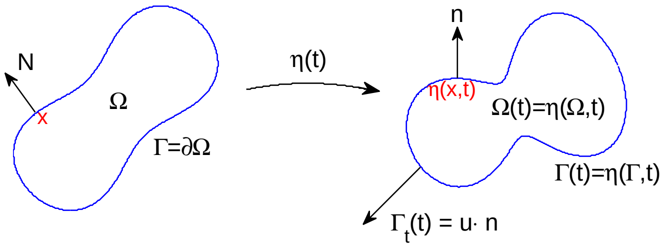
- Take limit $\Omega_0 = \lim_{\epsilon \to 0} \eta^\epsilon(-T, \Omega^\epsilon)$ and prove that Ω_0 is connected, with connected boundary, open, and smooth.
- Run the problem forward-in-time from Ω_0 as initial domain to find singularity at $t = T$.

Lagrangian variables (or coordinates)



Let $\eta(\cdot, t) : \Omega \to \Omega(t)$ be the solution of

$$\eta_t = u \circ \eta, \quad \eta(x, 0) = e$$

so locally $\eta_{,1}(x, t)$ and $\eta_{,2}(x, t)$ span $T_{\eta(x,t)}\Gamma(t)$.
Set

$$v(x, t) := u(\eta(x, t), t), \quad q(x, t) := p(\eta(x, t), t),$$
$$A(x, t) := [D\eta(x, t)]^{-1} = \text{inverse deformation}\,, \quad J = \det D\eta = 1\,.$$

The Euler equations in Lagrangian variables

$$
\begin{aligned}
\partial_t v + A^T Dq &= 0 && \text{in } \Omega \times (0,T]\,,\\
\operatorname{div}_\eta v = (A^j_i v^i{,_j}) &= 0 && \text{in } \Omega \times (0,T]\,,\\
q &= 0 && \text{on } \Gamma \times (0,T]\,,\\
(\eta, v) &= (\mathrm{Id}, u_0) && \text{on } \Omega \times \{t=0\}\,,
\end{aligned}
$$

Definition (Taylor Sign Condition)

$-\frac{\partial q}{\partial N}\Big|_{t=0} > 0$ Required for Well-Posedness (Always true for irrotational flows)

Theorem (Coutand-S (2007))

Simple: Local W-P with norm $E(t) = \|\eta(t)\|^2_{4.5} + \|v(t)\|^2_4$.
Optimal: Local W-P with norm $E(t) = \|\eta(t)\|^2_{3.5} + \|v(t)\|^2_3$.

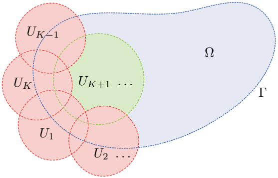
A standard domain and its norm

Cover Γ with K open sets and Ω with L open sets:

$$B = B(0,1), \quad B^+ = B \cap \{x_3 > 0\}, \quad B^0 = \overline{B} \cap \{x_3 = 0\}$$

$\theta_l \colon B \to U_l$ is an $H^{4.5}$ diffeomorphism,

$$\theta_l(B^+) = U_l \cap \Omega, \quad \theta_l(B^0) = U_l \cap \Gamma\,.$$



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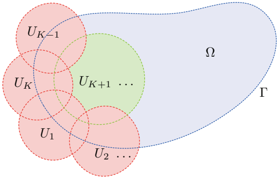

Definition

A domain Ω is of class $H^{4.5}$ if

$$\left(\sum_{l=1}^{K} \|\theta_l\|^2_{4.5,B^+} + \sum_{l=K+1}^{L} \|\theta_l\|^2_{4.5,B} \right)^2 < \infty\,.$$

A priori estimates: control for the free-boundary

- **Higher-order energy or norm for incompressible problem:**

$$E(t) = \|\eta(t)\|^2_{4.5} + \|v(t)\|^2_4, \quad f(T) = \sup_{t\in[0,T]} E(t)$$

- **Notation.** $\bar{\partial}$ = tangential derivative near boundary Γ
- **Energy estimates using tangential derivatives.**

$$\int_0^T \int_\Omega \bar{\partial}^4 (v^i_t + a^k_i q,_k)\, \bar{\partial}^4 v^i \; dxdt = 0$$

so

$$\int_0^T \frac{1}{2}\frac{d}{dt}\|\bar{\partial}^4 v(t)\|^2_0 + \int_\Omega \bar{\partial}^4 a^k_i \; q,_k \, \bar{\partial}^4 v^i dx + \int_\Omega a^k_i \, \bar{\partial}^4 q,_k \; \bar{\partial}^4 v^i dxdt = l.o.t.$$

- First two terms contribute two the energy $E(t)$ while the last term is an error term controlled using a techinical lemma.

