Almost Global Existence Of Classical Discontinuous Solutions To General Quasilinear Hyperbolic Systems Of Conservation Laws With Small BV Initial Data

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14th International Conference on Hyperbolic Problems
University of Padova, June 25-29, 2012

Outline

Consider the following quasilinear hyperbolic system of conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \qquad x \in \mathbf{R}, \quad t > 0, \tag{1}$$

where $u = (u_1, \ldots, u_n)^T$ is the unknown vector-valued function of (t, x), $f : \mathbf{R}^n \to \mathbf{R}^n$ is a given C^3 vector function of u.

The system is assumed to be strictly hyperbolic, i.e., the Jacobian $A(u) = \nabla f(u)$ has n real distinct eigenvalues:

$$\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u),$$

and each characteristic field is either genuinely nonlinear in the sense of Lax:

$$\nabla \lambda_i(u) r_i(u) \neq 0$$

or linearly degenerate in the sense of Lax:

$$\nabla \lambda_i(u) r_i(u) \equiv 0$$

for any given u on the domain under consideration.

We are interested in the generalized Riemann problem for system (1), which is a Cauchy problem with a piecewise C^1 initial data of the form:

$$t = 0 : u = \begin{cases} u_0^-(x), & x \leq 0, \\ u_0^+(x), & x \geq 0, \end{cases} \quad (2)$$

where $u_0^-(x)$ and $u_0^+(x)$ are C^1 vector functions defined for $x \leq 0$ and $x \geq 0$ respectively with

$$u_0^-(0) \neq u_0^+(0).$$

Problem (1) and (2) may be regarded as a perturbation of the corresponding Riemann problem (1) and

$$t = 0 : u = \begin{cases} \widehat{u}_-, & x \leq 0, \\ \widehat{u}_+, & x \geq 0, \end{cases} \quad (3)$$

in which

$$\widehat{u}_\pm = u_0^\pm(0).$$

Let

$$\theta = |\widehat{u}_- - \widehat{u}_+|.$$

When $\theta > 0$ is suitably small, by Lax [1957], the Riemann problem (1) and (3) admits admits a unique self-similar solution composed of $n+1$ constant states $\widehat{u}^{(0)} = \widehat{u}_-, \widehat{u}^{(1)}, \dots, \widehat{u}^{(n-1)}, \widehat{u}^{(n)} = \widehat{u}_+$ separated by shocks, centered rarefaction waves (corresponding characteristics are genuinely nonlinear) or contact discontinuities (corresponding characteristics are linearly degenerate).

- As in kong [JDE 2003], this kind of solution is simply called the Lax's Riemann solution of the system (1).

Previous Works

- For the self-similar solution of the Riemann problem of general quasilinear hyperbolic systems of conservation laws, the local nonlinear structure stability has been proved by Li and Yu [1985] for one-dimensional case, and by Majda [1983] for multidimensional case.

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- For the self-similar solution of the Riemann problem of general quasilinear hyperbolic systems of conservation laws, the local nonlinear structure stability has been proved by Li and Yu [1985] for one-dimensional case, and by Majda [1983] for multidimensional case.
- If the system (1) is strictly hyperbolic and linearly degenerate, Li and Kong [CPDE 1999] proved the global structure stability of the self-similar solution with small amplitude under perturbation (2) satisfying (3). In this case the self-similar solution contains only n contact discontinuities.

Theorem 1(T. Li, D.-X. Kong [1999])

Suppose that $u_0^-(x)$ and $u_0^+(x)$ are all C^1 vector functions on $x \leq 0$ and on $x \geq 0$ respectively, satisfying that there exists a constant $\mu > 0$ such that

$$\theta \overset{def}{=} \sup_{x\leq 0}\{(1+|x|)^{1+\mu}(|u_0^-(x)| + |{u_0^-}'(x)|)\}$$

$$+\sup_{x\geq 0}\{(1+|x|)^{1+\mu}(|u_0^+(x)| + |{u_0^+}'(x)|)\} < +\infty,$$

and under the assumptions mentioned above. Then there exists $\theta_0 > 0$ so small that for any given $\theta \in (0, \theta_0]$, the generalized Riemann problem (1) and (2) admits a unique global piecewise C^1 solution $u = u(t,x)$ containing only n contact discontinuities with small amplitude $x = x_i(t)$ $(i = 1, \ldots, n)$ on $t \geq 0$. This solution has a global structure similar to that of the self-similar solution $u = U(\frac{x}{t})$ of the corresponding Riemann problem (1) and (3).

