Entropy decreasing in resonant contact discontinuities

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Hyperbolic PDE's with an interface in 1D

Two hyperbolic systems to couple through an interface $x = 0$:

$\partial_t u + \partial_x f_L(u) = 0$	$\partial_t u + \partial_x f_R(u) = 0$

$x < 0 \qquad x = 0 \qquad x > 0$

Jump relations at the interface

Conservative case: $f_L(u(t,0^-)) = f_R(u(t,0^+))$

Nonconservative case: $f_L(u(t,0^-)) \neq f_R(u(t,0^+))$

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Classical case (Kruzhkov, Lax...):

$f_L \equiv f_r$ and entropy inequality at $x = 0$

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Saturation in a porous medium with a discontinuous permeability k:

$$f_L(u) = k_L f(u) \text{ and } f_R(u) = k_R f(u)$$

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Traffic flow with a tollgate:

$$f_L(u) = f_r(u) \text{ but with } f_L(u(t, 0^-)) = f_R(u(t, 0^+)) \leqslant F(t)$$

Hyperbolic PDE's with an interface in 1D

Two hyperbolic systems to couple through an interface $x = 0$:

$x < 0$	$x = 0$	$x > 0$
$\partial_t u + \partial_x f_L(u) = 0$	‖	$\partial_t u + \partial_x f_R(u) = 0$

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Conservative case: $f_L(u(t,0^-)) = f_R(u(t,0^+))$

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Shallow-water equations with a discontinuous bathymetry:

$$\partial_t \begin{bmatrix} h \\ hu \end{bmatrix} + \partial_x \begin{bmatrix} hu \\ hu^2 + gh^2/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -gh(a_R - a_L)\delta_0(x) \end{bmatrix}$$

Hyperbolic PDE's with an interface in 1D

Two hyperbolic systems to couple through an interface $x = 0$:

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Jump relations at the interface

Conservative case: $f_L(u(t, 0^-)) = f_R(u(t, 0^+))$

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Gas dynamics in a discontinuous nozzle:

$$\partial_t \begin{bmatrix} \rho \\ \rho u \end{bmatrix} + \partial_x \begin{bmatrix} \rho u \\ \rho u^2 + P(\rho) \end{bmatrix} = \begin{bmatrix} -\rho u \frac{S_R - S_L}{(1-H(x))S_L + H(x) S_R} \delta_0(x) \\ 0 \end{bmatrix}$$

Hyperbolic PDE's with an interface in 1D

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Jump relations at the interface

Conservative case: $f_L(u(t,0^-)) = f_R(u(t,0^+))$

Nonconservative case: $f_L(u(t,0^-)) \neq f_R(u(t,0^+))$

Gas dynamics in a pipe with a grid (laminar friction):

$$\partial_t \begin{bmatrix} \rho \\ \rho u \end{bmatrix} + \partial_x \begin{bmatrix} \rho u \\ \rho u^2 + P(\rho) \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda \rho u \delta_0(x) \end{bmatrix}$$

Hyperbolic PDE's with an interface in 1D

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Jump relations at the interface

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Gas dynamics in a pipe with a grid (turbulent friction):

$$\partial_t \begin{bmatrix} \rho \\ \rho u \end{bmatrix} + \partial_x \begin{bmatrix} \rho u \\ \rho u^2 + P(\rho) \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda \rho u |u| \delta_0(x) \end{bmatrix}$$

Hyperbolic PDE's with an interface in 1D

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Nonconservative case: $f_L(u(t, 0^-)) \neq f_R(u(t, 0^+))$

and many other examples...

General forms invovling a discontinuity

The conservative case

Equivalent form:

$$\partial_t u + \partial_x f(u,x) = 0 \qquad t > 0, x \in \mathbb{R}$$

with $f(u,x) = (1 - \mathbf{H}(x)) f_L(u) + \mathbf{H}(x) f_R(u)$ ($\mathbf{H}$ Heaviside function)

The nonconservative case

Equivalent form:

$$\partial_t u + \partial_x f(u,x) + g(u)\, \delta_0(x) = 0 \qquad t > 0, x \in \mathbb{R}$$

with $f(u,x) = (1 - \mathbf{H}(x)) f_L(u) + \mathbf{H}(x) f_R(u)$ ($\mathbf{H}$ Heaviside function)
and $g(u)$ to encode the nonconservative contribution of the interface

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Use of an additional "unknown"

Equivalent form:

$$\begin{aligned}
&\partial_t u + \partial_x f(u,\alpha) = 0 \\
&\partial_t \alpha = 0 \\
&\alpha(0,x) = \mathbf{H}(x)
\end{aligned}$$

with $f(u,\alpha) = (1-\alpha)f_L(u) + \alpha f_R(u)$

General forms invovling a discontinuity

The nonconservative case

Equivalent form:

$$\partial_t u + \partial_x f(u,x) + g(u)\,\delta_0(x) = 0 \qquad t>0, x\in\mathbb{R}$$

with $f(u,x) = (1-\mathbf{H}(x))f_L(u) + \mathbf{H}(x)f_R(u)$ ($\mathbf{H}$ Heaviside function)
and $g(u)$ to encode the nonconservative contribution of the interface

Use of an additional “unknown”

Equivalent form:

$$\begin{aligned}
&\partial_t u + \partial_x f(u,\alpha) + g(u)\partial_x\alpha = 0\\
&\partial_t\alpha = 0\\
&\alpha(0,x) = \mathbf{H}(x)
\end{aligned}$$

with $f(u,\alpha) = (1-\alpha)f_L(u) + \alpha f_R(u)$

The quasilinear form

The conservative case

$$\partial_t \begin{bmatrix} u \\ \alpha \end{bmatrix} + \begin{bmatrix} \partial_u f(u,\alpha) & \partial_\alpha f(u,\alpha) \\ 0 & 0 \end{bmatrix} \partial_x \begin{bmatrix} u \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The nonconservative case

$$\partial_t \begin{bmatrix} u \\ \alpha \end{bmatrix} + \begin{bmatrix} \partial_u f(u,\alpha) & \partial_\alpha f(u,\alpha) + g(u) \\ 0 & 0 \end{bmatrix} \partial_x \begin{bmatrix} u \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- The system admits an additional eigenvalue: 0 (standing wave)
- Since this eigenvalue is constant, it characteristic field is linearly degenerate
- BUT some eigenvalues of $\partial_u f(u,\alpha)$ may vanish

  $\rightsquigarrow$ interaction of waves with the standing wave

  superposition of a shock wave on the interface wave

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In most of cases, $\partial_u f(u,\alpha)$ is $\mathbb{R}$-diagonalizable

But in general, there is resonance, i.e. $\mathcal{R} := \{\bar{u}$ s.t. $0 \in \mathrm{sp}[\partial_u f(\bar{u},\alpha)]\} \neq \emptyset$:

The quasilinear form

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But in general, there is resonance, i.e. $\mathcal{R} := \{\bar{u} \text{ s.t. } 0 \in \mathrm{sp}[\partial_u f(\bar{u},\alpha)]\} \neq \emptyset$:

- Coalescence of eigenvalues (loss of strict hyperbolicity)
- What about the associated eigenvectors?

Temple, Liu, Keyfitz, Shearer, LeFloch, Chen, Glimm, Isaacson, Marchesin...

The quasilinear form

The conservative case

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But in general, there is resonance, i.e. $\mathcal{R} := \{\bar{u} \text{ s.t. } 0 \in \mathrm{sp}[\partial_u f(\bar{u},\alpha)]\} \neq \emptyset$:

- Conservative case: 0 double eigenvalue

  Either $\mathcal{R} \subset \mathrm{Ker}(\partial_\alpha f(\cdot,\alpha))$ $\rightsquigarrow$ still $\mathbb{R}$-diagonalizable

  Or $\mathcal{R} \not\subset \mathrm{Ker}(\partial_\alpha f(\cdot,\alpha))$ $\rightsquigarrow$ Jordan block

  Rankine-Hugoniot relations still valid, but entropy conditions to be defined

The quasilinear form

The conservative case

$$\partial_t \begin{bmatrix} u \\ \alpha \end{bmatrix} + \begin{bmatrix} \partial_u f(u,\alpha) & \partial_\alpha f(u,\alpha) \\ 0 & 0 \end{bmatrix} \partial_x \begin{bmatrix} u \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The nonconservative case

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In most of cases, $\partial_u f(u,\alpha)$ is $\mathbb{R}$-diagonalizable

But in general, there is resonance, i.e. $\mathcal{R} := \{\bar{u} \text{ s.t. } 0 \in \mathrm{sp}[\partial_u f(\bar{u},\alpha)]\} \neq \emptyset$:

- Nonconservative case: 0 double eigenvalue

  Either $\mathcal{R} \subset \mathrm{Ker}(\partial_\alpha f(\cdot,\alpha) + g)$ $\rightsquigarrow$ still $\mathbb{R}$-diagonalizable

  Or $\mathcal{R} \not\subset \mathrm{Ker}(\partial_\alpha f(\cdot,\alpha) + g)$ $\rightsquigarrow$ Jordan block

  No Rankine-Hugoniot jump relations available (neither entropy condition)

The quasilinear form

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In most of cases, $\partial_u f(u,\alpha)$ is $\mathbb{R}$-diagonalizable
The nonconservative and non-resonant case, i.e.

$$\mathcal{R} := \{\bar{u} \text{ s.t. } 0 \in \mathrm{sp}[\partial_u f(\bar{u},\alpha)]\} = \emptyset,$$

has been extensively study by Gosse and co-workers

Examples

The scalar case

- The conservative case [Andreianov, Karlsen, Risebro '11]...
  Tollgate in traffic flow
- The nonconservative case
  - $\mathbb{R}$-diagonalizable case: Burgers equation with pointwise friction
  - Resonant case: [Isaacson, Temple '95] $\rightsquigarrow$ $\exists\ 1$ to 3 solutions to the RP

The system case

- The conservative case: [Isaacson, Temple '92]...
  Gas dynamics with pointwise linear friction [Aguillon '12]
- The nonconservative case
  - $\mathbb{R}$-diagonalizable case ($\mathcal{R} \subset \mathrm{Ker}(\partial_\alpha f(\cdot, \alpha) + g)$): probably meaningless...
  - Resonant case: gas dynamics in a nozzle, shallow water equations with bathymetry... [Chen, Glimm '95], [Goatin, LeFloch '04]...

The scalar case

The conservative case

$$\partial_t u + \partial_x f(u, \alpha) = 0$$
$$\alpha(t, x) = \mathbf{H}(x)$$

The Rankine-Hugoniot jump relations are valid:

$$f(u(t, 0^-), 0) = f(u(t, 0^+), 1)$$

Entropy condition at $x = 0$?

- $\partial_\alpha f \equiv 0$ and classical condition (Lax, Oleinik, Kruzhkov, Liu...):
  $\rightsquigarrow$ recover the classical entropy solution
- $f(\cdot, 0) \not\equiv f(\cdot, 1)$: need to provide an adapted entropy condition
  Risebro, Karlsen, Adimurthi, Mishra, Gowda, S., Vovelle, Andreianov, Bachmann, Audusse, Perthame...
- $\partial_\alpha f \equiv 0$ but with "pointwise effects"
  Ex.: Traffic flow with a tollgate or thin layer in a porous medium

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Entropy condition at $x = 0$?

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  $\leadsto$ recover the classical entropy solution
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  Risebro, Karlsen, Adimurthi, Mishra, Gowda, S., Vovelle, Andreianov, Bachmann, Audusse, Perthame...
- $\partial_\alpha f \equiv 0$ but with "pointwise effects"
  Ex.: Traffic flow with a tollgate or thin layer in a porous medium

The scalar case

The nonconservative case

$$(\text{NC}) \qquad \begin{array}{l} \partial_t u + \partial_x f(u,\alpha) + g(u)\partial_x \alpha = 0 \\ \alpha(t,x) = \mathbf{H}(x) \end{array}$$

The standing wave corresponds to a LD field if there is no resonance

Proposition (jump relations by Riemann invariants)

If there is no resonance and if smooth solutions of (NC) satisfy an additional conservation law

$$\partial_t w(u,\alpha) + \partial_x \varphi(u,\alpha) = 0,$$

then weak solutions of (NC) satisfy $\varphi(u(t,0^+),1) = \varphi(u(t,0^-),0)$.