A priori estimates: control for the free-boundary

- **Cofactor identities** $A_i^k = a_i^k$:

$$\bar\partial a_i^k = -a_r^k \bar\partial \eta^r{,_s}\, a_i^s \text{ and } a_i^k(t) N_k = \sqrt{g}\, n_i(t)$$

- Since on Γ $q{,_k} = \frac{\partial q}{\partial N} N_k$ (since $q=0$ on Γ),

$$\begin{aligned}
\int_\Omega \bar\partial^4 a_i^k\, q{,_k}\, \bar\partial^4 v^i\, dx &= -\int_\Omega a_r^k \bar\partial^4 \eta^r{,_s}\, a_i^s\, q{,_k}\, \bar\partial^4 v^i\, dx \\
&= -\int_\Gamma a_r^k \bar\partial^4 \eta^r a_i^s N_s\, q{,_k}\, \bar\partial^4 v^i\, dS + \text{errors} \\
&= -\int_\Gamma \frac{\partial q}{\partial N} \bar\partial^4 \eta^r a_r^k N_k\;\, a_i^s N_s\, \bar\partial^4 v^i\, dS + \text{errors} \\
&= -\int_\Gamma \frac{\partial q}{\partial N} \sqrt{g}^2 (\bar\partial^4 \eta \cdot n)\, (\bar\partial^4 v \cdot n) dS + \text{errors} \\
&= \frac{1}{2} \frac{d}{dt} \int_\Gamma (-\partial q/\partial N) |\bar\partial^4 \eta \cdot N|^2\, dS + \text{errors}
\end{aligned}$$

A priori estimates: control for the free-boundary

- Elliptic equation for $q(t)$:

$$-\partial_{x_j}[A^j_i A^k_i \partial_{x_k} q(t)] = \partial_{x_j} v^i A^j_r \partial_{x_s} v^r A^s_i \quad \text{in} \quad \Omega\,,$$
$$q(t) = 0 \quad \text{on} \quad \Gamma\,.$$

- Elliptic estimate for $q(t)$: (Sobolev-class coefficients)

$$\|q(t)\|^2_{4.5} \le CE(t)$$

- **Lemma.** For $f \in H^{1/2}(\Omega)$, $\|\bar\partial f\|_{[H^{1/2}(\Omega)]'} \le C\|f\|_{H^{1/2}(\Omega)}$
- Analysis of error term $\int_0^T \int_\Omega a^k_i\, \bar\partial^4 q\, \bar\partial^4 v^i dxdt$: Integrate by parts

$$-\int_0^T \int_\Omega \bar\partial^4 q\, a^k_i \bar\partial^4 v^i{}_{,k}\, dxdt = \int_0^T \int_\Omega \bar\partial^4 q \bar\partial^4 a^k_i\, v^i{}_{,k}\, dxdt + \text{l.o.t.}$$
$$\le \int_0^T \|q\|_{4.5}\|a\|_{3.5}\|\nabla v\|_{L^\infty} dt \le CTP(\sup_{t\in[0,T]} E(t))\,.$$

Splash and splat domains: Main Theorems

Theorem (2-D, Cordoba, Fefferman et al.)

Finite-time splash singularity for irrotational 2-D water waves equations using complex analysis, analytic functions, and conformal transformations

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Theorem (3-D, Coutand-S)

There exist initial domains Ω_0 of class $H^{4.5}$ and initial velocity fields $u_0 \in H^4(\Omega_0)$, which satisfy the Taylor sign condition, such that after a finite time $T > 0$, the solution to the Euler equation $\eta(t)$ (with such data) maps Ω_0 onto the splash domain Ω_s, with final velocity u_s. This final velocity u_s satisfies the local Taylor sign condition on the splash domain Ω_s. The splash velocity u_s has a specified relative velocity on the boundary of the splash domain.

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Applications. Water-waves, incompressible Euler, compressible Euler in vacuum, incompressible and compressible MHD, surface tension, fluid-structure interaction, etc.

Splash Domain Ω_s

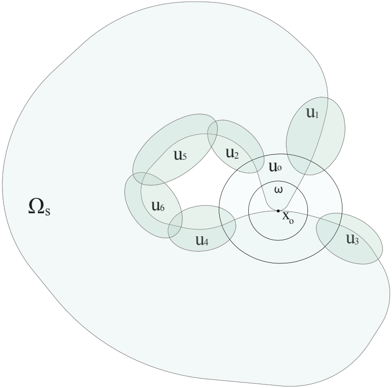

Figure: Splash domain is a generalized $\mathbf{H^{4.5}}$-class domain

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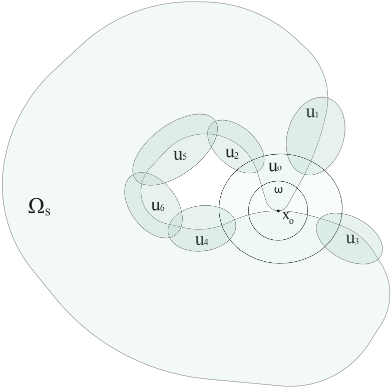

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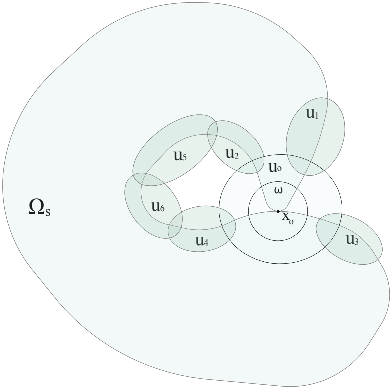

