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Weak singularities in the Multi-dimensional Riemann Problem

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HYP 2012

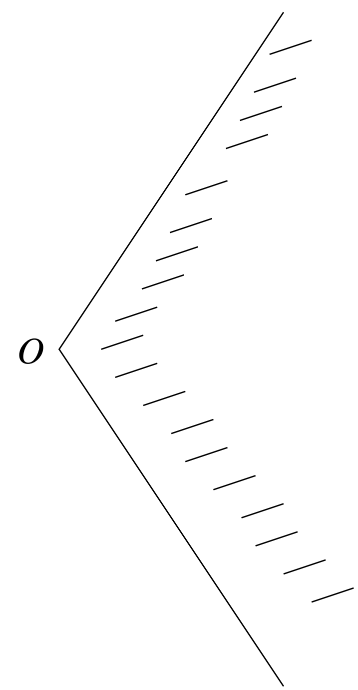
Padova, 25–29 june 2012

Paradigm : 2D-Shock reflexion in Gas dynamics

$$\partial_t \rho + \mathrm{div}(\rho v) = 0, \qquad \partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) + \nabla p(\rho) = 0.$$

Suppose that the domain is the complement of a wedge

$$x_1 \tan\alpha < |x_2|.$$



A fluid at rest ($v \equiv 0$, $\rho \equiv \rho_0$) is invaded by a planar shock. The upstream state is uniform : $v \equiv q_1 \vec{e}_1$, $\rho \equiv \rho_1$. For $t < 0$, the shock front is at $x_1 = Vt$. At $t_0 = 0$, it hits the tip of the wedge

... and everything is self-similar :

- the domain,
- the initial state $(\rho_0, v \equiv 0)$,
- the differential equations.

Hence one expects a self-similar solution

$$\rho = \rho(x/t), \qquad v = v(x/t).$$

This is a kind of

Multi-dimensional Riemann Problem

General framework for MdRP : hyperbolic systems of 1st-order conservation laws

$$\partial_t u + \sum_{\alpha=1}^{d} \partial_\alpha f^\alpha(u) = 0.$$

Self-similar solutions $u = u(x/t)$ satisfy

$$\sum_{\alpha=1}^{d} \partial_\alpha f^\alpha(u) = (x \cdot \nabla) u, \tag{1}$$

where x stands now for x/t (think to the trace of the solution at time $t = 1$).

Warning : System (1) is not everywhere hyperbolic. Its type depends upon both u and x.

Data for the Riemann Problem

The data $u(x,0) = a(x)$ of the Cauchy problem becomes a data at infinity :

$$\lim_{\mu \to +\infty} u(\mu x) = a(x).$$

Typically, a is piecewise constant.

For instance in 1D,

$$a(x) = \begin{cases} u_\ell, & x < 0, \\ u_r, & x > 0. \end{cases}$$

imposes

$$u(-\infty) = u_\ell, \qquad u(+\infty) = u_r.$$

Singularities in Riemann Problems

One expects

- Discontinuities across hypersurfaces (shocks, contact discontinuities),
- Discontinuities of ∇u ; u is Lipschitz. *Weak singularities*
- Possibly higher-order singularities ($\nabla^k u$ discontinuous),
- Possibly singularities of higher codimension (*e.g.* concentration of mass in Gas Dynamics),
- ...

This talk focuses on Lipschitz singularities.

Example : The 1-D Riemann Problem

In one space dimension (1D-RP), a weak singularity is a transition towards a *Rarefaction wave* : non-constant Lipschitz solutions of

$$\frac{d}{dx}f(u) = x\frac{du}{dx}$$

satisfy

$$f'(u) = x, \qquad \text{hence} \qquad f''(u)\frac{du}{dx} = 1.$$

The transition from a constant state $\bar{u}$ to a rarefaction occurs at the point

$$\bar{x} := f'(\bar{u}).$$

Genuine nonlinearity

$$f''(\bar{u}) \neq 0$$

is needed.