No need of defining nonconservative products using additional informations [DalMaso, LeFloch, Murat '95], [LeFloch, Tzavaras '99]
What happens when resonance occurs?

The scalar case

The nonconservative case

$$\text{(NC)} \qquad \begin{aligned} &\partial_t u + \partial_x f(u, \alpha) + g(u)\partial_x \alpha = 0 \\ &\alpha(t, x) = \mathbf{H}(x) \end{aligned}$$

The standing wave corresponds to a LD field if there is no resonance

What happens when resonance occurs

- The above jump relation fails: $\varphi(u(t, 0^+), 1) \neq \varphi(u(t, 0^-), 0)$
- A shock wave can be superposed on the standing wave
- Additional informations must be used:
  1. Definition as the limit of regularization processes
     Diffusion [Sainsaulieu '96], kinetic relations [Abeyaratne, Knowles '91]...
  2. The inner structure must be studied: [Isaacson, Temple '95], [Dalmaso, LeFloch, Murat '95], [Lefloch, Tzavaras '99]...

A ROUGH review of the theory of conservation laws

Analysis and approximation of the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u_{|t=0} = u_0 \end{cases}$$

Definition ([Kruzhkov '70])

A function $u \in \mathbf{L}^\infty(\mathbb{R}_+ \times \mathbb{R})$ is a weak entropy solution if $\forall \kappa \in \mathbb{R}$

$$\partial_t |u - \kappa| + \partial_x \Phi(u, \kappa) \leqslant 0 \quad \text{(in } \mathscr{D}'\text{)}$$

where $\Phi(a,b) = \operatorname{sgn}(a-b)(f(a) - f(b))$.

- Constant κ are solutions as soon as $u_0 = \kappa$
- Entropies $|u - \kappa|$ measure the dissipation with respect to constant solutions

A ROUGH review of the theory of conservation laws

Analysis and approximation of the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u_{|t=0} = u_0 \end{cases}$$

Theorem ([Kruzhkov '70])

The weak entropy solution is unique.

- Comparison of two solutions u and $\hat{u}$ (Kato inequality)

$$\partial_t |u - \hat{u}| + \partial_x \Phi(u, \hat{u}) \leqslant 0$$

- Continuous dependence can be deduced

(Existence can be proved by convergence of approximate solutions.)

A ROUGH review of the theory of conservation laws

Analysis and approximation of the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u_{|t=0} = u_0 \end{cases}$$

Monotone numerical schemes ([Crandall, Majda '80])

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}(g_{i+1/2}^n - g_{i-1/2}^n)$$

where $g_{i+1/2}^n = g(u_i^n, u_{i+1}^n)$ such that $g(\nearrow, \searrow)$, $g(a,a) = f(a)$ + CFL condition

- The numerical scheme preserves constant solutions
- u_i^{n+1} is a non-decreasing function of u_{i-1}^n, u_i^n and u_{i+1}^n

Theorem ([Crandall, Majda '80])

A monotone numerical scheme converges to the weak entropy solution.

Scalar equations with pointwise contributions

The Cauchy problem with a contribution at $x = 0$

Conservation laws for $x \neq 0$ — Coupling conditions at $x = 0$

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & x < 0 \\ \partial_t u + \partial_x f(u) = 0 & x > 0 \\ u_{|t=0} = u_0 & x \in \mathbb{R} \end{cases} \qquad (u(t,0^-), u(t,0^+)) \in \mathcal{G}$$

The germ $\mathcal{G} \in \mathbb{R}^2$ is the set of admissible traces

Let us define $\kappa(x) = \begin{cases} \kappa_L & \text{if } x < 0 \\ \kappa_R & \text{if } x > 0 \end{cases}$ where $(\kappa_L, \kappa_R) \in \mathcal{G}$

Proposition

If $u_0(x) = \kappa(x)$ then $u(t,x) = \kappa(x)$ is a solution.

(The germ depends on the problem under study.)

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Coupling conditions at $x = 0$

$$(u(t, 0^-), u(t, 0^+)) \in \mathcal{G}$$

Definition ([Baiti, Jenssen '97] [Audusse, Perthame '05])

A function $u \in \mathbf{L}^\infty(\mathbb{R}_+ \times \mathbb{R})$ is a weak entropy solution if $\forall \kappa(x)$ defined as above

$$\partial_t |u - \kappa(x)| + \partial_x \Phi(u, \kappa(x)) \leqslant 0 \quad (\text{in } \mathscr{D}')$$

- Functions κ are solutions as soon as $u_0 = \kappa$
- Entropies $|u - \kappa(x)|$ measure the dissipation with respect to solutions $\kappa(x)$

Theory in [Andreianov, Karlsen, Risebro '11]: a subset $\mathcal{G}_0 \subset \mathcal{G}$ can be sufficient

Scalar equations with pointwise contributions

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Coupling conditions at $x = 0$

$$(u(t,0^-), u(t,0^+)) \in \mathcal{G}$$

Theorem

If the germ is dissipative: $\forall (u_-, u_+), (\hat{u}_-, \hat{u}_+) \in \mathcal{G}$

$$\Phi(u_+, \hat{u}_+) - \Phi(u_-, \hat{u}_-) \leqslant 0$$

then the weak entropy solution is unique.

- Dissipativity of $\mathcal{G}$: comparison of two solutions u and $\hat{u}$ (Kato inequality)

$$\partial_t |u - \hat{u}| + \partial_x \Phi(u, \hat{u}) \leqslant 0$$

- Lipshitz-continuous dependence can be deduced

Scalar equations with pointwise contributions

The Cauchy problem with a contribution at $x = 0$

Conservation laws for $x \neq 0$

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & x < 0 \\ \partial_t u + \partial_x f(u) = 0 & x > 0 \\ u_{|t=0} = u_0 & x \in \mathbb{R} \end{cases}$$

Coupling conditions at $x = 0$

$$(u(t, 0^-), u(t, 0^+)) \in \mathcal{G}$$

Design of adapted numerical schemes (well-balanced schemes for $\mathcal{G}_0$)

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}(g_{i+1/2}^n - g_{i-1/2}^n)$$

where $g_{i+1/2}^n = g(u_i^n, u_{i+1}^n)$ except at the interface $x_{1/2} = 0$

- The numerical scheme must preserve solutions $\kappa(x)$ with $(\kappa_L, \kappa_R) \in \mathcal{G}_0$
- u_i^{n+1} must be a non-decreasing function of u_{i-1}^n, u_i^n and u_{i+1}^n

An monotone adapted numerical scheme converges to the weak entropy solution.

Conservation law with a constrained flux

The formal problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & t > 0, x \in \mathbb{R} \\ f(u(t,0^-)) = f(u(t,0^+)) \leqslant F & t > 0 \end{cases}$$

where $f(a) = a(1-a)$ and $F \leqslant \max_{a\in[0,1]} f(a)$

How to contruct solutions to this problem?

- Break the classical entropy condition at $x = 0$ $(u(t,0^-) \leqslant u(t,0^+))$
- $u_{|x\neq 0}$ is a classical entropy solution
- $u(t,0^-)$ is an admissible boundary trace for the left-hand half-problem
- $u(t,0^+)$ is an admissible boundary trace for the right-hand half-problem
- constraint: $f(u(t,0^-)) = f(u(t,0^+)) \leqslant F$

Such solutions do not enter in the Kruzhkov frame... But we have well-posedness !

Conservation law with a constrained flux

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where $f(a) = a(1-a)$ and $F \leqslant \max_{a\in[0,1]} f(a)$

[Colombo, Goatin '07] [Andreianov, Goatin, S. '10] [Cancès, S. '12]

- Construct the set of admissible traces $(u(t,0^-), u(t,0^+))$:

$$\mathcal{G} = \{(a,b) \mid f(a) = f(b) = F, a > b\} \cup \{(a,a) \mid f(a) \leqslant F\} \cup \{(a,b) \mid f(a) = f(b) \leqslant F, a < b\}$$

- Dissipative property:

$$\forall (u_-, u_+), (\hat{u}_-, \hat{u}_+) \in \mathcal{G} \qquad \Phi(u_+, \hat{u}_+) - \Phi(u_-, \hat{u}_-) \leqslant 0$$

$\Longrightarrow$ Uniqueness + Lipshitz-continuous dependence

Conservation law with a constrained flux

The formal problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & t > 0, x \in \mathbb{R} \\ f(u(t, 0^-)) = f(u(t, 0^+)) \leqslant F & t > 0 \end{cases}$$

where $f(a) = a(1 - a)$ and $F \leqslant \max_{a \in [0,1]} f(a)$

[Colombo, Goatin '07] [Andreianov, Goatin, S. '10] [Cancès, S. '12]

- A sufficient dissipative germ

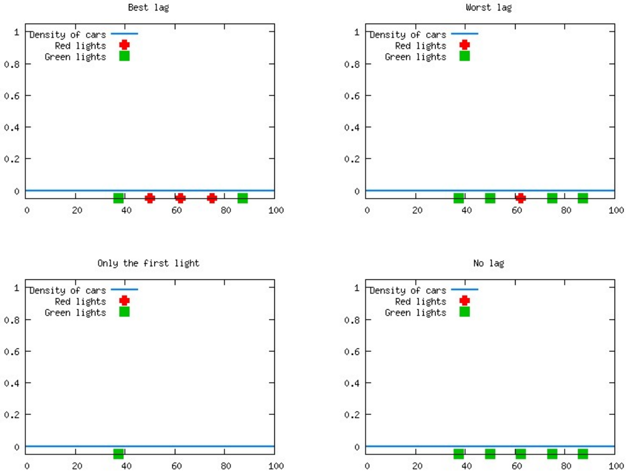
$$\mathcal{G}_0 = \{(a, b) \mid f(a) = f(b) = F, a > b\} \cup \{(a, a) \mid f(a) \leqslant F\}$$

- Adapted numerical schemes
  At the interface $x_{1/2} = 0$, take the numerical flux $g^n_{1/2} = \min(g(u^n_0, u^n_1), F)$
  - It exactly preserves solutions of $\mathcal{G}_0$
  - The constrained numerical scheme is monotone (CFL condition)
  - If g is the Godunov numerical flux, $\mathbf{L}^\infty \cap \mathrm{BV}$ bounds

$\Longrightarrow$ Strong convergence + Existence + Error estimates

The green wave in traffic flow

B. Andreianov, P. Goatin, N. Seguin. Finite volume schemes for locally constrained conservation laws. Numerische Mathematik.