Figure: Splash domain is a generalized $\mathbf{H^{4.5}}$-class domain

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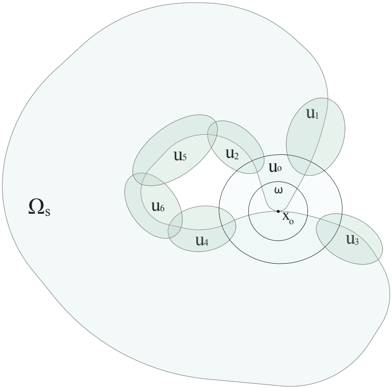

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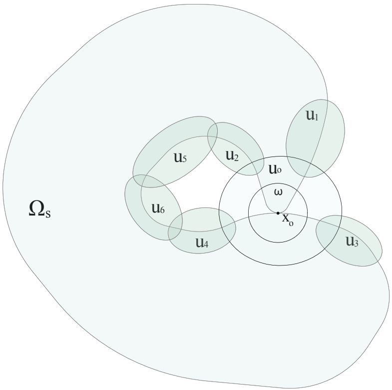

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- x_0 is unique self-intersection point with tangent plane, the horizontal plane $x_3 - (x_0)_3 = 0$
- U_0 is open set about x_0
- Define two $H^{4.5}$-class diffeomorphisms θ_+ and θ_- of B onto U_0 with the following properties:

$$\theta_+(B^+) = U_0^+ , \qquad \theta_-(B^+) = U_0^- ,$$

$$\theta_+(B^0) = \overline{U_0^+} \cap \Gamma_s , \quad \theta_-(B^0) = \overline{U_0^-} \cap \Gamma_s ,$$

$$\{x_0\} = \theta_+(B^0) \cap \theta_-(B^0) ,$$

$$\theta_+(0) = \theta_-(0) = x_0 .$$

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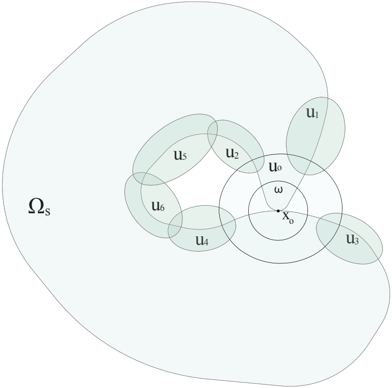

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$$\theta_+(B^0) = \overline{U_0^+} \cap \Gamma_s \,, \quad \theta_-(B^0) = \overline{U_0^-} \cap \Gamma_s \,,$$

$$\{x_0\} = \theta_+(B^0) \cap \theta_-(B^0) \,,$$

$$\theta_+(0) = \theta_-(0) = x_0 \,.$$

- $\{U_l\}_{l=0}^L$ cover Ω_s with charts $\theta_\pm$ and θ_l, $l = 1, .., L$

The splash velocity $u_s : \Omega_s \to \mathbb{R}^3$

Definition (Splash velocity u_s)

A velocity field u_s on an $\mathbf{H^{4.5}}$-class splash domain Ω_s is called a *splash velocity* if it satisfies the following properties:

1. $\zeta u_s \circ \theta_\pm \in H^{4.5}(B^+)$, $\zeta u_s \circ \theta_l \in H^{4.5}(B^+)$ for each $1 \leq l \leq K$ and $u_s \in H^{4.5}(\omega)$ for each $\overline{\omega} \subset \Omega_s$;
2. $u_s^3 \circ \theta_- > C_-$, $-u_s^3 \circ \theta_+ > C_+$ in B^+ and $C_- + C_+ > 0$

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Definition (Splash pressure p_s)

p_s is called a *splash pressure* if it satisfies the following properties:

1. $p_s \in H^{4.5}(\Omega_s)$ is the unique solution of
$$-\Delta p_s = -\frac{\partial u_s^i}{\partial x_j}\frac{\partial u_s^j}{\partial x_i} \qquad \text{in } \Omega_s \,,$$
$$\zeta p_s \circ \theta_\pm = 0 \;\; \zeta p_s \circ \theta_l = 0 \;\; \text{on } B^0 \;\; \text{for} \;\; l = 1, ..., K\,;$$
2. p_s satisfies the local version of the Taylor sign condition:
$$\frac{\partial}{\partial x_3}(\zeta p_s \circ \theta_\pm) > 0 \text{ and } \frac{\partial}{\partial x_3}(\zeta p_s \circ \theta_l) > 0 \;\; \text{on} \;\; B^0 \;\; \text{for} \;\; l = 1, ..., K\,.$$

Approximation of Ω_s by standard domains Ω^ϵ

For $\epsilon > 0$, approximate $\theta_\pm$ by $\theta_\pm^\epsilon$ such that

$$\theta_-^\epsilon \to \theta_- \quad \text{and} \quad \theta_+^\epsilon \to \theta_+ \quad \text{as} \quad \epsilon \to 0\,.$$

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Let $0 \le \psi \in \mathcal{D}(B(0,\mathfrak{r}))$ with $\psi(0) = 1$ ($\mathfrak{r} > 0$ taken sufficiently small)

$$\theta_-^\epsilon(x) = \theta_-(x) - \epsilon\,\psi(x)\,\mathbf{e_3}\,,$$
$$\theta_+^\epsilon(x) = \theta_+(x) + \epsilon\,\psi(x)\,\mathbf{e_3}\,,$$