Back to multi-D Riemann Problems

Rule : Away from some compact subset, the self-similar solution behaves locally like low-D Riemann Problems. For instance, 1D-RPs involve constant states separated by planar shocks, rarefactions waves or contact discontinuities.

The reasons are

- 1D-RP is a special case of multi-D RP,
- the solution at (x_0, t_0) of a hyperbolic Cauchy problem depends only upon the restriction of the data to a ball $|x - x_0| < V t_0$, where V is a bound of the wave velocities.

But multi-D RPs contain more than just 1-D patterns ...

• Genuinely 2D-patterns are produced by the interaction of nearby 1D-patterns. Complicated !

• In 3D-RPs, the transitions from constants and 1D-waves to genuinely 2D-waves are followed by transitions towards genuinely-3D patterns.

• Transition from k-dimensional waves to $(k+1)$-dimensional waves occurs across either *shocks* (discontinuities) or characteristic hypersurfaces (Lipschitz singularities). Usually, the type of System (1) changes across such fronts.

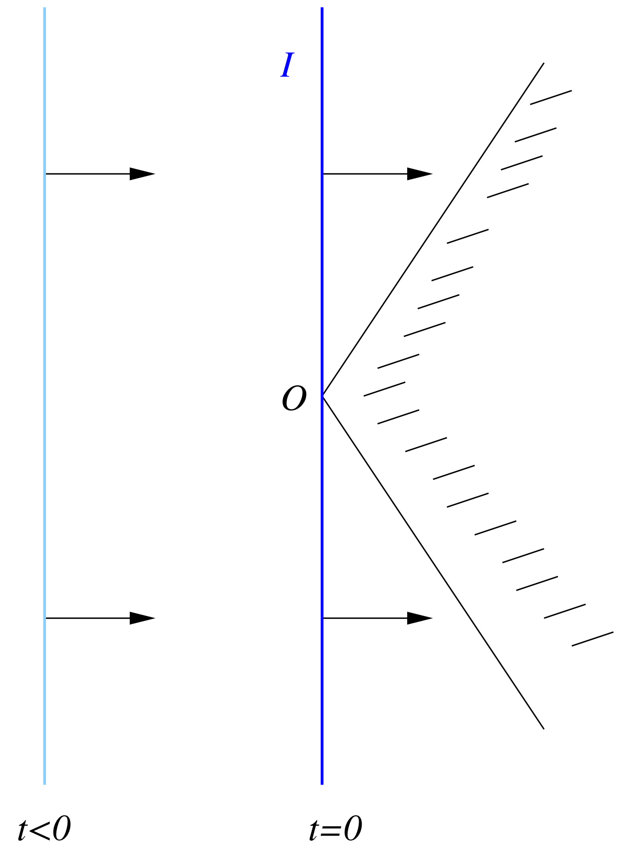
Goal of this work :

To quantify the jump of ∇u across weak singularities (Lipschitz).