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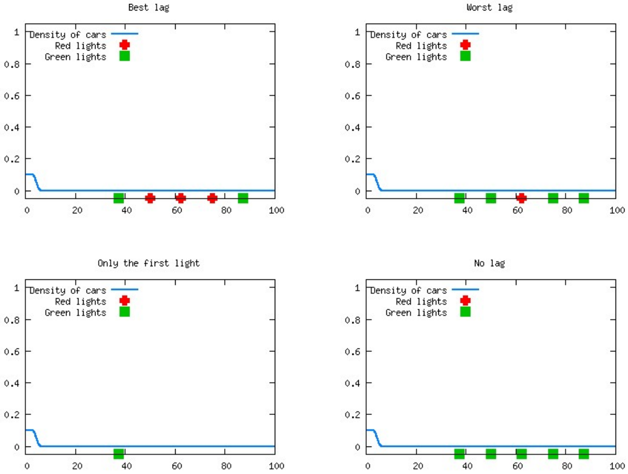

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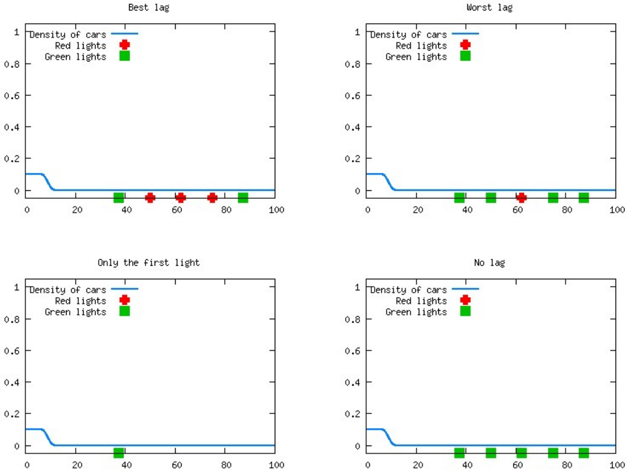

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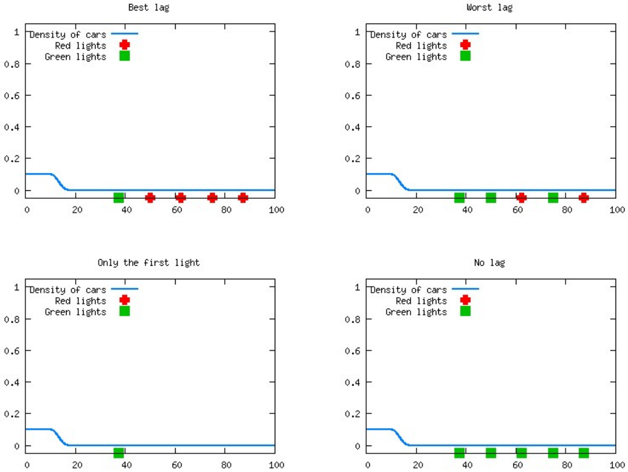

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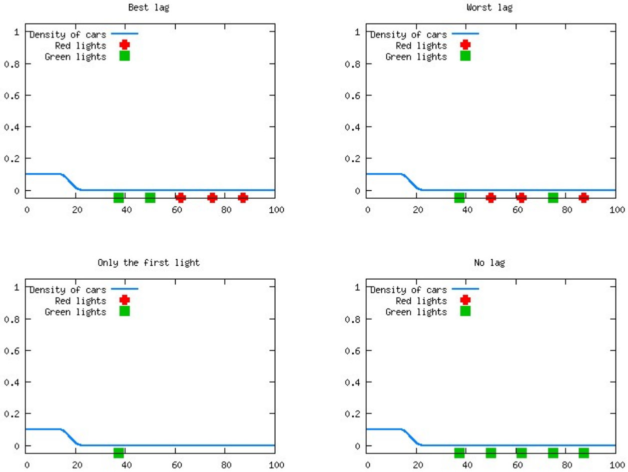

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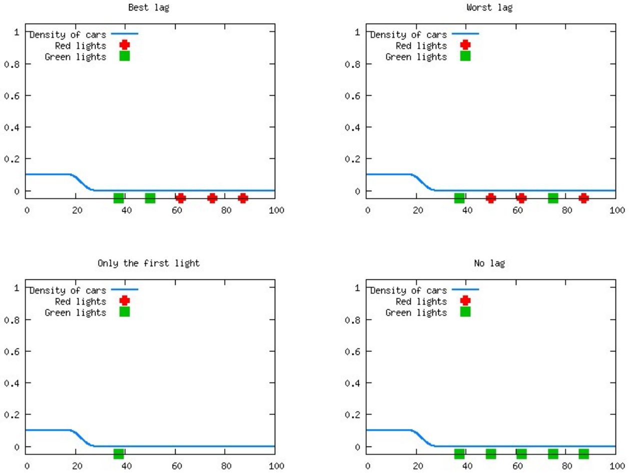

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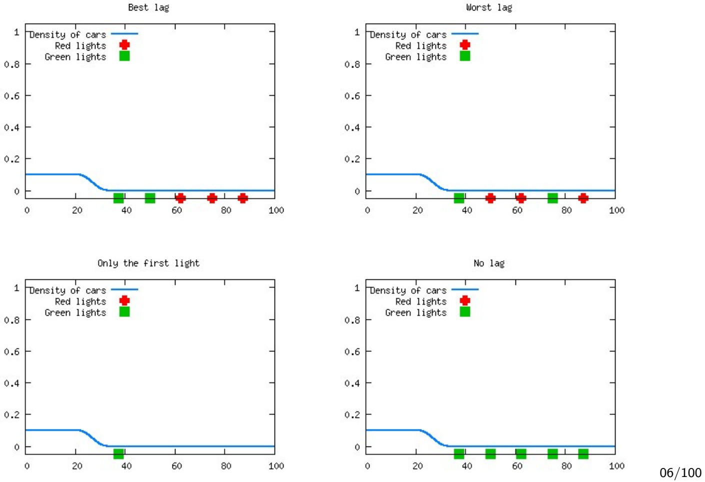

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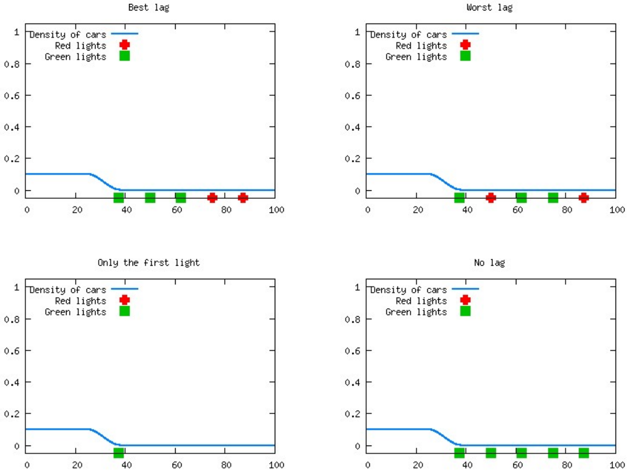

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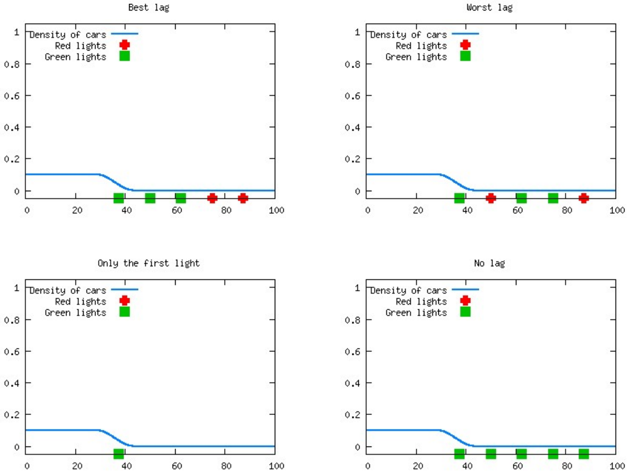

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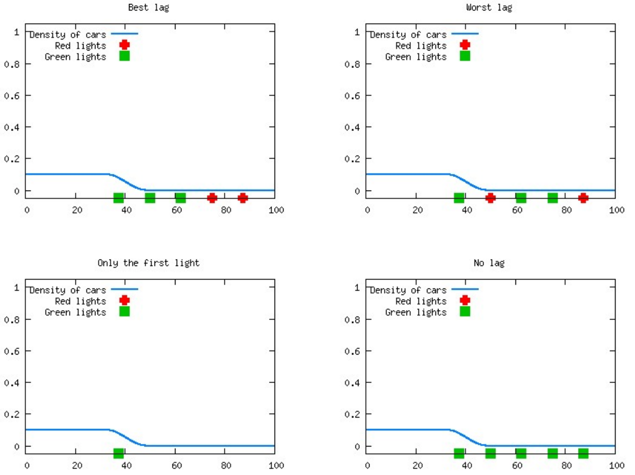

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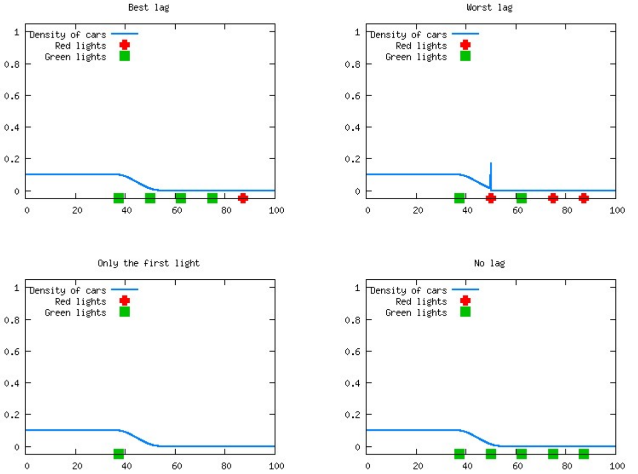

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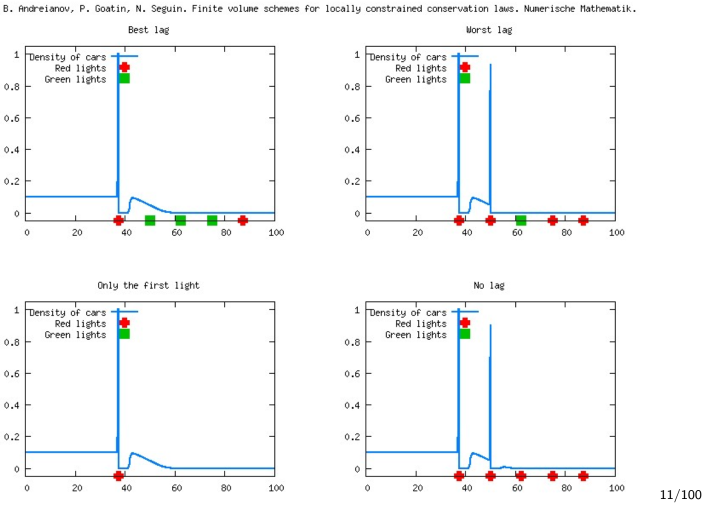

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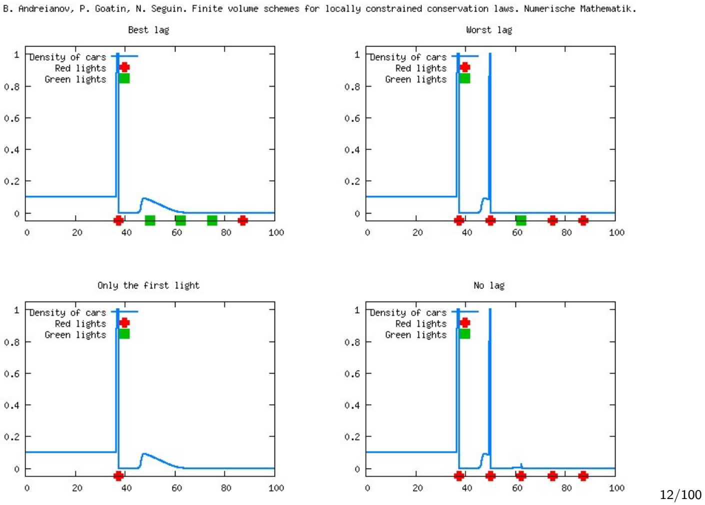

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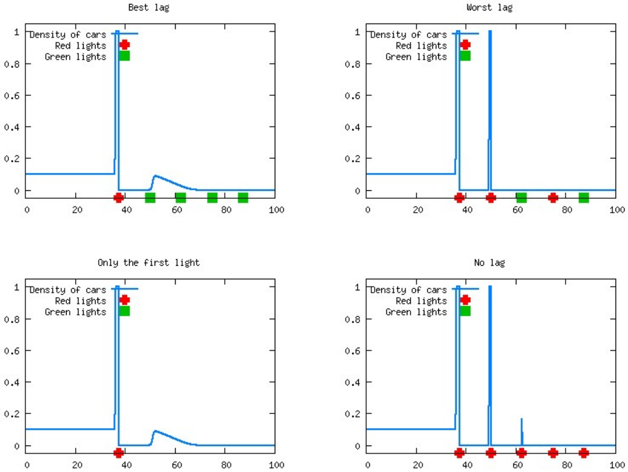