Precisely speaking,

$$u = u(t,x) = \begin{cases} u^{(0)}(t,x), x \le x_1(t), \\ u^{(i)}(t,x),\ x_i(t) \le x \le x_{i+1}(t)(i = 1, \dots, n-1), \\ u^{(n)}(t,x), x \ge x_n(t), \end{cases}$$

where $u^{(i)}(t,x)$ $(i = 0, 1, \dots, n)$ are all C^1 solutions to system (1) on each corresponding domains respectively and for $i = 1, \dots, n$, $u^{(i-1)}(t,x)$ and $u^{(i)}(t,x)$ are connected to each other by the i-th contact discontinuity $x = x_i(t)$. Moreover,

$$x_i'(0) = \widehat{\lambda}_i \quad (i = 1, \dots, n)$$

and

$$u^{(i)}(0,0) = \widehat{u}^{(i)} \quad (i = 0, 1, \dots, n).$$

Remark Recently, under the assumption that the Lax's Riemann solution of the system (1) only contains non-degenerate shocks and contact discontinuities but no centered rarefaction waves and other weak discontinuities, Kong [JDE 2003, JDE 2005] proved the global structure stability of this kind of Lax's Riemann solution with small amplitude.

However, it is well known that the BV space is a suitable framework for one-dimensional Cauchy problem for the hyperbolic systems of conservation laws (see Bressan [Oxford University Press, 2000]), the result in Bressan [Indiana Univ. Math. J., 1988] suggests that one may achieve global smoothness even if the C^1 norm of the initial data is large.

Problem: can we obtain the global existence and uniqueness of piecewise C^1 solution containing only shocks and contact discontinuities to a class of the generalized Riemann problem, which can be regarded as a small BV perturbation of the corresponding Riemann problem, for system (1) with the following piecewise C^1 initial data:

$$t = 0 : u = \begin{cases} \hat{u}_- + u_-(x), & x \leq 0, \\ \hat{u}_+ + u_+(x), & x \geq 0, \end{cases} \tag{4}$$

where $u_\pm(x) \in C^1$ with both bounded C^1 norm and small bounded variation?

- It is important to mention that the global existence of weak solutions to a strictly hyperbolic system of conservation laws in one space dimension when the initial data is a small BV perturbation of a solvable Riemann problem has been proved by Schochet [JDE 1991], unfortunately his method is not useful to show that the solutions are still either contact discontinuities or shocks.

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- An analogous result on stability of contact discontinuities (resp. a strong shock wave) under perturbations of small bounded variation is stated by Corli and Sable-Tougeron [Rend. Sem. Mat. Univ. Padova, 1997](resp. [JDE 1997]).

Aim of this work

In this paper we exploit to some extent the ideas of Bressan [Indiana Univ. Math. J., 1988], we will develop the method of using continuous Glimm's functional to solve this problem globally and to provide a new, concise proof of an estimate on the lifespan of the piecewise C^1 solution to the generalized Riemann problem under consideration mentioned above.

The basic idea here is to combine the techniques employed by Li-Kong [CPDE 1999], especially both the decomposition of waves and the global behavior of waves on the discontinuity curves, with the method of using continuous Glimm's functional. However, we must modify Glimm's functional in order to take care of the presence of contact discontinuities or shocks. This makes our new analysis more complicated than those for the C^1 solutions of the Cauchy problem for linearly degenerate quasilinear hyperbolic systems in Bressan [Indiana Univ. Math. J., 1988], Zhou [Chin. Ann. Math. Ser. B, 2004], Dai and Kong [JDE 2007].

To do so, i.e., To get a lower bound of the lifespan of the piecewise C^1 solution to the generalized Riemann problem, we consider the generalized Riemann problem for system (1) with the following piecewise C^1 initial data:

$$t = 0 : u = \begin{cases} \widehat{u}_- + \varepsilon u_-(x), & x \le 0, \\ \widehat{u}_+ + \varepsilon u_+(x), & x \ge 0, \end{cases} \tag{5}$$

where ε $(0 < \varepsilon \ll | \widehat{u}_+ - \widehat{u}_- |)$ is a small parameter, $u_-(x)$ and $u_+(x)$ are C^1 vector functions defined on $x \le 0$ and $x \ge 0$ respectively, which satisfy

$$u_-(0) = u_+(0) = 0, \tag{6}$$

$$||u_-(x)||_{C^1},\ ||u_+(x)||_{C^1} \le K_1 \tag{7}$$

and

$$\int_0^{+\infty} |u_+'(x)|dx,\ \int_{-\infty}^0 |u_-'(x)|dx \le K_2, \tag{8}$$

where K_1 and K_2 are positive constants independent of ε.

Theorem 2(Shao [JMAA 2012])

Suppose that $u_-(x)$ and $u_+(x)$ are all C^1 vector functions on $x \leq 0$ and on $x \geq 0$ respectively satisfying (6)-(8). Suppose furthermore that the self-similar solution $u = U(\frac{x}{t})$ of the Riemann problem (1) and (3) consists of k shock waves and $n-k$ contact discontinuities for some integer k $(1 \leq k \leq n)$. Suppose finally that

$$\theta = |\widehat{u}_+ - \widehat{u}_-| = |u_0^+(0) - u_0^-(0)| > 0$$

is suitably small, and under the assumptions mentioned above. Then for small $\theta > 0$, there exists a constant $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, the lifespan $\widetilde{T}(\varepsilon)$ of the piecewise C^1 solution to the generalized Riemann problem (1) and (5) satisfies

$$\widetilde{T}(\varepsilon) \geq K_3 \varepsilon^{-1},$$

where K_3 is a positive constant independent of ε. Moreover, when $u = u(t, x)$ blows up in a finite time, $u = u(t, x)$ itself is bounded on the domain $[0, \widetilde{T}(\varepsilon)) \times \mathbf{R}$, while the first-order derivatives of $u = u(t, x)$ tend to be unbounded as $t \nearrow \widetilde{T}(\varepsilon)$.