Example : 2D-Shock Reflexion

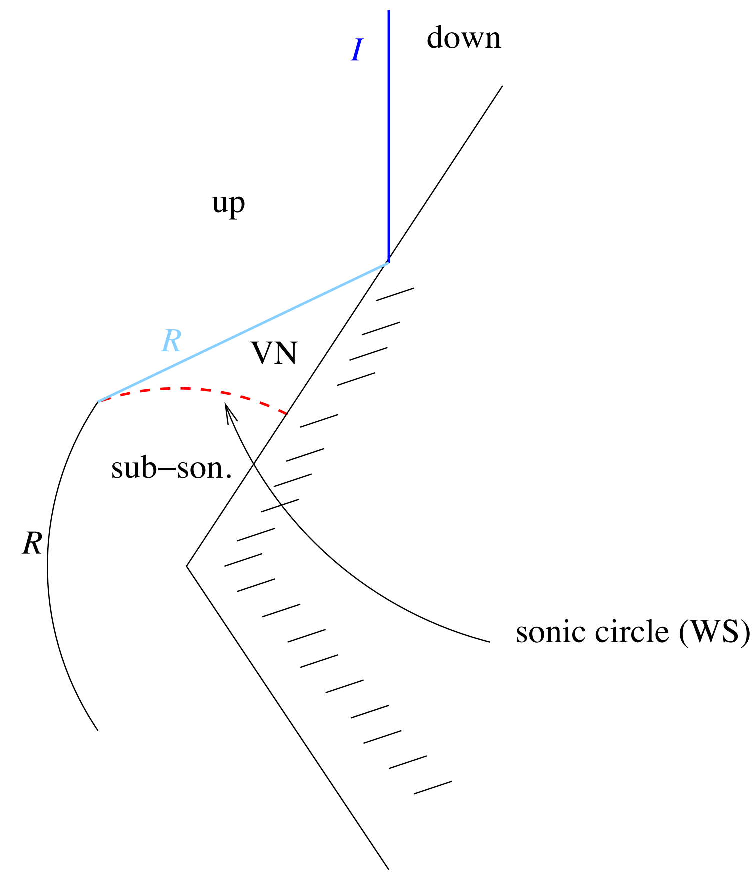
The planar shock travels at constant speed from left to right, until it reaches the tip of the wedge.

Then the interaction fluid / solid takes place, self-similarly.



Regular Reflexion

1. The incoming shock keeps travelling. It reflects along the ramp.
2. Constant state behind the reflected shock (calculated by Von Neumann ; *shock polar analysis*),
3. Until one reaches a *sonic line*, where a weak singularity occurs. Below the sonic line, the flow is sub-sonic.
4. After meeting the sonic line, the reflected shock bends.
5. The jump of ∇u across (WS) has been calculated by G.-C. Chen & coll.



Assumptions

H1 (hyperbolicity). The original system is hyperbolic : the symbol

$$A(u;\xi) := \sum_{\alpha=1}^{d} \xi_\alpha A^\alpha(u), \qquad A^\alpha(u) = \mathrm{D}f^\alpha(u)$$

is diagonalizable with real eigenvalues $\lambda_1(u;\xi) \leq \cdots \leq \lambda_n(u;\xi)$.

H2. For $\xi \neq 0$, $\lambda_n(u;\xi)$ is a simple eigenvalue.

H3 (Dispersion). The rank of $\mathrm{D}^2_\xi \lambda_n(u;\xi)$ is $d-1$.

Example : in gas dynamics, $\lambda_3 = v \cdot \xi + c(\rho)|\xi|$.

H3 (Genuine nonlinearity). If $r_n(u;\xi)$ is an associated eigenvector, then

$$\mathrm{d}_u \lambda_n \cdot r_n \neq 0.$$

Comments

• Recall the

Theorem (Gårding). The largest eigenvalue $\xi \mapsto \lambda_n(u;\xi)$ is convex.

$\diamondsuit$

Hence (**H2**) rewrites as :

$$\text{the Sylvester index of } \mathrm{D}^2_\xi \lambda_n(u;\xi) \text{ is } (d-1,1,0).$$

• In isentropic gas dynamics, GNL (**H4**) means that

$$\frac{d^2}{d\rho^2}(\rho p(\rho)) \neq 0.$$

Characteristic hypersurfaces

Lipschitz transition from a constant state $\bar{u}$ towards a non-constant solution must occur across a *characteristic* hypersurface Σ. Its normal ν at x satisfies

$$\det(A(\bar{u};\nu)-(x\cdot\nu)I_n)=0,\qquad \left(\text{recall}\quad A(u;\xi):=\sum_\alpha \xi_\alpha \mathrm{D}f^\alpha(u)\right). \tag{2}$$

In other words,

$$x\cdot\nu=\lambda_j(\bar{u};\nu),$$

for some index $1\le j\le n$.

> **Example** : If $\xi\in\mathbf{S}^{d-1}$, then $\Pi(\bar{u};\xi):=\{x\in\mathbb{R}^d\,|\,x\cdot\xi=\lambda_j(\bar{u};\xi)\}$ is a characteristic hyperplane. It occurs in *planar* rarefaction waves.