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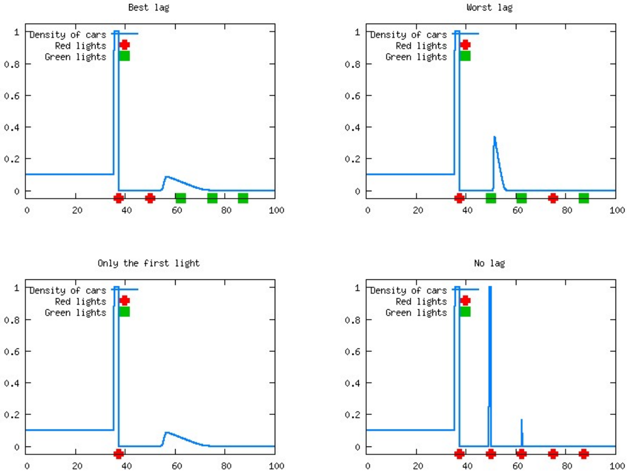

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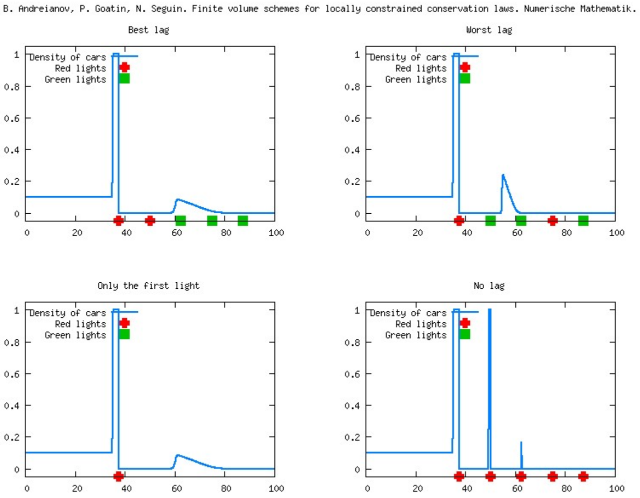

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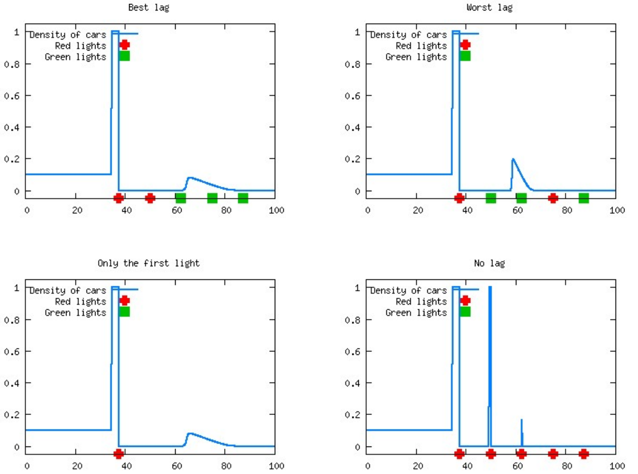

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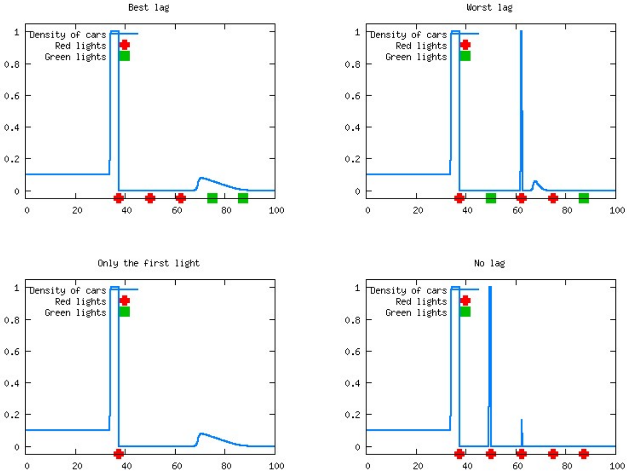

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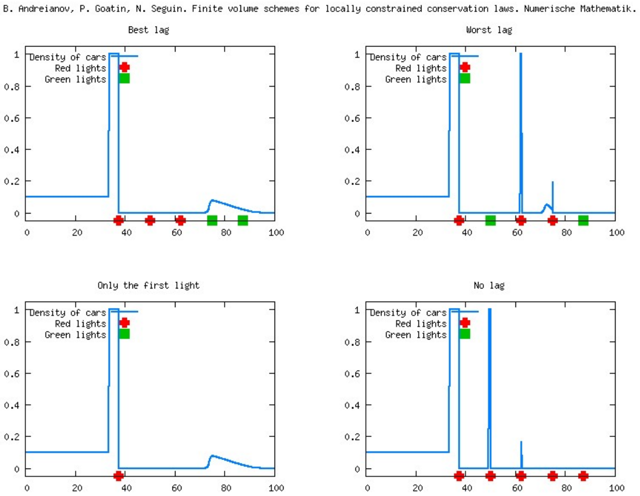

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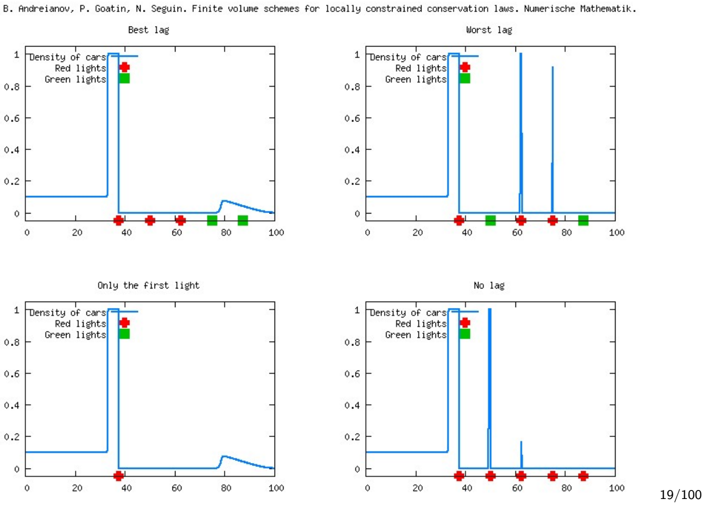

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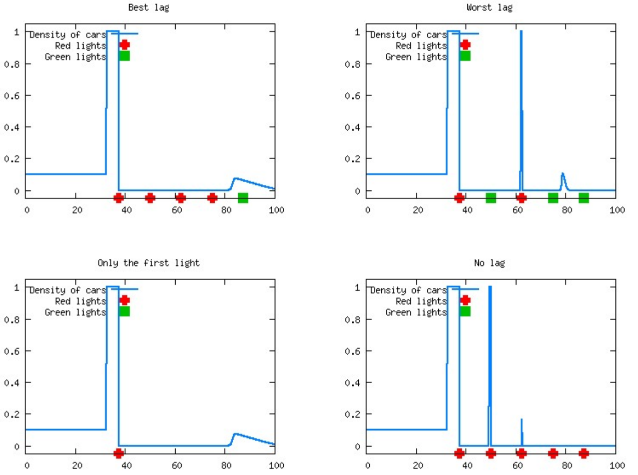

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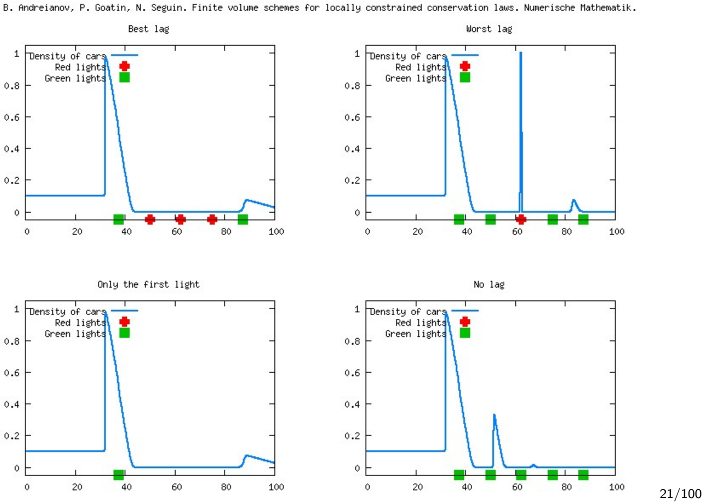

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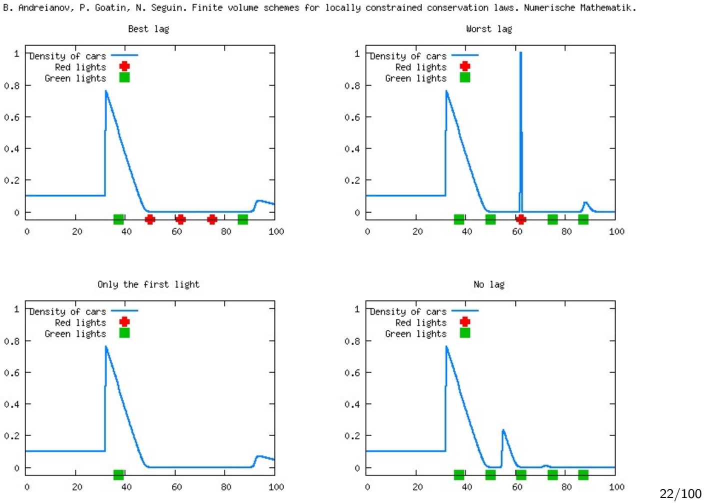

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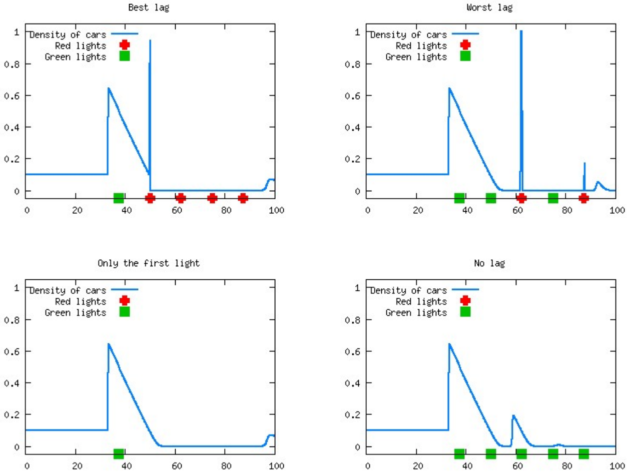

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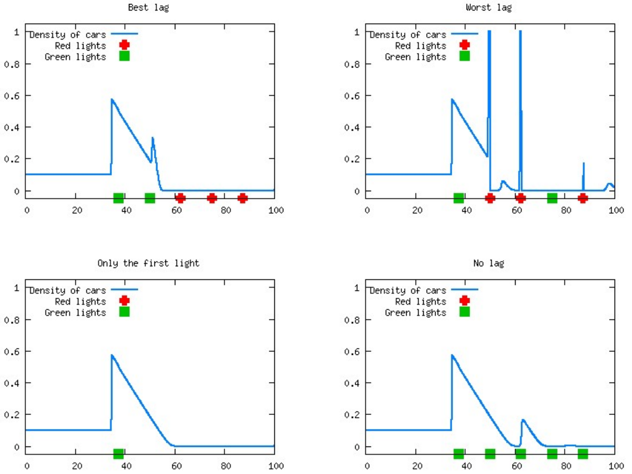