Remark Our result implies that classical discontinuous solutions to the generalized Riemann problem under consideration exist almost globally in time.

Remark Suppose that (1) is a non-strictly hyperbolic system with characteristics with constant multiplicity, say, on the domain under consideration,

$$\lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u) \quad (1 \leq p \leq n).$$

Then the conclusion of Theorem still holds.

Known results(Partial list)

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- Liu and Zumbrun investigated the nonlinear stability of an undercompressive shock for complex Burgers equation.

Let

$$v_i = l_i(u)u \quad (i = 1, \dots, n) \tag{9}$$

and

$$w_i = l_i(u)u_x \quad (i = 1, \dots, n), \tag{10}$$

where $l_i(u) = (l_{i1}(u), \dots, l_{in}(u))$ denotes the i-th left eigenvector.

It is easy to see that

$$u = \sum_{k=1}^{n} v_k r_k(u)$$

and

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{11}$$

Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x}$$

be the directional derivative along the i-th characteristic. We have

$$\frac{dv_i}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k \quad (i = 1, \dots, n),$$

where

$$\beta_{ijk}(u) = (\lambda_k(u) - \lambda_i(u)) l_i(u) \nabla r_j(u) r_k(u).$$

Hence, we have

$$\beta_{iji}(u) \equiv 0, \quad \forall\ i, j.$$

On the other hand, we have (cf. [John, CPAM 1974])

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k \quad (i = 1, \dots, n), \tag{12}$$

where

$$\gamma_{ijk}(u) = \frac{1}{2}\{(\lambda_j(u) - \lambda_k(u)) l_i(u) \nabla r_k(u) r_j(u) \\ - \nabla \lambda_i(u) r_j(u) \delta_{ik} + (j|k)\},$$

in which $(j|k)$ denotes all the terms obtained by changing j and k in the previous terms. We have

$$\gamma_{ijj}(u) \equiv 0, \quad \forall\, j \neq i \qquad (i, j = 1, \ldots, n)$$

and

$$\gamma_{iii}(u) \equiv -\nabla\lambda_i(u) r_i(u) \quad (i = 1, \ldots, n).$$

Moreover, noting (11), by (12) we have

$$\frac{\partial w_i}{\partial t} + \frac{\partial(\lambda_i(u) w_i)}{\partial x} = \sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k \overset{def}{=} G_i(t, x),$$

equivalently,

$$d[w_i(dx - \lambda_i(u)dt)] = \sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k dt \wedge dx,$$

where

$$\Gamma_{ijk}(u) = \frac{1}{2}(\lambda_j(u) - \lambda_k(u))l_i(u)[\nabla r_k(u)r_j(u) - \nabla r_j(u)r_k(u)]. \quad (13)$$

Hence, we have

$$\Gamma_{ijj}(u) \equiv 0, \qquad \forall\ i, j.$$

Lemma 1(Generalized Hörmander Lemma)

Suppose that $u = u(t,x)$ is a piecewise C^1 solution to system (1), τ_1 and τ_2 are two C^1 arcs which are never tangent to the i-th characteristic direction, and $\mathcal{D}$ is the domain bounded by τ_1, τ_2 and two i-th characteristic curves L_i^- and L_i^+. Suppose furthermore that the domain $\mathcal{D}$ contains m C^1 curves of discontinuity of u, denoted by $\widehat{C}_j : x = x_j(t)(j = 1, \ldots, m)$, which are never tangent to the i-th characteristic direction. Then we have

$$\int_{\tau_1} |w_i(dx - \lambda_i(u)dt)| \le \int_{\tau_2} |w_i(dx - \lambda_i(u)dt)|$$

$$+\sum_{j=1}^{m} \int_{\widehat{C}_j} |[w_i]dx - [w_i\lambda_i(u)]dt| + \int\int_{\mathcal{D}} |\sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k| dtdx,$$

where $\Gamma_{ijk}(u)$ is given by (13) and $[w_i] = w_i^+ - w_i^-$ denotes the jump of w_i over the curve of discontinuity $\widehat{C}_j(j = 1, \ldots, m)$, etc.

The proof can be found in Li and Kong [CPDE 1999].