We are really interested in “higher-dimensional” patterns : non-flat Σ.

The extreme characteristic field

In practice, the first transition from low-D waves to higher-dimensional patterns are due to the extreme field : From now on, $j = n$.

$$x \cdot \nu = \lambda_n(\bar{u}; \nu).$$

That is, every tangent hyperplane to Σ is one of the hyperplanes $\Pi(\bar{u}; \xi)$: Σ is generated by tangent subspaces to the enveloppe $K(\bar{u}) \subset \mathbb{R}^d$ of all $\Pi(\bar{u}; \xi)$.

Because of (**H3**) and Gårding's Theorem :

Proposition : $K(\bar{u})$ is a compact convex subset of $\mathbb{R}^d$, with non-void interior.

♠

Example : In gas dynamics, $K(\bar{\rho}, \bar{v})$ is the ball $B(\bar{v}; \bar{c})$.

Definition : Consider a weak wave across a front Σ. Let assume a constant state $\bar{u}$ on one side of Σ. We say that the wave is k-dimensional, if Σ is the enveloppe of a $(k-1)$-dimensional family of hyperplanes $\Pi(\bar{u};\xi)$.

♣

Examples : – If $k=1$, one has a planar wave, – if $k=d$, then Σ coincides with a part of $\partial K(\bar{u})$.

We anticipate that the jump of ∇u across the front Σ depends upon the *dimension* $1 \leq k \leq d$ of the wave.

In practice, $k=1$ or 2.

The case $k=3$ is exceptional ; in most cases, u is not constant on one side of Σ, because of previous interactions. Then the results below don't apply.

Rk. Genuine nonlinearity must be at work ; already the case in 1-D.

Goal : We are interested in d-dimensional waves

Proposition. Across a genuinely d-dimensional front ($\Sigma = \partial K(\bar{u})$), the system governing self-similar solutions

$$\sum_{\alpha=1}^{d} \partial_\alpha f^\alpha(u) = (x \cdot \nabla)u,$$

has a transition from hyperbolic to non-hyperbolic.

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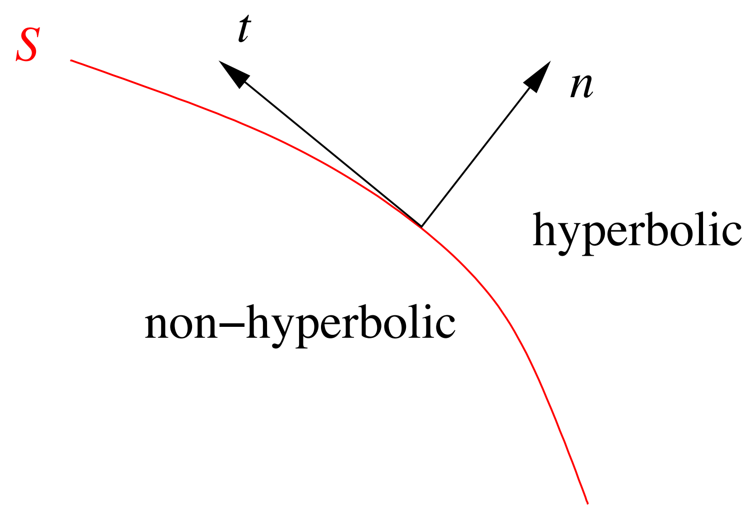
In particular, the domain of hyperbolicity of the linearized system about $\bar{u}$ is the complement of $K(\bar{u})$.

The picture in 2-D :

In the hyperbolic region, $u \equiv \bar{u}$.

$$\frac{d\tau}{ds} = -\kappa\nu, \quad \frac{d\nu}{ds} = \kappa\tau.$$

Denote u_{in} the trace along Σ from the non-hyperbolic side.