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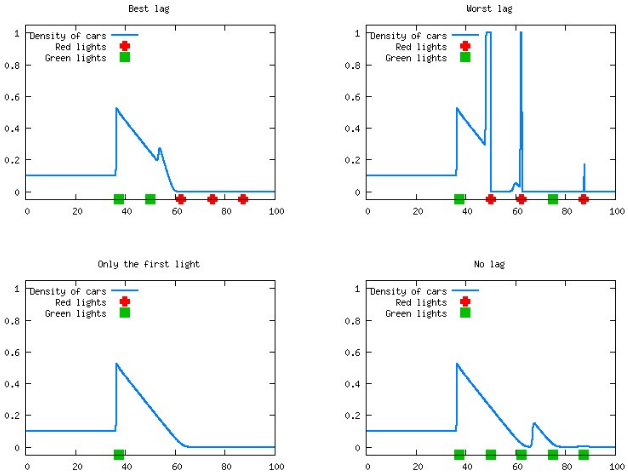

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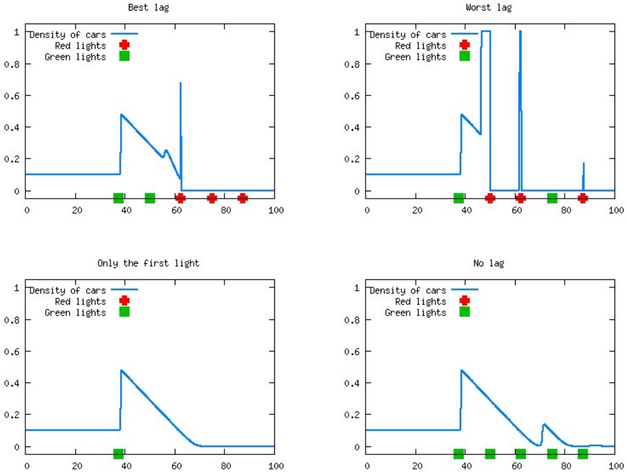

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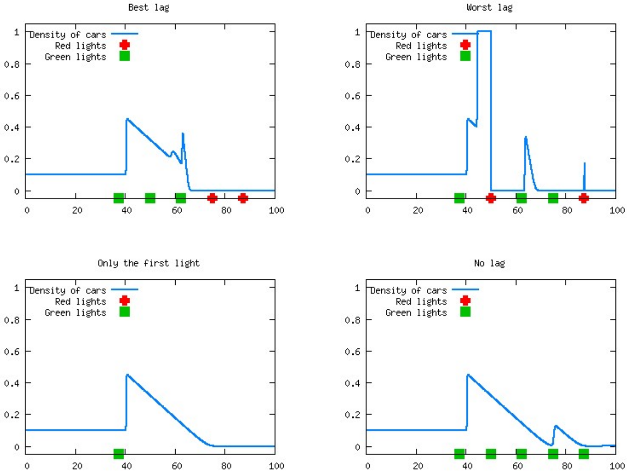

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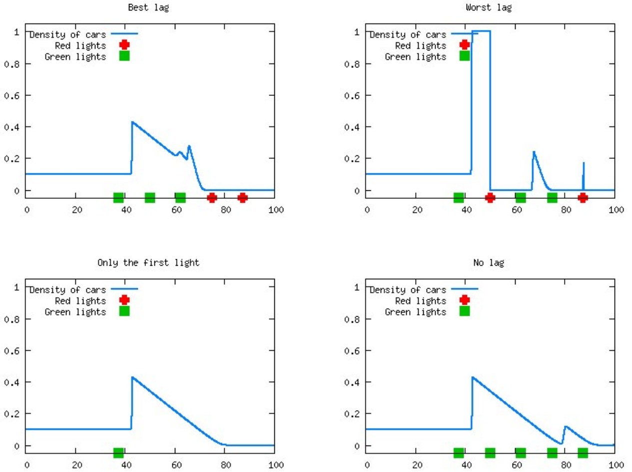

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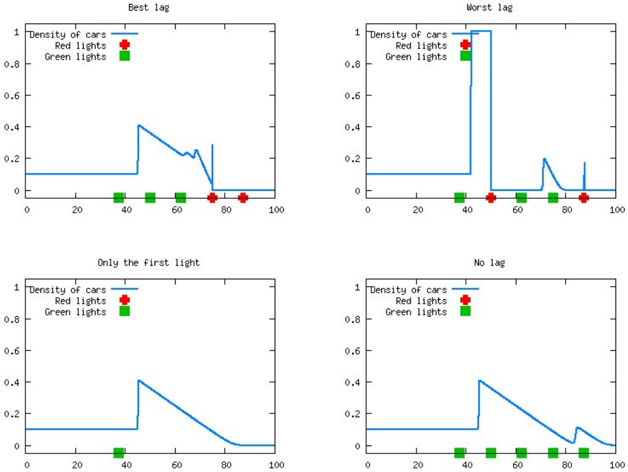

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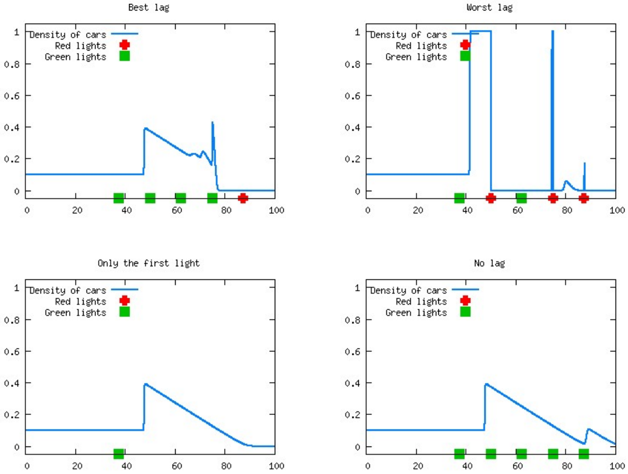

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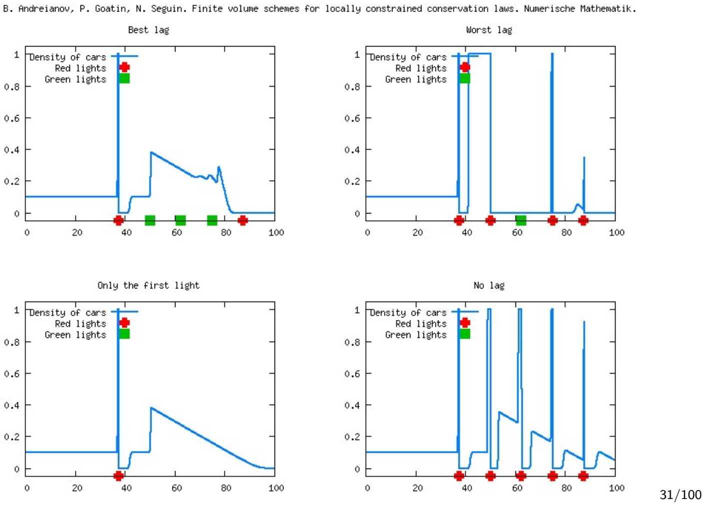

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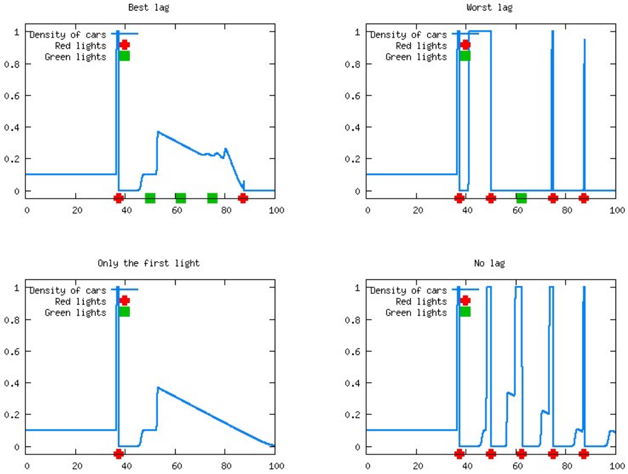

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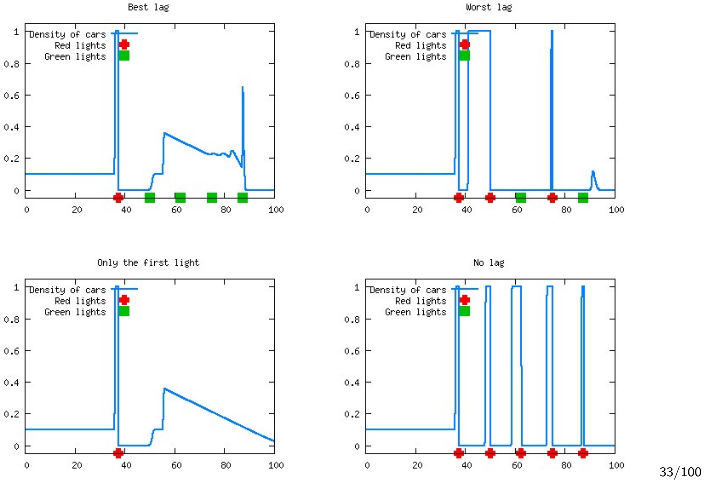

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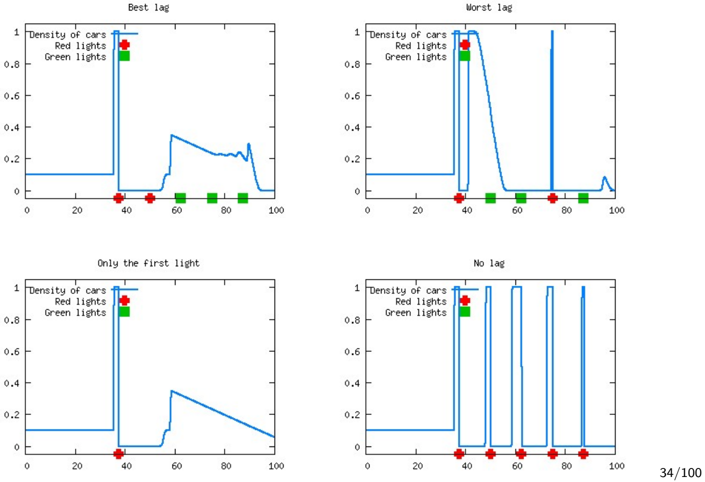

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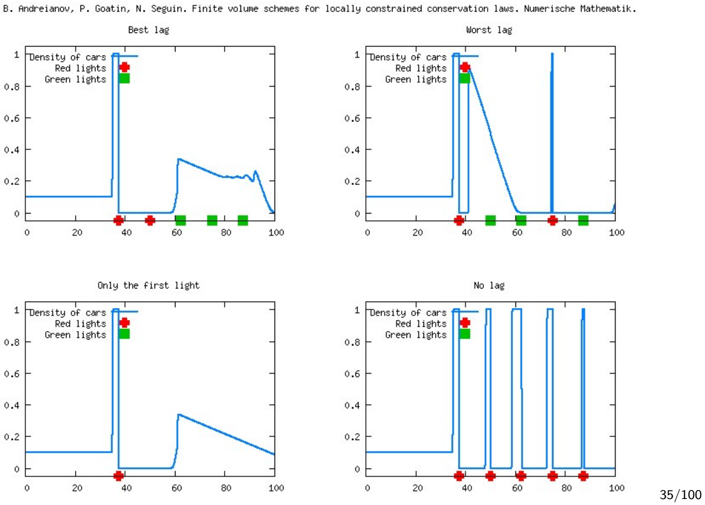