Definition 1

A piecewise C^1 vector function $u = u(t,x)$ is called a piecewise C^1 solution containing a k-th shock $x = x_k(t)(x_k(0) = 0)$ for system (1), if $u = u(t,x)$ satisfies system (1) away from $x = x_k(t)$ in the classical sense and satisfies on $x = x_k(t)$ the following Rankine-Hugoniot condition:

$$f(u^+) - f(u^-) = s(u^+ - u^-)$$

and the Lax entropy condition:

$$\lambda_k(u^+) < s < \lambda_k(u^-), \quad \lambda_{k+1}(u^+) > s > \lambda_{k-1}(u^-), \tag{14}$$

where $u^\pm = u^\pm(t, x_k(t)) \stackrel{\triangle}{=} u(t, x_k(t) \pm 0)$ and $s = \frac{dx_k(t)}{dt}$ (when $k = 1$ (resp. k=n), the term $\lambda_{k-1}(u^-)$ (resp. $\lambda_{k+1}(u^+)$) disappears in (14)).

Definition 2

A piecewise C^1 vector function $u = u(t, x)$ is called a piecewise C^1 solution containing a k-th contact discontinuity $x = x_k(t)$ $(x_k(0) = 0)$ for system (1), if $u = u(t, x)$ satisfies system (1) away from $x = x_k(t)$ in the classical sense and satisfies on $x = x_k(t)$ the Rankine-Hugoniot condition:

$$f(u^+) - f(u^-) = s(u^+ - u^-),$$

and

$$s = \lambda_k(u^+) = \lambda_k(u^-),$$

where $u^\pm = u^\pm(t, x_k(t)) \overset{\triangle}{=} u(t, x_k(t) \pm 0)$ and $s = \frac{dx_k(t)}{dt}$.

Lemma 2

Suppose that $|u^{\pm}|$ $(u^{\pm} = u(t, x_k(t) \pm 0))$ are suitably small. Then, on the k-th contact discontinuity $x = x_k(t)$ we have

$$v_i^+ = v_i^- + O(|v^{\pm}|^2) \quad (i = 1, \dots, k-1, k+1, \dots, n)$$

and

$$w_i^+ = w_i^- + O\left(|u^+ - u^-| \cdot \sum_{j \neq k} |w_j^{\pm}|\right)$$

$$(i = 1, \dots, k-1, k+1, \dots, n),$$

where $v = (v_1, \dots, v_n)^T$ is defined by (9) and $v^{\pm} \triangleq v(t, x_k(t) \pm 0), etc.$

The proof can be found in Li and Kong [CPDE 1999].

Corollary 1

On the k-th contact discontinuity $x = x_k(t)$, it holds that

$$(w_i\lambda_i(u))^+ = (w_i\lambda_i(u))^- + O\left(|u^+ - u^-| \cdot \sum_{j\neq k} |w_j^{\pm}|\right)$$

$$(i = 1, \ldots, k-1, k+1, \ldots, n),$$

provided that $|u^{\pm}|$ is small.

For the sake of simplicity and without loss of generality, we may suppose that

$$0 < \lambda_1(0) < \lambda_2(0) < \ldots < \lambda_n(0) \tag{15}$$

and

$$|\widehat{u}_\pm| \leq \theta.$$

By the existence and uniqueness of local classical discontinuous solutions of quasilinear hyperbolic systems of conservation laws, when $\theta > 0$ is suitably small, the generalized Riemann problem (1) and (5) admits a unique piecewise C^1 solution $u = u(t,x)$ containing only shocks and (or) contact discontinuities (denoted by $x = x_i(t)\,(i = 1, \ldots, n)$) on the strip $[0, h] \times \mathbf{R}$, where $h > 0$ is a small number; moreover, this solution has a local structure similar to the one of the self-similar solution to the corresponding Riemann problem.

In order to prove Theorem 2, we first establish some uniform a priori estimates on u and u_x on the domain of existence of the piecewise C^1 solution $u = u(t, x)$.

By (15), there exist sufficiently small positive constants δ and δ_0 such that

$$\lambda_{i+1}(u) - \lambda_i(v) \geq \delta_0, \quad \forall\ |u|,\ |v| \leq \delta\ \ (i = 1, \dots, n-1). \tag{16}$$

For the time being it is supposed that on the domain of existence of the piecewise C^1 solution $u = u(t, x)$ to the generalized Riemann problem (1) and (5), we have

$$|u(t, x)| \leq \delta. \tag{17}$$

At the end of the proof of Lemma 7, we will explain that this hypothesis is reasonable.