Continuity across Σ (that is $u_{\text{in}} \equiv \bar{u}$) yields

$$(\nabla u)_{\text{in}} = \nu \otimes X.$$

Problem : To determine X.

From the differential equations (System (1)), one gets

$$(A(\bar{u};\nu)-x\cdot\nu)X=0, \qquad \text{where we recall} \quad x\cdot\nu=\lambda_n(\bar{u};\nu).$$

Whence

$$X \parallel r_n(\bar{u};\nu).$$

Therefore

$$(\nabla u)_{\rm in}=\rho\nu\otimes r_n(\bar{u};\nu), \qquad \rho\in\mathbb{R}. \tag{3}$$

There remains to determine ρ.

Notice that $\rho=0$ is always one of the possible solutions, because of the everywhere constant solution.

Strategy :

– Differentiate the PDEs in the normal direction. One obtains

$$\text{Linear}(\mathrm{D}^2 u) + \text{Quadratic}(\nabla u) = 0.$$

– Differentiate (3) along Σ. The curvature of Σ comes into play !
– Along Σ, the linear part is characteristic : elimination of $\mathrm{D}^2 u$ is possible. There remains one scalar information about $(\nabla u)_{\mathrm{in}}$.
– At the end, the curvature essentially disappears. Only the ”wave dimension” k remains. This is a quantization of the jump of ∇u.

For instance, d-dimensional weak waves obey :

Theorem. Let u be a continuous, piecewise-C^2 solution of the d-dimensional system (1), which is constant ($u \equiv \bar{u}$) on one side of its singular set Σ. We assume (H1,2,3), and that Σ coincides locally with $\partial K(\bar{u})$.

Then along Γ, we have either $\nabla u_{in} = 0$ or

$$(\nabla u)_{in} = \frac{3-d}{2\,\mathrm{d}_u\lambda_n \cdot r_n(\bar{u};\nu)}\nu \otimes r_n(\bar{u};\nu). \tag{4}$$

♡

Corollary. Across a genuinely 3-dimensional weak singularity bounding a constant state, one has

$$\nabla u_{in} = 0 \qquad !!!$$

♢

In this case, u is even C^1 across Σ ...

Back to Regular Reflexion

- The existence of a solution was proved for irrotational flows, if the aperture of the wedge is large enough, by G.-Q. Chen & M. Feldman (2005).
- Irrotationality is only an approximation across the reflected shock.
- But irrotationality is OK across the sonic line, because of

  $$(\partial_t + v \cdot \nabla)\frac{\omega}{\rho} = 0,$$

  plus $\omega \equiv 0$ in the VN side, and Lipschitz continuity.

  Therefore the values of $(\nabla u)_{\text{in}}$ in the irrotational and non-irrotational situations coincide (even if u_{irr} and $u_{\text{non}-\text{irr}}$ don't coincide near the sonic line).
- This value was computed by M. Bae, G.-Q. Chen & M. Feldman (2009).
- Constrast with the acoustic approximation : the singularity is Lipschitz instead of $C^{1/2}$. A fundamental role is played by Genuine Nonlinearity.

Thanks for your attention !

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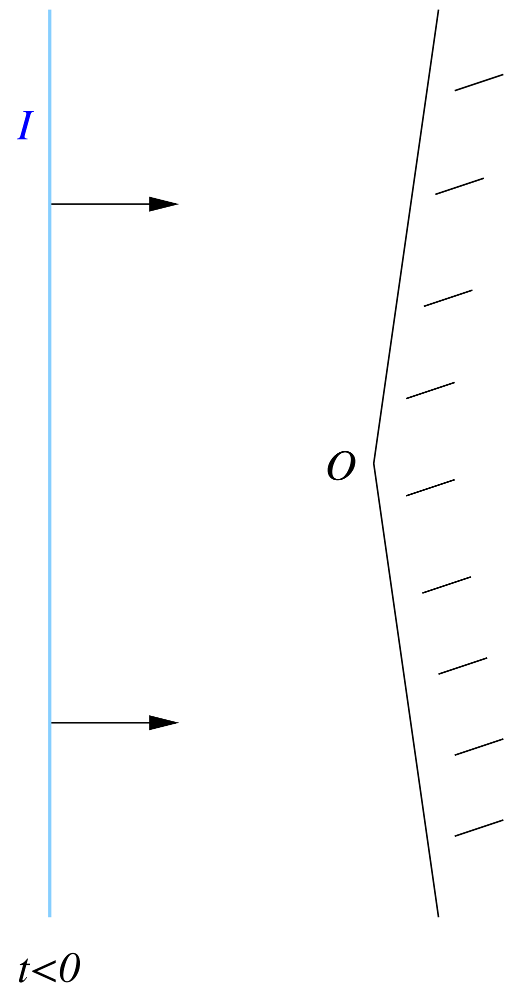
A boundary layer phenomenon