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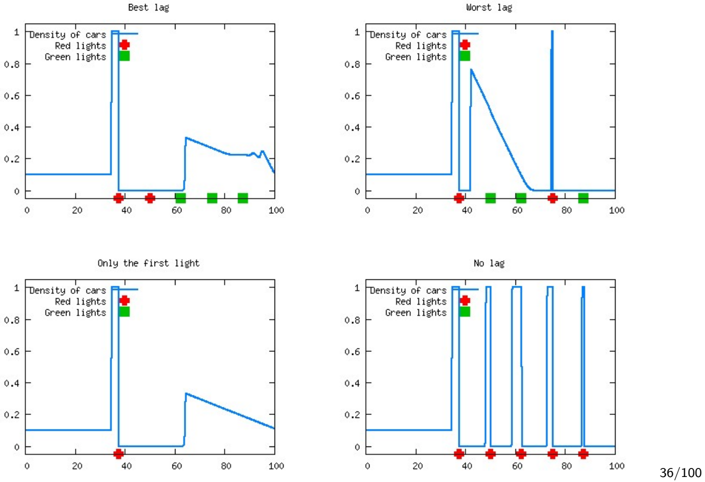

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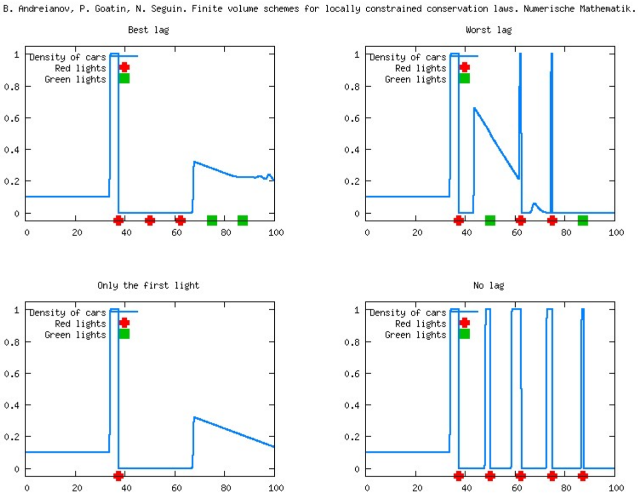

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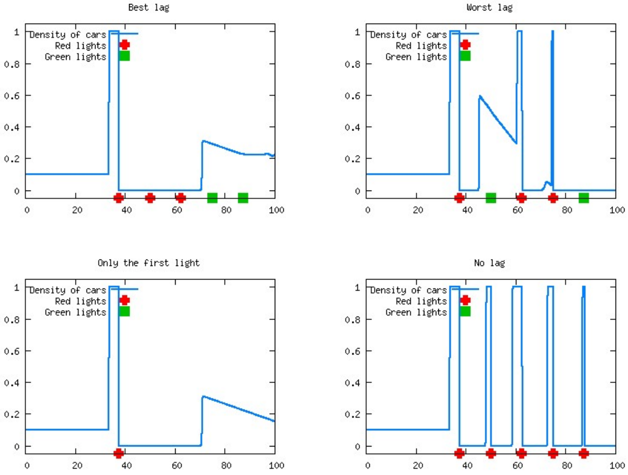

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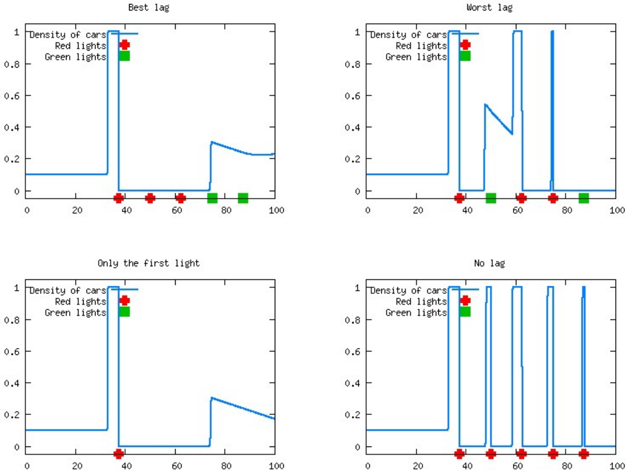

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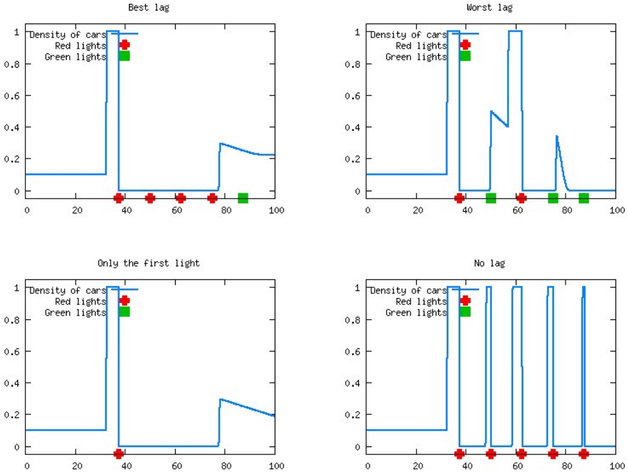

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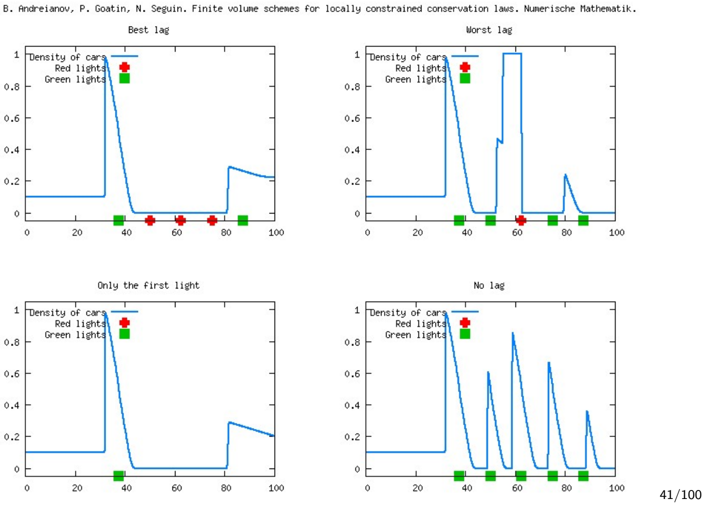

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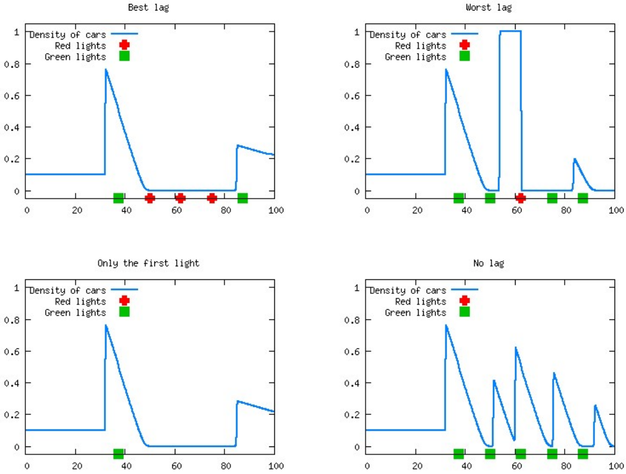

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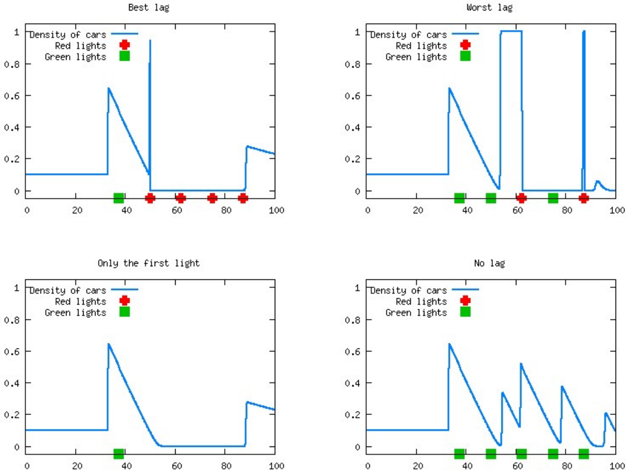

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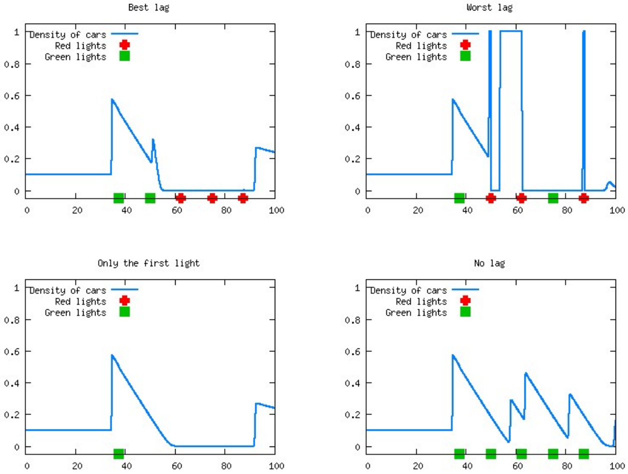
The green wave in traffic flow

B. Andreianov, P. Goatin, N. Seguin. Finite volume schemes for locally constrained conservation laws. Numerische Mathematik.



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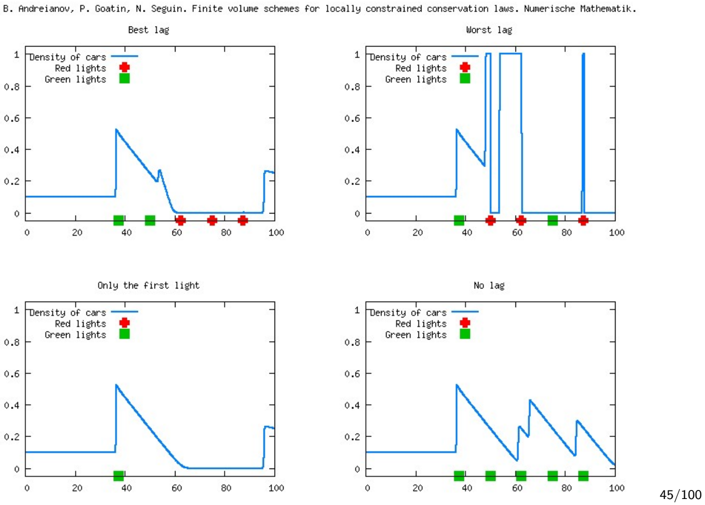
The green wave in traffic flow



The green wave in traffic flow

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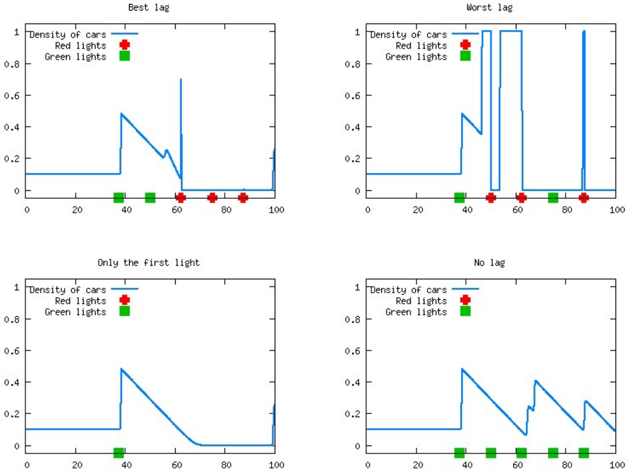

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The green wave in traffic flow