For any fixed $T > 0$, let

$$U_\infty(T) = \sup_{0\le t\le T}\sup_{x\in\mathbf{R}} |u(t,x)|, \tag{18}$$

$$V_\infty(T) = \sup_{0\le t\le T}\sup_{x\in\mathbf{R}} |v(t,x)|, \tag{19}$$

$$W_\infty(T) = \sup_{0\le t\le T}\sup_{x\in\mathbf{R}} |w(t,x)|, \tag{20}$$

$$\widetilde{W}_1(T) = \max_{i=1,\dots,n}\max_{j\neq i}\sup_{\widetilde{C}_j}\int_{\widetilde{C}_j} |w_i(t,x)|dt,$$

$$W_1(T) = \max_{j\in J_S}\int_0^T |(x_j'(t) - \lambda_j(u(t,x_j(t)\pm 0)))w_j(t,x_j(t)\pm 0)|dt,$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbf{R}^n$, $v = (v_1, \ldots, v_n)^T$ and $w = (w_1, \ldots, w_n)^T$ in which v_i and w_i are defined by (9) and (10) respectively, while $\widetilde{C}_j$ stands for any given j-th characteristic on the domain $[0, T] \times \mathbf{R}$. In (17)-(20), on any contact discontinuity or shock $x = x_k(t)$ the values of $u(t,x)$, $v(t,x)$ and $w(t,x)$ are taken to be $u^{\pm}(t,x) = u(t, x_k(t) \pm 0)$, $v^{\pm}(t,x) = v(t, x_k(t) \pm 0)$ and $w^{\pm}(t,x) = w(t, x_k(t) \pm 0)$. Clearly, $V_\infty(T)$ is equivalent to $U_\infty(T)$.

First we recall some basic L^1 estimates. They are essentially due to Schatzman [Indiana Univ. Math. J., 34, 1985; Lectures in Applied Mathematics, Vol. 23, 1986] and Zhou [Chin. Ann. Math. Ser. B, 25, 2004].

Lemma 3.

Let $\phi = \phi(t,x) \in C^1$ satisfy

$$\phi_t + (\lambda(t,x)\phi)_x = F(t,x),\ 0 \le t \le T, x \in R,$$

$$\phi(0,x) = g(x),$$

where $\lambda \in C^1$. Then

$$\int_{-\infty}^{+\infty} |\phi(t,x)| dx \le \int_{-\infty}^{+\infty} |g(x)| dx$$

$$+ \int_0^T \int_{-\infty}^{+\infty} |F(t,x)| dxdt,\ \ \forall\ t \le T, \tag{21}$$

provided that the right-hand side of the inequality is bounded.

Lemma 4

Let $\phi = \phi(t,x)$ and $\psi = \psi(t,x)$ be C^1 functions satisfying

$$\phi_t + (\lambda(t,x)\phi)_x = F_1(t,x),\ 0 \le t \le T, x \in R,$$

$$\phi(0,x) = g_1(x)$$

and

$$\psi_t + (\mu(t,x)\psi)_x = F_2(t,x),\ 0 \le t \le T, x \in R,$$

$$\psi(0,x) = g_2(x),$$

respectively, where $\lambda, \mu \in C^1$ such that there exists a positive constants δ_0 independent of T verifying

$$\mu(t,x) - \lambda(t,x) \ge \delta_0,\ 0 \le t \le T, x \in R.$$

Then

$$\int_0^T \int_{-\infty}^{+\infty} |\phi(t,x)||\psi(t,x)| dxdt \leq C\Big(\int_{-\infty}^{+\infty} |g_1(x)| dx + \int_0^T \int_{-\infty}^{+\infty} |F_1(t,x)| dxdt \Big) \times \Big(\int_{-\infty}^{+\infty} |g_2(x)| dx + \int_0^T \int_{-\infty}^{+\infty} |F_2(t,x)| dxdt \Big), \tag{22}$$

provided that the two factors on the right-hand side of the inequality is bounded.

In the present situation, similar to the above basic L^1 estimates (21)-(22), we have

Lemma 5. Under the assumptions of Theorem 2, on any given domain of existence $[0, T] \times \mathbf{R}$ of the piecewise C^1 solution $u = u(t, x)$ to the generalized Riemann problem (1) and (5), there exists a positive constant k_1 independent of ε, T and M such that

$$\int_{-\infty}^{+\infty} |w_i(t,x)|dx \le k_1 \Big\{ \varepsilon + W_1(T) + V_\infty(T)(\widetilde{W}_1(T) + W_1(T)) + \int_0^T \int_{-\infty}^{+\infty} |G_i(t,x)|dxdt \Big\}, \quad \forall\ t \le T, \tag{23}$$

provided that the right-hand side of the inequality is bounded.

Lemma 6

Under the assumptions of Theorem 2, on any given domain of existence $[0, T] \times \mathbf{R}$ of the piecewise C^1 solution $u = u(t, x)$ to the generalized Riemann problem (1) and (5), there exists a positive constant k_2 independent of ε, T and M such that

$$\int_0^T \int_{-\infty}^{+\infty} |w_i(t,x)||w_j(t,x)|dxdt$$

$$\leq k_2\Big(\varepsilon+W_1(T)+V_\infty(T)(\widetilde{W}_1(T)+W_1(T))+\int_0^T \int_{-\infty}^{+\infty} |G_i(t,x)|dxdt\Big)$$

$$\times\Big(\varepsilon+W_1(T)+V_\infty(T)(\widetilde{W}_1(T)+W_1(T))+\int_0^T \int_{-\infty}^{+\infty} |G_j(t,x)|dxdt\Big),$$

$$\forall\ i \neq j \quad (i,j=1,\ldots,n), \tag{24}$$

provided that the right-hand side of the inequality is bounded.