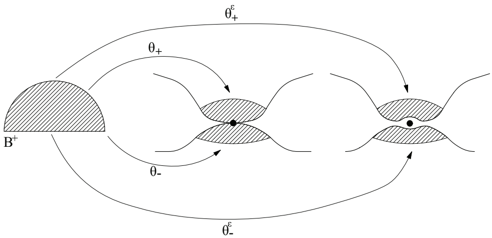

Figure: Perform surgery: $|\theta_+^\epsilon(x) - \theta_-^\epsilon(y)| > \epsilon$ for all $x, y \in B^+$

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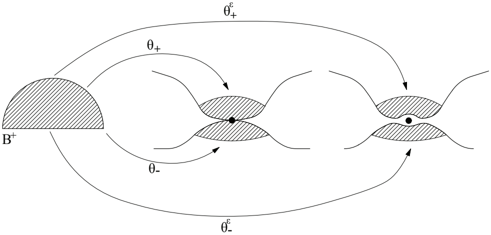

Figure: Perform surgery: $|\theta_+^\epsilon(x) - \theta_-^\epsilon(y)| > \epsilon$ for all $x, y \in B^+$

Then, $\theta_-^\epsilon(\overline{B^+}) \cap \theta_+^\epsilon(\overline{B^+}) = \emptyset$ and

$$\theta_\pm^\epsilon(B^+ \cap B(0,1/2)) \cap \theta_l(B^+) = \emptyset \text{ for } l = 1, \ldots, K$$

Approximate splash velocity $u_s^\epsilon : \Omega^\epsilon \to \mathbb{R}^3$

u_s^ϵ is modified on the charts $\theta_\pm^\epsilon$:

$$u_s^\epsilon \circ \theta_l = u_s \circ \theta_l\,,\ \text{in}\ \ B^+\,,\ \text{for}\ l = 1, ..., K\,;$$
$$u_s^\epsilon \circ \theta_l = u_s \circ \theta_l\,,\ \text{in}\ \ B\,,\ \text{for}\ l = K+1, ..., L\,;$$
$$u_s^\epsilon \circ \theta_-^\epsilon = u_s \circ \theta_-\,,\ \text{and}\,,\ u_s^\epsilon \circ \theta_+^\epsilon = u_s \circ \theta_+\,,\ \text{in}\ B^+\,.$$

Approximate splash velocity $u_s^\epsilon : \Omega^\epsilon \to \mathbb{R}^3$

u_s^ϵ is modified on the charts $\theta_\pm^\epsilon$:

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$$u_s^\epsilon \circ \theta_l = u_s \circ \theta_l \,, \text{ in } \; B \,, \text{ for } l = K+1, ..., L \,;$$
$$u_s^\epsilon \circ \theta_-^\epsilon = u_s \circ \theta_- \,, \text{ and} \,, \; u_s^\epsilon \circ \theta_+^\epsilon = u_s \circ \theta_+ \,, \text{ in } B^+ \,.$$

We then have the existence of constants $A > 0$, $B > 0$ such that

$$\|u_s^\epsilon\|_{4.5,\Omega^\epsilon} \leq A \big(\|\zeta u_s^\epsilon \circ \theta_-^\epsilon\|_{4,B^+} + \|\zeta u_s^\epsilon \circ \theta_+^\epsilon\|_{4,B^+} + \sum_{l=1}^{K} \|\zeta u_s^\epsilon \circ \theta_l^\epsilon\|_{4.5,B^+} + \sum_{l=K+1}^{L} \|\zeta u_s^\epsilon \circ \theta_l^\epsilon\|_{4.5,B} \Bigg) \leq B \|u_s\|_{4.5,\Omega_s} \,.$$

Approximate splash pressure $p_s^\epsilon : \Omega^\epsilon \to \mathbb{R}$

- $p_s^\epsilon \in H^{4.5}(\Omega^\epsilon)$ is the unique solution of

$$
\begin{aligned}
-\Delta p_s^\epsilon &= \frac{\partial u_s^{\epsilon i}}{\partial x_j} \frac{\partial u_s^{\epsilon j}}{\partial x_i} && \text{in } \Omega^\epsilon\,,\\
p_s^\epsilon &= 0 && \text{on } \partial\Omega^\epsilon\,.
\end{aligned}
$$

- $\theta_\pm^\epsilon \to \theta_\pm$ and $\theta_l^\epsilon \to \theta_l$ in $H^{4.5}(B^+)$
- u_s^ϵ only modified on charts $\theta_\pm^\epsilon$
- Then $\zeta p_s^\epsilon \circ \theta_\pm^\epsilon \to \zeta p \circ \theta_\pm$ and $\zeta p_s^\epsilon \circ \theta_l^\epsilon \to \zeta p \circ \theta_l$ in $H^{4.5}(B^+)$
- Conclusion: uniformly for $\epsilon > 0$ small enough, local TSC satisfied:

$$
\frac{\partial}{\partial x_3}(\zeta p_s^\epsilon \circ \theta_\pm^\epsilon) > 0 \text{ and } \frac{\partial}{\partial x_3}(\zeta p_s^\epsilon \circ \theta_l^\epsilon) > 0\,, \text{ on } B^0 \text{ for each } 1 \le l \le K\,.
$$