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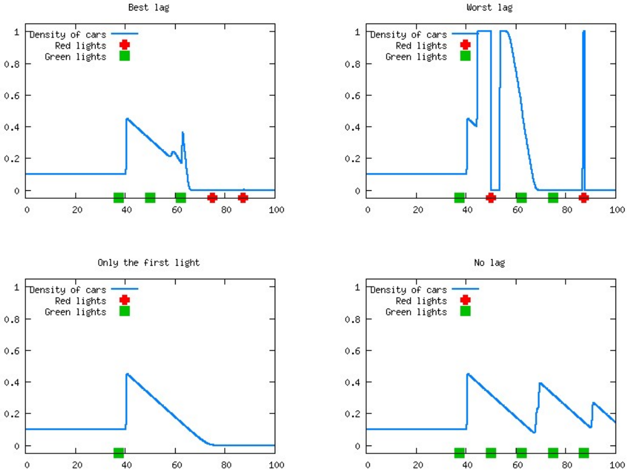

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The green wave in traffic flow

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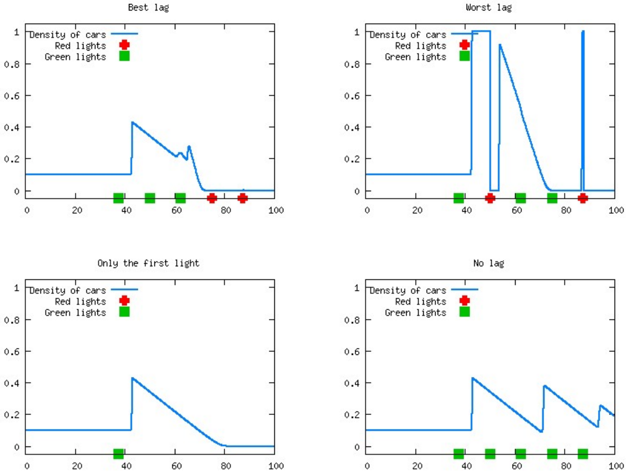

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The green wave in traffic flow

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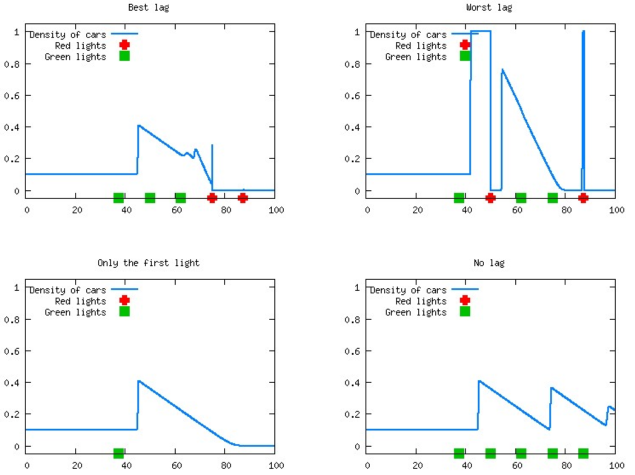

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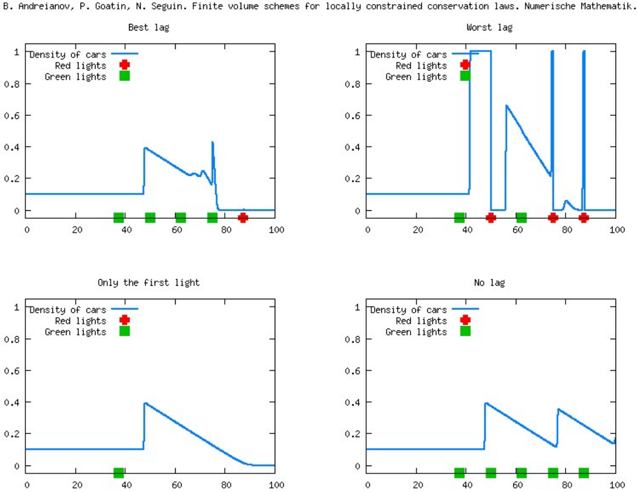
The green wave in traffic flow



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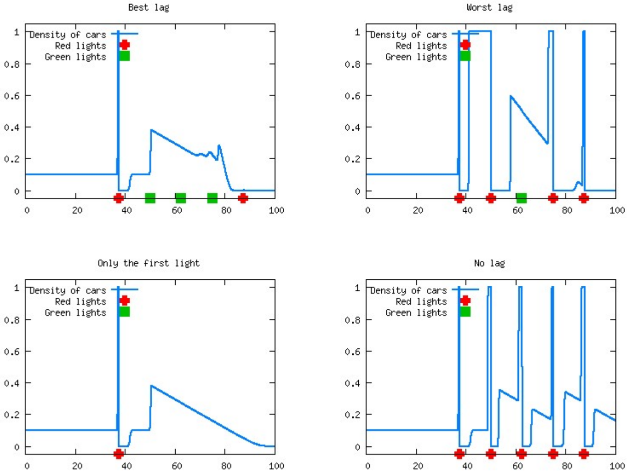

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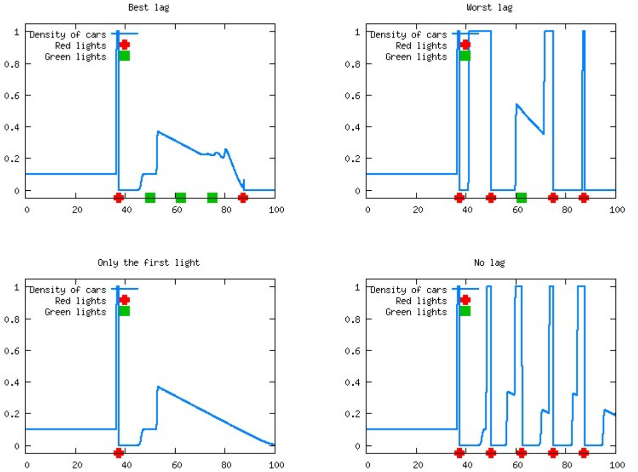

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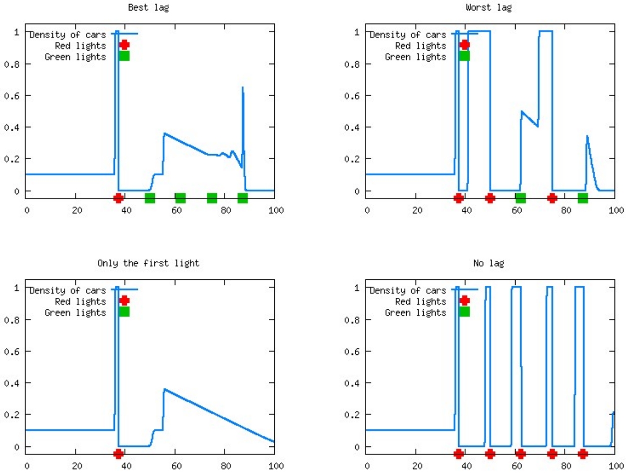

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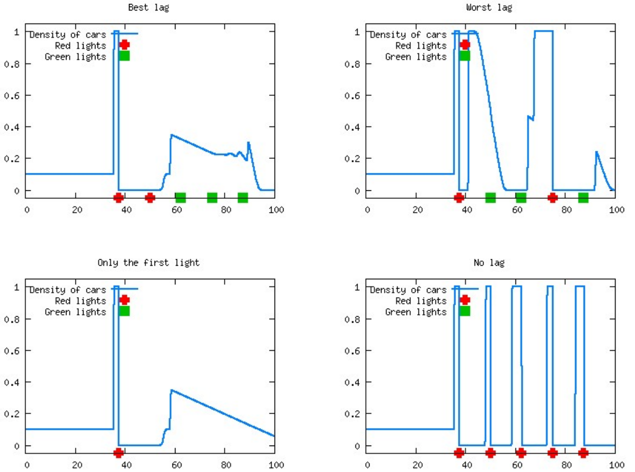

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The green wave in traffic flow

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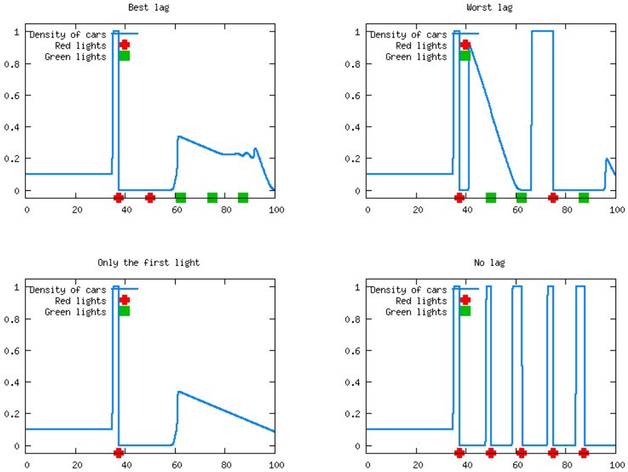

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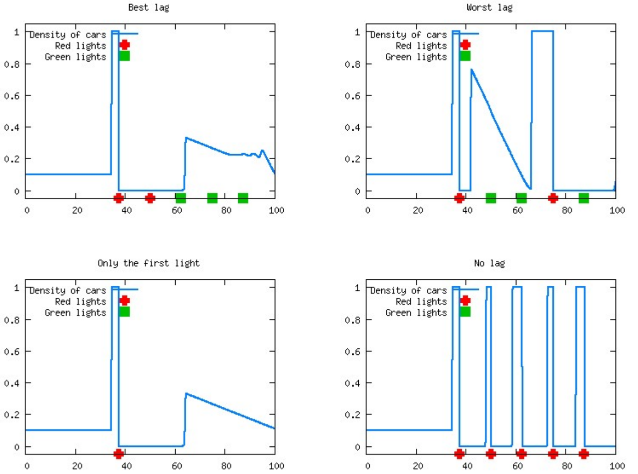

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The green wave in traffic flow

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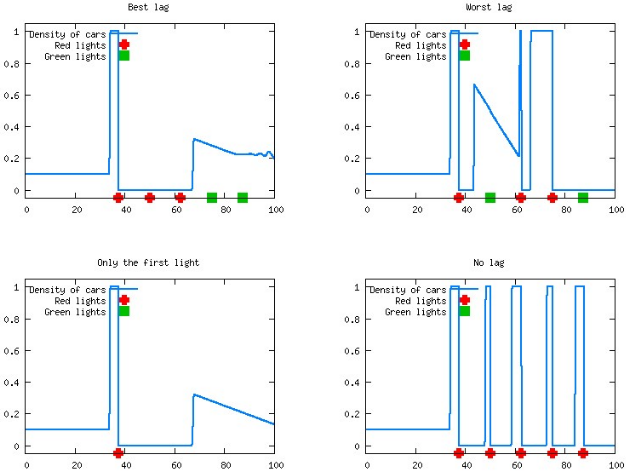

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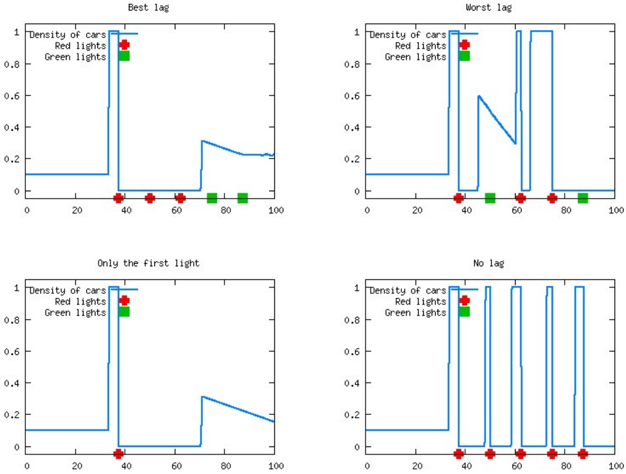