Proof. To estimate

$$\int_0^T \int_{-\infty}^{+\infty} |w_i(t,x)||w_j(t,x)|dxdt,$$

it is enough to estimate

$$\int_0^T \int_{-L}^{L} |w_i(t,x)||w_j(t,x)|dxdt$$

for any given $L > 0$ and then let $L \to +\infty$.

For $i, j \in \{1, \dots, n\}$ and $i \neq j$, without loss of generality, we suppose that $i < j$. Let $x = x_i(t, L)$ $(0 \leq t \leq T)$ be the ith forward characteristic passing through point $(0, L)\,(L > x_n(T))$. Then, we draw the ith backward characteristic $x = s_i(t)$ $(0 \leq t \leq T)$ passing through point $(T, a)\,(a > x_i(T, L))$. In the meantime, passing through the point $(T, -L)$, we draw the jth characteristic $x = s_j(t)$ $(0 \leq t \leq T)$ which intersects the x-axis at a point.

We introduce the "continuous Glimm's functional"

$$Q(t) = \int\int_{s_j(t)<x<y<s_i(t)} |w_j(t,x)||w_i(t,y)|dxdy.$$

Because of the piecewise C^1 solution $u = u(t,x)$ containing only n contact discontinuities $x = x_k(t)\,(x_k(0) = 0)\,(k = 1,\dots,n)$, we divide the bounded domain $\widetilde{\Omega} \triangleq \{(x,y)|s_j(t) < x < y < s_i(t)\}$ by the straight lines $y = x_k(t)\,(k = 1,\dots,n)$ into some parts. Then, the straightforward calculations on all parts of the domain $\widetilde{\Omega}$ reveal that

$$\frac{dQ(t)}{dt} = s_i'(t)|w_i(t,s_i(t))|\int_{s_j(t)}^{s_i(t)} |w_j(t,x)|dx$$

$$-s_j'(t)|w_j(t,s_j(t))|\int_{s_j(t)}^{s_i(t)} |w_i(t,x)|dx$$

$$+\sum_{k=1}^{n} x_k'(t)\{|w_i(t,x_k(t)-0)|-|w_i(t,x_k(t)+0)|\}$$

$$\times \int_{s_j(t)}^{x_k(t)} |w_j(t,x)|dx$$

$$+\int\int_{s_j(t)<x<y<s_i(t)} \frac{\partial}{\partial t}(|w_j(t,x)|)|w_i(t,y)|dxdy$$

$$+\int\int_{s_j(t)<x<y<s_i(t)} |w_j(t,x)|\frac{\partial}{\partial t}(|w_i(t,y)|)dxdy$$

$$=-\int_{s_j(t)}^{s_i(t)} (\lambda_j(u(t,x))-\lambda_i(u(t,x)))|w_i(t,x)||w_j(t,x)|dx$$

$$+(s_i'(t)-\lambda_i(u(t,s_i(t))))|w_i(t,s_i(t))|\int_{s_j(t)}^{s_i(t)} |w_j(t,x)|dx$$

$$+(\lambda_j(u(t,s_j(t))) - s_j'(t))|w_j(t,s_j(t))| \int_{s_j(t)}^{s_i(t)} |w_i(t,x)|dx$$

$$+(x_i'(t) - \lambda_i(u(t,x_i(t)-0)))|w_i(t,x_i(t)-0)|$$

$$\times \int_{s_j(t)}^{x_i(t)} |w_j(t,x)|dx$$

$$+(\lambda_i(u(t,x_i(t)+0)) - x_i'(t))|w_i(t,x_i(t)+0)|$$

$$\times \int_{s_j(t)}^{x_i(t)} |w_j(t,x)|dx$$

$$+ \sum_{k=1,k\neq i}^{n} x_k'(t)\{|w_i(t,x_k(t)-0)| - |w_i(t,x_k(t)+0)|\}$$

$$\times \int_{s_j(t)}^{x_k(t)} |w_j(t,x)|dx$$

$$
\begin{aligned}
&+\sum_{k=1,k\neq i}^{n}\{\lambda_i(u(t,x_k(t)+0))|w_i(t,x_k(t)+0)| \\
&-\lambda_i(u(t,x_k(t)-0))|w_i(t,x_k(t)-0)|\} \\
&\times\int_{s_j(t)}^{x_k(t)}|w_j(t,x)|dx \\
&+\int\int_{s_j(t)<x<y<s_i(t)} sgn(w_j)G_j(t,x)|w_i(t,y)|dxdy \\
&+\int\int_{s_j(t)<x<y<s_i(t)}|w_j(t,x)|\ sgn(w_i)G_i(t,y)dxdy. \qquad (21)
\end{aligned}
$$