Reminder: approximate domain Ω^ϵ in chart U_0

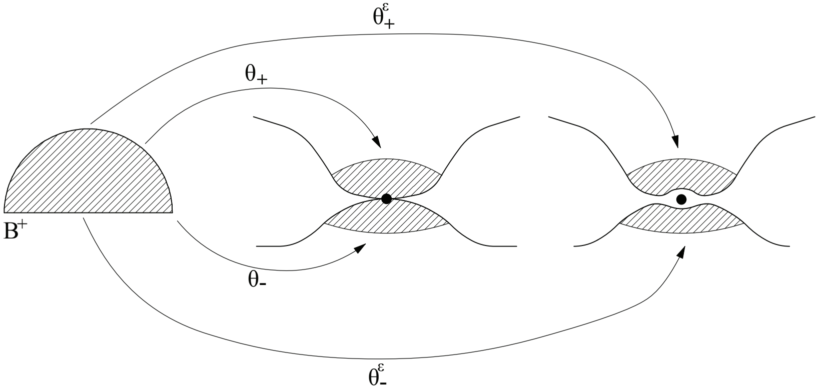

Figure: U_0^+ and U_0^- are approximated by non self-intersecting charts

Solving the Euler equations backwards-in-time from the *final* states Ω^ϵ and u_s^ϵ

- from Ω^ϵ and u_s^ϵ solve backward-in-time:

$$\begin{aligned}
\eta^\epsilon(t) &= e + \int_0^t v^\epsilon && \text{in } \Omega^\epsilon \times [-T^\epsilon, 0]\,, \\
v_t^\epsilon + [A^\epsilon]^T D q^\epsilon &= 0 && \text{in } \Omega^\epsilon \times [-T^\epsilon, 0)\,, \\
\operatorname{div}_{\eta^\epsilon} v^\epsilon &= 0 && \text{in } \Omega^\epsilon \times [-T^\epsilon, 0]\,, \\
q^\epsilon &= 0 && \text{on } \Gamma^\epsilon \times [-T^\epsilon, 0]\,, \\
(\eta^\epsilon, v^\epsilon) &= (e, u_s^\epsilon) && \text{in } \Omega^\epsilon \times \{t = 0\}\,,
\end{aligned}$$

where $A^\epsilon(x,t) = [D\eta^\epsilon(x,t)]^{-1}$.

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  where $A^\epsilon(x,t) = [D\eta^\epsilon(x,t)]^{-1}$.

- By local well-posedness theorem, $\exists T_\epsilon > 0$ and a unique solution on $[-T^\epsilon, 0]$ with

$$E^\epsilon(t) = \|\eta^\epsilon(t)\|_{4.5}^2 + \|v^\epsilon(t)\|_4^2 \le 2M_0\,.$$

Solving the Euler equations backwards-in-time from the *final* states Ω^ϵ and u_s^ϵ

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- GOAL: show T^ϵ is independent of $\epsilon > 0$

Time of existence $-T$ is independent of $\epsilon > 0$

- fundamental theorem of calculus: with $x = \theta_-^\epsilon(X)$, $y = \theta_+^\epsilon(Y)$,

$$\eta^\epsilon(x,t) - \eta^\epsilon(y,t) = x - y + \int_0^t [v^\epsilon(x,s) - v^\epsilon(y,s)] ds \,.$$

Time of existence $-T$ is independent of $\epsilon > 0$

- fundamental theorem of calculus: with $x = \theta^\epsilon_-(X)$, $y = \theta^\epsilon_+(Y)$,

$$\eta^\epsilon(x,t) - \eta^\epsilon(y,t) = x - y + \int_0^t [v^\epsilon(x,s) - v^\epsilon(y,s)]ds\,.$$

- if we do not have at the same time $x \in \theta^\epsilon_-(B^+)$ and $y \in \theta^\epsilon_+(B^+)$, then independently of $\epsilon > 0$,

$$|\eta^\epsilon(x,t) - \eta^\epsilon(y,t) - (x-y)| \le C_1|t| \sup_{[-T^\epsilon,0]} E^\epsilon(t)\,|x-y|\,,$$

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- For $x \in \theta^\epsilon_-(B^+)$ and $y \in \theta^\epsilon_+(B^+)$,

$$\begin{aligned}|\eta^\epsilon(x,t) - \eta^\epsilon(y,t)| &\ge |(\eta^\epsilon(x,t) - \eta^\epsilon(y,t)) \cdot \mathbf{e_3}| \\ &\ge \epsilon + (C_- + C_+)|t| - t^2 P_1(\sup_{[-T^\epsilon,0]} E^\epsilon)\end{aligned}$$

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- A priori estimate: on $[-T^\epsilon, 0]$,

$$\sup_{t\in[-T^\epsilon,0]} E^\epsilon(t) \leq M_0^\epsilon + tP_2(\sup_{[-T^\epsilon,0]} E^\epsilon(t))$$

where the constant $M_0^\epsilon = P(E^\epsilon(0))$, where $M_0^\epsilon < M_0$, independent of ϵ.