Suppose that the wedge is almost fully open (aperture angle $= \pi - \varepsilon$)

Q. How does the solution behave as $\varepsilon \to 0$?

Formally, the limit is the *normal reflexion* problem.



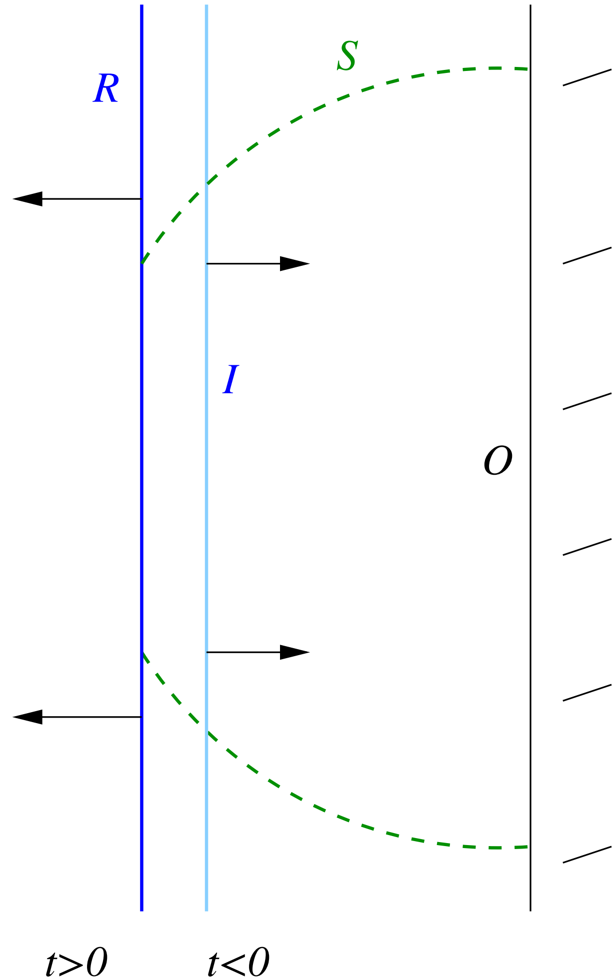
The Normal Reflexion ($\varepsilon = 0$)

For $t < 0$, the flow is piecewise constant (upstream & downstream data).

For $t > 0$, it is still piecewise constant, but different.

The type of (1) changes across the sonic line, even if u remains constant across it.

Here $(\nabla u)_{\text{in}} = 0$.



However, the value of $(\nabla u)_{\rm in}$ computed in the Theorem (or as well by M. B., G.-Q. C. & M. F.) does not tend to zero as $\varepsilon \to 0$:

$$\frac{1}{2\,\mathrm{d}_u\lambda_n \cdot r_n(\bar{u};\nu)}\nu \otimes r_n(\bar{u};\nu) = \lim_{\varepsilon\to 0}\lim_{d(x;S)\to 0}\nabla u \neq \lim_{d(x;S)\to 0}\lim_{\varepsilon\to 0}\nabla u = 0.$$

This boundary layer does not prevent convergence (L^1 is proved by BCF).

Even uniform convergence remains possible.