Noting (10)-(11) and using (12), we get from (21) that

$$\frac{dQ(t)}{dt} \leq -\delta_0 \int_{s_j(t)}^{s_i(t)} |w_i(t,x)||w_j(t,x)|dx$$

$$+\sum_{k\neq i} x_k'(t)\{|w_i(t,x_k(t)-0)$$

$$-w_i(t,x_k(t)+0)|\} \times \int_{s_j(t)}^{x_k(t)} |w_j(t,x)|dx$$

$$+\sum_{k\neq i}\{|\lambda_i(u(t,x_k(t)+0))w_i(t,x_k(t)+0)$$

$$-\lambda_i(u(t,x_k(t)-0))w_i(t,x_k(t)-0)|\}$$

$$\times \int_{s_j(t)}^{x_k(t)} |w_j(t,x)|dx$$

$$+\int_{s_j(t)}^{s_i(t)} |G_j(t,x)|dx \int_{s_j(t)}^{s_i(t)} |w_i(t,x)|dx$$

$$
\begin{aligned}
&\leq -\delta_0 \int_{s_j(t)}^{s_i(t)} |w_i(t,x)||w_j(t,x)|dx \\
&+ \sum_{k\neq i} x_k'(t)\{|w_i(t,x_k(t)-0) \\
&-w_i(t,x_k(t)+0)|\} \times \int_{-\infty}^{+\infty} |w_j(t,x)|dx \\
&+ \sum_{k\neq i}\{|\lambda_i(u(t,x_k(t)+0))w_i(t,x_k(t)+0) \\
&-\lambda_i(u(t,x_k(t)-0))w_i(t,x_k(t)-0)|\} \times \int_{-\infty}^{+\infty} |w_j(t,x)|dx \\
&+ \int_{-\infty}^{+\infty} |G_j(t,x)|dx \int_{-\infty}^{+\infty} |w_i(t,x)|dx \\
&+ \int^{+\infty} |G_i(t,x)|dx \int^{+\infty} |w_j(t,x)|dx.
\end{aligned}
$$

It then follows from Lemma 5 that

$$\frac{dQ(t)}{dt} + \delta_0 \int_{s_j(t)}^{s_i(t)} |w_i(t,x)||w_j(t,x)|dx$$

$$\leq k_1 \int_{-\infty}^{+\infty} |G_j(t,x)|dx \Big(\int_{-\infty}^{0} |u'_-(x)|dx$$

$$+ \int_{0}^{+\infty} |u'_+(x)|dx + V_\infty(T)\widetilde{W}_1(T) +$$

$$\int_0^T \int_{-\infty}^{+\infty} |G_i(t,x)|dxdt \Big)$$

$$+k_1 \Big(\sum_{k \neq i} x'_k(t) \{ |w_i(t, x_k(t)-0) - w_i(t, x_k(t)+0)| \}$$

$$+ \sum_{k \neq i} \{ |\lambda_i(u(t, x_k(t)+0)) w_i(t, x_k(t)+0)$$

$$-\lambda_i(u(t,x_k(t)-0))w_i(t,x_k(t)-0)|\}$$

$$+\int_{-\infty}^{+\infty}|G_i(t,x)|dx\Big)$$

$$\times\Big(\int_{-\infty}^{0}|u'_-(x)|dx+\int_{0}^{+\infty}|u'_+(x)|dx+V_\infty(T)\widetilde{W}_1(T)$$

$$+\int_0^T\int_{-\infty}^{+\infty}|G_j(t,x)|dxdt\Big).$$

Therefore

$$\delta_0\int_0^T\int_{s_j(t)}^{s_i(t)}|w_i(t,x)||w_j(t,x)|dxdt$$

$$\leq Q(0)+k_1\int_0^T\int_{-\infty}^{+\infty}|G_j(t,x)|dxdt$$

$$\times\Big(\int_{-\infty}^{0}|u'_-(x)|dx+\int_{0}^{+\infty}|u'_+(x)|dx+V_\infty(T)\widetilde{W}_1(T)$$

$$+\int_0^T\int_{-\infty}^{+\infty}|G_i(t,x)|dxdt\Big)$$

$$+k_1\Big(\sum_{k\neq i}\int_{\widehat{C}_k}|[w_i]|\lambda_k(u^\pm)dt$$

$$+\sum_{k\neq i}\int_{\widehat{C}_k}|[w_i\lambda_i(u)]|dt+\int_0^T\int_{-\infty}^{+\infty}|G_i(t,x)|dxdt\Big)$$

$$\times\Big(\int_{-\infty}^{0}|u'_-(x)|dx+\int_{0}^{+\infty}|u'_+(x)|dx$$

$$+V_\infty(T)\widetilde{W}_1(T)+\int_0^T\int_{-\infty}^{+\infty}|G_j(t,x)|dxdt\Big).$$