$-T$ is independent of $\epsilon > 0$ (Continued)

- $\theta^{\epsilon}_{-1} = \theta^{\epsilon}_{-}$ and $\theta^{\epsilon}_{0} = \theta^{\epsilon}_{+}$.
- We set
$$T = \min\left(\frac{1}{4C_1 M_0}, \frac{C_- + C_+}{2P_1(2M_0)}, \frac{M_0}{2P_2(2M_0)}\right),$$

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- We see that on $[-T, 0]$,

$$|\eta^\epsilon(x,t) - \eta^\epsilon(y,t)| \geq \frac{1}{2}|x-y|$$

$$\text{for } (x,y) \in \theta^\epsilon_l(\mathcal{B}) \times \theta^\epsilon_k(\mathcal{B}), (l,k) \notin \{(-1,0),(-1,0)\}$$

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- and
$$|\eta^\epsilon(x,t) - \eta^\epsilon(y,t)| \geq \epsilon + (C_- + C_+)\frac{|t|}{2} \quad \text{for all } (x,y) \in \theta^\epsilon_-(\mathcal{B}) \times \theta^\epsilon_+(\mathcal{B}).$$

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- thus, the domain $\eta^\epsilon(t, \Omega^\epsilon)$ does not self-intersect for each $t \in [-T, 0]$

Technical Lemma

- Smooth cut-off function: $0 \leq \zeta \in \mathcal{D}(B(0,1))$ is a smooth cut-off function on $\mathcal{B} = B^+$ or B, with $\zeta(0) = 1$ and $\operatorname{spt}(\zeta) \subset B(0, \varsigma)$
- ζ replaces a partition-of-unity

Lemma (Equivalence-of-norms lemma)

With the smooth cut-off function $0 \leq \zeta \in \mathcal{D}(B)$, there exist constants $\tilde{C}_1 > 0$ and $\tilde{C}_2 > 0$ such that for any $\epsilon > 0$ and $f \in H^s(\Omega)$ with $0 \leq s \leq 4.5$,

$$\tilde{C}_1 \sum_{l=-1}^{L} \|\zeta f \circ \theta_l^\epsilon\|_{s,\mathcal{B}}^2 \leq \|f\|_{s,\Omega^\epsilon}^2 \leq \tilde{C}_2 \sum_{l=-1}^{L} \|\zeta f \circ \theta_l^\epsilon\|_{s,\mathcal{B}}^2 . \tag{4}$$

Asymptotics as $\epsilon \to 0$ on the time-interval $[-T, 0]$

- From our a priori estimate, we have that $\|\eta^\epsilon(-T)\|^2_{4.5}$ is bounded, so that

$$\sum_{l=-1}^{L} \|\zeta\, \eta^\epsilon(-T, \theta^\epsilon_l)\|^2_{4.5,\mathcal{B}} \le \frac{2}{C} M_0 \,.$$

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- By compactness, for $l = -1, 0, 1, 2, ..., L$, there exists limits Θ_l such that

$$\eta^{\epsilon'}(-T, \cdot) \circ \theta_l^{\epsilon'} \rightharpoonup \Theta_l \,, \text{ as } \epsilon' \to 0 \,, \text{ in } H^{4.5}(\mathcal{B}_\varsigma) \,,$$

$$\eta^{\epsilon'}(-T, \cdot) \circ \theta_l^{\epsilon'} \to \Theta_l \,, \text{ as } \epsilon' \to 0 \,, \text{ in } H^{3.5}(\mathcal{B}_\varsigma) \,,$$

- The set Ω_0 is the union of the sets $\Theta_l(\mathcal{B}_\varsigma)$ $(-1 \leq l \leq L)$

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- The set Ω_0 is the union of the sets $\Theta_l(\mathcal{B}_\varsigma)$ $(-1 \leq l \leq L)$
- For $(x, y) \in \mathcal{B}_\varsigma \times \mathcal{B}_\varsigma\,, ((l, k) \notin \{(-1, 0), (0, -1)\})$

$$|\Theta_l(x) - \Theta_k(y)| \geq \frac{1}{2}|\theta_l(x) - \theta_k(y)|$$

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$$|\Theta_l(x) - \Theta_k(y)| \geq \frac{1}{2}|\theta_l(x) - \theta_k(y)|$$

- and $\forall(x, y) \in \mathcal{B}_\varsigma \times \mathcal{B}_\varsigma$

$$|\Theta_-(x) - \Theta_+(y)| \geq (C_- + C_+)\frac{T}{2}$$

Asymptotics as $\epsilon \to 0$ on the time-interval $[-T, 0]$

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$$|\Theta_l(x) - \Theta_k(y)| \geq \frac{1}{2} |\theta_l(x) - \theta_k(y)|$$

- and $\forall (x, y) \in \mathcal{B}_\varsigma \times \mathcal{B}_\varsigma$

$$|\Theta_-(x) - \Theta_+(y)| \geq (C_- + C_+) \frac{T}{2}$$

- Ω_0 is a connected, $H^{4.5}$-class domain, which is locally on one side of its boundary.

Asymptotics as $\epsilon \to 0$ (Continued)

- From our a priori estimate, we have that $\|v^\epsilon(-T)\|_4^2$ is bounded, so that

$$\sum_{l=-1}^{L} \|\zeta \; v^\epsilon(-T, \theta_l^\epsilon)\|_{4,\mathcal{B}}^2 \leq \frac{2}{C} M_0 \,.$$

Asymptotics as $\epsilon \to 0$ (Continued)

- From our a priori estimate, we have that $\|v^\epsilon(-T)\|_4^2$ is bounded, so that

$$\sum_{l=-1}^{L} \|\zeta \; v^\epsilon(-T, \theta_l^\epsilon)\|_{4,\mathcal{B}}^2 \le \frac{2}{C} M_0 \, .$$

- By compactness, for $l = -1, 0, 1, 2, ..., L$, there exists limits V_l such that

$$v^{\epsilon'}(-T, \cdot) \circ \theta_l^{\epsilon'} \rightharpoonup \; V_l \, , \text{ as } \epsilon' \to 0 \, , \text{ in } H^4(\mathcal{B}_\varsigma) \, ,$$
$$v^{\epsilon'}(-T, \cdot) \circ \theta_l^{\epsilon'} \to V_l \, , \text{ as } \epsilon' \to 0 \, , \text{ in } H^3(\mathcal{B}_\varsigma) \, ,$$

Asymptotics as $\epsilon \to 0$ (Continued)

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- We now define initial velocity u_0 on Ω_0 as follows:

$$\forall l \in \{-1, 0, 1, 2, ..., L\} \,, \; u_0(\Theta_l) = V_l \text{ on } \mathcal{B}_\varsigma \,.$$

- Check that if $\Theta_l(x) = \Theta_j(y)$, for x and y in $\mathcal{B}_\varsigma$, then $V_l(x) = V_l(y)$. YES.
- We now have good data (Ω_0, u_0) at time $t = -T$.

Asymptotic Euler equations

- It remains for us to prove that
$$u_f(t,x) = u(t-T,x)\,, \quad 0 \le t \le T$$
is indeed a solution of the free-surface Euler equations on the moving domain
$$\Omega_f(t) = \Omega(t-T)\,,$$
which evolves the initial velocity u_0 and initial domain Ω_0 onto the final data at time $t = T$ given by u_s and Ω_s.

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- This will, in turn, establish the fact that after a finite time T, the free-surface of the 3-D Euler equations develops a splash singularity.
- To this end, we first define the forward in time quantities for $0 \le t \le T$ by
$$\Omega_f^\epsilon(t) = \Omega^\epsilon(t-T)\,,$$
$$\begin{aligned}
u_f^\epsilon(\cdot,t) &= u^\epsilon(\cdot,t-T) && \text{in } \Omega_f^\epsilon(t)\,,\\
\eta_f^\epsilon(\cdot,t) &= \eta^\epsilon(\cdot,t-T)\circ\eta^\epsilon(\cdot,-T)^{-1} && \text{in } \Omega_f^\epsilon(0)\,,\\
v_f^\epsilon(\cdot,t) &= v^\epsilon(\cdot,t-T)\circ\eta^\epsilon(\cdot,-T)^{-1} && \text{in } \Omega_f^\epsilon(0)\,,\\
p_f^\epsilon(\cdot,t) &= p^\epsilon(\cdot,t-T) && \text{in } \Omega_f^\epsilon(t)\,,\\
q_f^\epsilon(\cdot,t) &= q^\epsilon(\cdot,t-T)\circ\eta^\epsilon(\cdot,-T)^{-1} && \text{in } \Omega_f^\epsilon(0)\,.
\end{aligned}$$

Asymptotic Euler equations (Continued)

- It follows that

$$\begin{aligned} \operatorname{div} u_f^\epsilon &= 0 && \text{in } \Omega_f^\epsilon(t)\,, \\ v_f^\epsilon = u_f^\epsilon \circ \eta_f^\epsilon &= \partial_t \eta_f^\epsilon && \text{in } \Omega_f^\epsilon(0)\,, \\ \eta_f^\epsilon(\cdot,0) &= e && \text{in } \Omega_f^\epsilon(0)\,. \end{aligned}$$

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- $(u_f^\epsilon, p_f^\epsilon)$ is a solution of Euler on $[0, T]$ with initial domain $\Omega_f^\epsilon(0)$ and initial velocity $u_f^\epsilon(0)$, with the domain and velocity at time $t = T$ equal to Ω^ϵ and u_s^ϵ, respectively.

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- In order to pass to limit $\epsilon \to 0$, we consider Euler in each local chart: We set

$$\tilde\theta_l^\epsilon = \eta^\epsilon(-T, \theta_l^\epsilon) \;\text{ and }\; \tilde b_l^\epsilon = [\nabla(\eta_f^\epsilon \circ \tilde\theta_l^\epsilon)]^{-1} \text{ for }\; l = -1, 0, 1, 2, ..., L\,.$$

Asymptotic Euler equations (Continued)