Using Lemma 2 and Corollary 1 and noting (13), we obtain

$$\begin{aligned}
&\delta_0 \int_0^T \int_{s_j(t)}^{s_i(t)} |w_i(t,x)||w_j(t,x)| dxdt \\
\leq\ & Q(0) + c_3 \Big(\int_{-\infty}^0 |u'_-(x)| dx + \int_0^{+\infty} |u'_+(x)| dx \\
&+ V_\infty(T) \widetilde{W}_1(T) + \int_0^T \int_{-\infty}^{+\infty} |G_i(t,x)| dxdt \Big) \\
&\times \Big(\int_{-\infty}^0 |u'_-(x)| dx + \int_0^{+\infty} |u'_+(x)| dx \\
&+ V_\infty(T) \widetilde{W}_1(T) + \int_0^T \int_{-\infty}^{+\infty} |G_j(t,x)| dxdt \Big).
\end{aligned}$$

Noting

$$Q(0) \leq \int_{-\infty}^{+\infty} |w_i(0,x)| dx \int_{-\infty}^{+\infty} |w_j(0,x)| dx,$$

we get

$$
\begin{aligned}
&\delta_0 \int_0^T \int_{s_j(t)}^{s_i(t)} |w_i(t,x)||w_j(t,x)| dxdt \\
&\leq c\Big(\int_{-\infty}^0 |u'_-(x)| dx + \int_0^{+\infty} |u'_+(x)| dx \\
&+ V_\infty(T)\widetilde{W}_1(T) + \int_0^T \int_{-\infty}^{+\infty} |G_i(t,x)| dxdt \Big) \\
&\times \Big(\int_{-\infty}^0 |u'_-(x)| dx + \int_0^{+\infty} |u'_+(x)| dx \\
&+ V_\infty(T)\widetilde{W}_1(T) + \int_0^T \int_{-\infty}^{+\infty} |G_j(t,x)| dxdt \Big).
\end{aligned}
$$

It then follows

$$\int_0^T \int_{-L}^{L} |w_i(t,x)||w_j(t,x)|dxdt$$

$$\leq \frac{c}{\delta_0}\Big(\int_{-\infty}^{0} |u'_-(x)|dx + \int_0^{+\infty} |u'_+(x)|dx$$

$$+ V_\infty(T)\widetilde{W}_1(T) + \int_0^T \int_{-\infty}^{+\infty} |G_i(t,x)|dxdt\Big)$$

$$\times \Big(\int_{-\infty}^{0} |u'_-(x)|dx + \int_0^{+\infty} |u'_+(x)|dx$$

$$+ V_\infty(T)\widetilde{W}_1(T) + \int_0^T \int_{-\infty}^{+\infty} |G_j(t,x)|dxdt\Big)$$

and the desired conclusion follows by taking $L \to +\infty$. The proof of Lemma 6 is finished.

Lemma 7

Under the assumptions of Theorem 2, for small $\theta > 0$ there exists a constant $\varepsilon > 0$ so small that on any given domain of existence $[0, T] \times \mathbf{R}$ of the piecewise C^1 solution $u = u(t, x)$ to the generalized Riemann problem (1) and (5), there exist positive constants k_i $(i = 3, \cdots, 7)$ independent of θ, ε and T, such that the following uniform a priori estimates hold:

$$W_1(T) \le k_3\varepsilon, \tag{25}$$

$$\widetilde{W}_1(T) \le k_4\varepsilon, \tag{26}$$

$$U_\infty(T), V_\infty(T) \le k_5\theta \tag{27}$$

and

$$W_\infty(T) \le k_6\varepsilon, \tag{28}$$

where T satisfies

$$T\varepsilon \le k_7. \tag{29}$$

Proof of Theorem 2

Under the assumptions of Theorem 2, from (27)-(28), we know that for small $\theta > 0$ there exists $\varepsilon > 0$ suitably small such that the generalized Riemann problem (1) and (5) admits a unique piecewise C^1 solution $u = u(t, x)$ containing shocks and contact discontinuities on the strip $[0, T] \times \mathbf{R}$, where T satisfies (29). Therefore, the lifespan $\widetilde{T}(\varepsilon)$ of the piecewise C^1 solution satisfies

$$\widetilde{T}(\varepsilon) \geq K_3 \varepsilon^{-1}, \tag{30}$$

where $K_3 (= k_7)$ is a positive constant independent of ε. Moreover, by Lemma 7, when the piecewise C^1 solution $u = u(t, x)$ blows up in a finite time, $u = u(t, x)$ itself must be bounded on the domain $[0, \widetilde{T}(\varepsilon)) \times \mathbf{R}$. Hence, the first-order derivative u_x of $u = u(t, x)$ should tend to be unbounded as $t \nearrow \widetilde{T}(\varepsilon)$. The proof of Theorem 2 is finished.

1-Dimensional compressible Euler equations in Eulerian coordinates

System of traffic flow on a road network using the Aw-Rascle model

It is well-known that these systems are strictly hyperbolic and that their fields are either genuinely nonlinear or linearly degenerate, so the theorems are obviously applicable.

Thank you very much!

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