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- In order to pass to limit $\epsilon \to 0$, we consider Euler in each local chart: We set

$$\tilde{\theta}_l^\epsilon = \eta^\epsilon(-T, \theta_l^\epsilon) \quad \text{and} \quad \tilde{b}_l^\epsilon = [\nabla(\eta_f^\epsilon \circ \tilde{\theta}_l^\epsilon)]^{-1} \quad \text{for} \quad l = -1, 0, 1, 2, ..., L\,.$$

$$\begin{aligned} \eta_f^\epsilon \circ \tilde{\theta}_l^\epsilon &= \tilde{\theta}_l^\epsilon + \int_0^t v_f^\epsilon \circ \tilde{\theta}_l^\epsilon && \text{in } \mathcal{B}_\varsigma \times (0, T]\,, \\ \partial_t v_f^\epsilon \circ \tilde{\theta}_l^\epsilon + [\tilde{b}_l^\epsilon]^T \, \nabla(q_f^\epsilon \circ \tilde{\theta}_l^\epsilon) &= 0 && \text{in } \mathcal{B}_\varsigma \times (0, T)\,, \\ \operatorname{div}_{\eta_f^\epsilon \circ \tilde{\theta}_l^\epsilon} v_f^\epsilon \circ \tilde{\theta}_l^\epsilon &= 0 && \text{in } \mathcal{B}_\varsigma \times (0, T)\,, \\ q_f^\epsilon \circ \tilde{\theta}_l^\epsilon &= 0 && \text{on } B_0 \times (0, T)\,, \\ (\eta_f^\epsilon, v_f^\epsilon) \circ \tilde{\theta}_l^\epsilon &= (e, u_f^\epsilon(0)) \circ \tilde{\theta}_l^\epsilon && \text{on } \mathcal{B}_\varsigma \times \{t = 0\}\,, \\ \eta_f^\epsilon(T, \Omega_f^\epsilon(0)) &= \Omega^\epsilon\,. \end{aligned}$$

Asymptotic Euler equations (Continued)

- Our a priori estimate shows that for each $l = -1, 0, 1, 2, ..., L$

$$\sup_{t\in[0,T]} \left(\|\eta_f^\epsilon(t) \circ \tilde{\theta}_l^\epsilon\|^2_{4.5,\mathcal{B}_\varsigma} + \|v_f^\epsilon(t) \circ \tilde{\theta}_l^\epsilon\|^2_{4,\mathcal{B}_\varsigma} + \|q_f^\epsilon(t) \circ \tilde{\theta}_l^\epsilon\|^2_{4.5,\mathcal{B}_\varsigma} \right) \leq 2\tilde{M}_0^\epsilon \,,$$

where $\tilde{M}_0^\epsilon$ is a constant that depends on the $H^{4.5}$-norms of $\tilde{\theta}_l^\epsilon$ and the H^4-norm of $u_f^\epsilon(0)$. Our construction shows that $\tilde{M}_0^\epsilon$ is bounded by a constant which is independent of ϵ.

- As such, we have the following convergence in two weak topologies and one strong topology:

$$v_f^\epsilon \circ \tilde{\theta}_l^\epsilon \rightharpoonup v_f \circ \Theta_l \,, \text{ in } L^2(0, T; H^4(\mathcal{B}_\varsigma)) \,,$$
$$\eta_f^\epsilon \circ \tilde{\theta}_l^\epsilon \to \eta_f \circ \Theta_l \,, \text{ in } L^2(0, T; H^3(\mathcal{B}_\varsigma)) \,,$$
$$q_f^\epsilon \circ \tilde{\theta}_l^\epsilon \rightharpoonup q_f \circ \Theta_l \,, \text{ in } L^2(0, T; H^{4.5}(\mathcal{B}_\varsigma)) \,,$$

and we have our previous bound

$$\eta^\epsilon(-T, \cdot) \circ \theta_l^\epsilon \to \Theta_l \,, \text{ as } \epsilon \to 0 \,, \text{ in } H^{3.5}(\mathcal{B}_\varsigma) \,,$$

Asymptotic Euler equations (Continued)

- For $l = -1, 0, 1, 2, ..., K$, the limit as $\epsilon \to 0$ of this sequence of solutions is indeed a solution of

$$
\begin{aligned}
\eta_f \circ \Theta_l &= \Theta_l + \int_0^t v_f \circ \Theta_l && \text{in } \mathcal{B}_\varsigma \times (0, T] \,, \\
\partial_t v_f \circ \Theta_l + [\mathfrak{b}_l]^T \, \nabla(q_f \circ \Theta_l) &= 0 && \text{in } \mathcal{B}_\varsigma \times (0, T) \,, \\
\operatorname{div}_{\eta_f \circ \Theta_l} v_f \circ \Theta_l &= 0 && \text{in } \mathcal{B}_\varsigma \times (0, T) \,, \\
q_f \circ \Theta_l &= 0 && \text{on } B_0 \times (0, T) \,, \\
(\eta_f, v_f) \circ \Theta_l &= (e, u_0) \circ \Theta_l && \text{on } \mathcal{B}_\varsigma \times \{t = 0\} \,, \\
\eta_f(T, \Omega_0) &= \Omega_s \,,
\end{aligned}
$$

  where $\mathfrak{b}_l = [\nabla(\eta_f \circ \Theta_l)]^{-1}$, and where v_f, q_f and η_f are the forward in time velocity, pressure and displacement fields.

- For details see the paper:
  http://arxiv.org/abs/1201.4919
  OR
  http://www.math.ucdavis.edu/~shkoller