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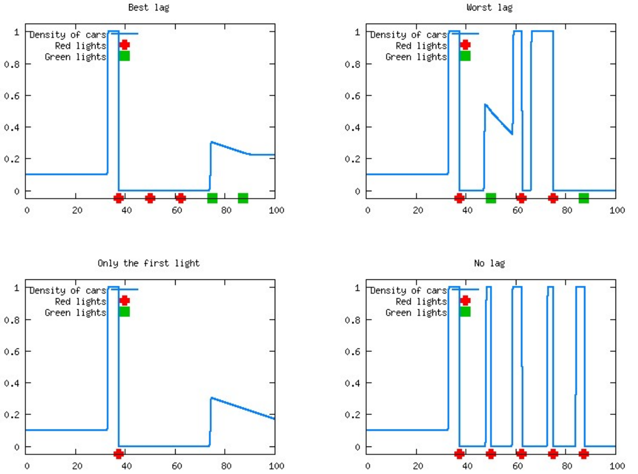

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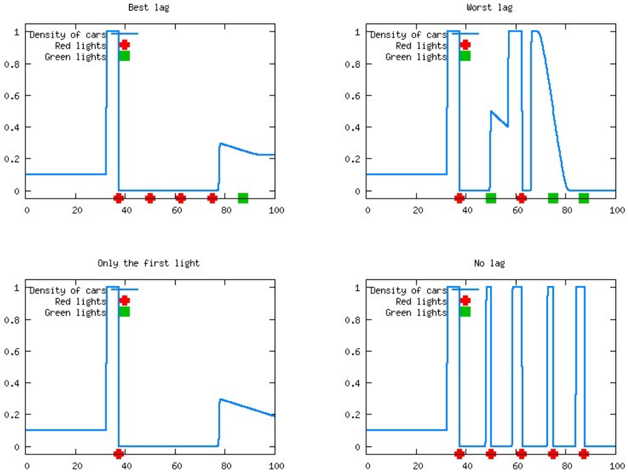

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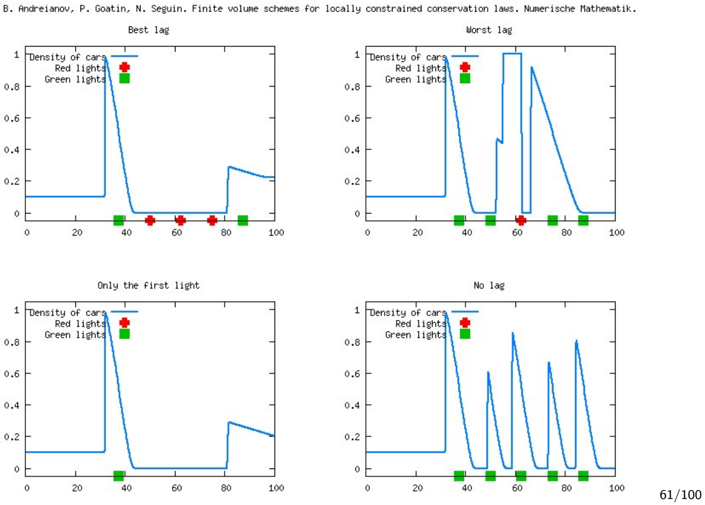

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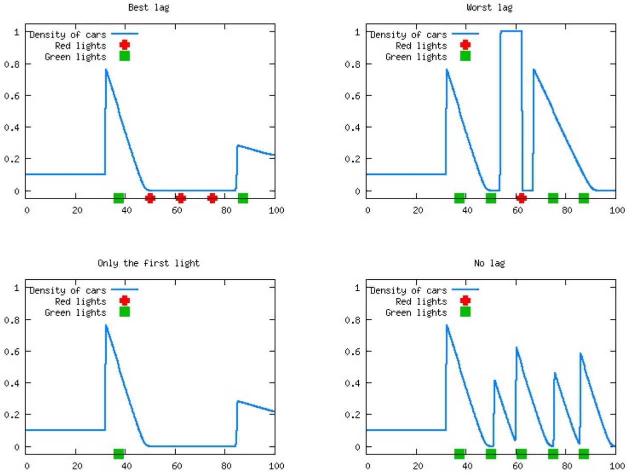

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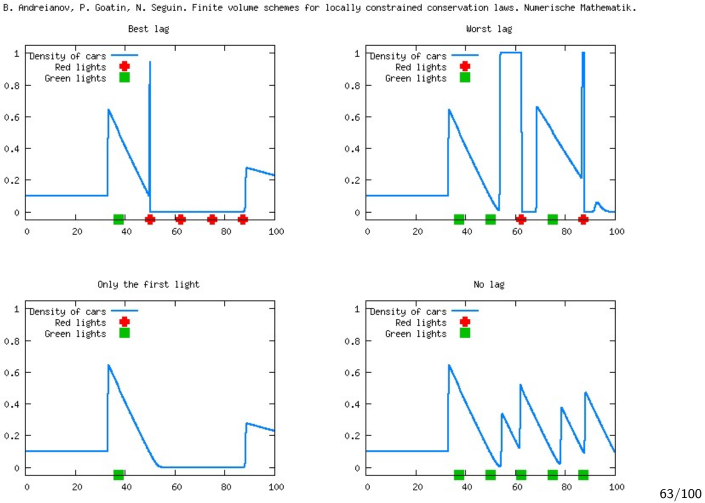

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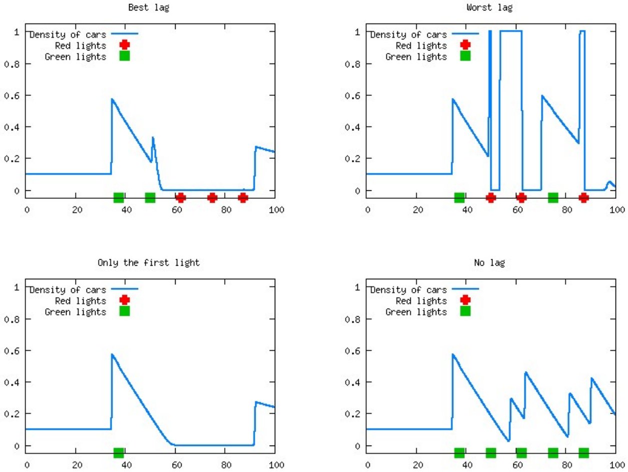

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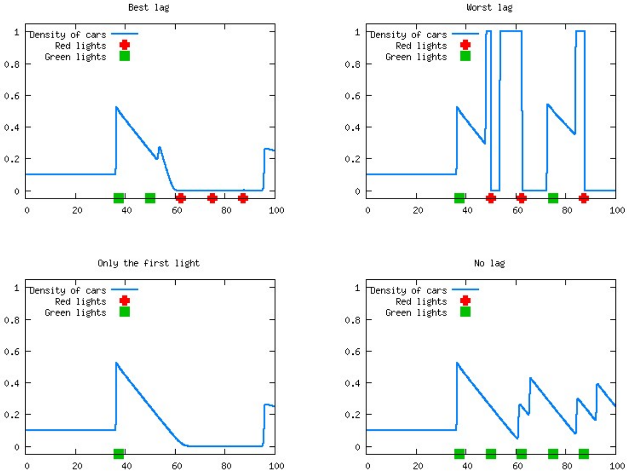

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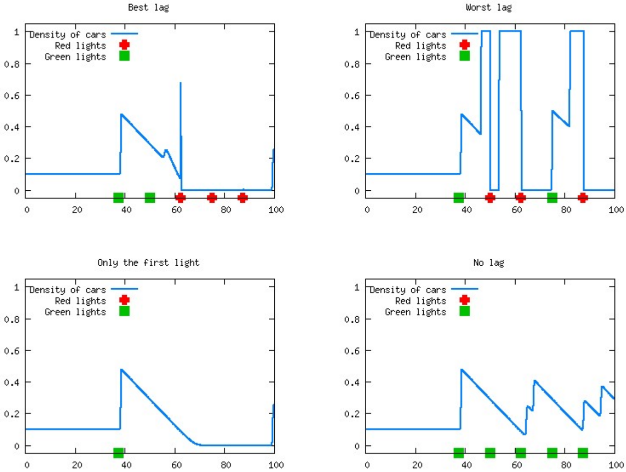

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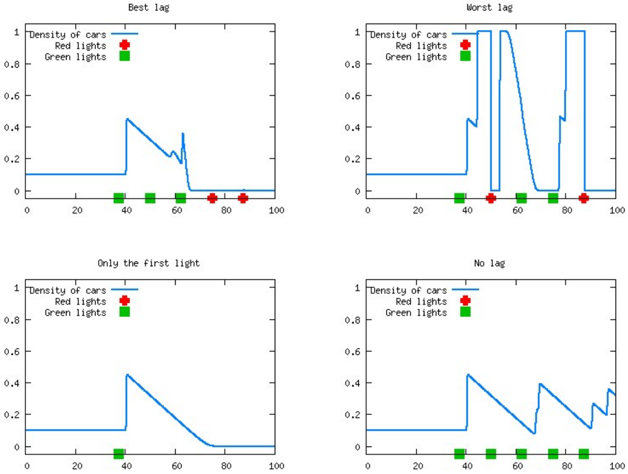

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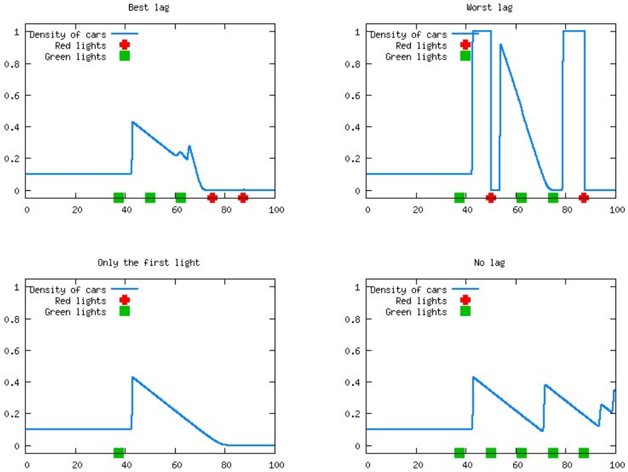

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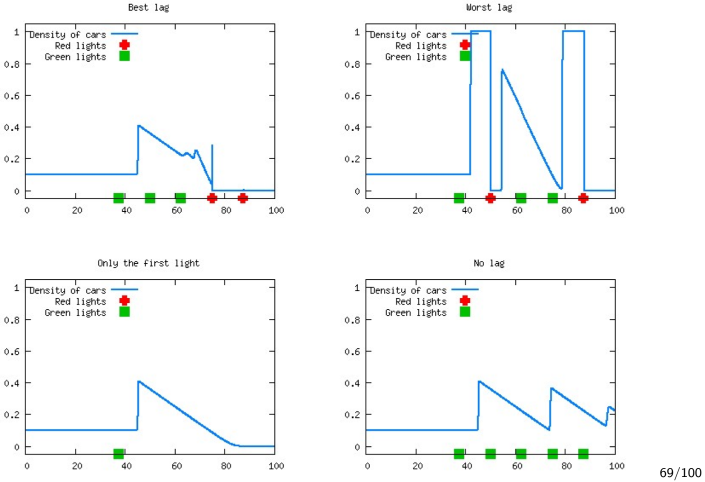

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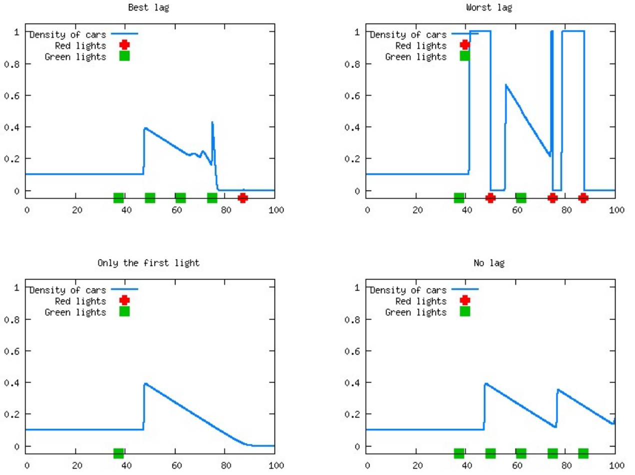

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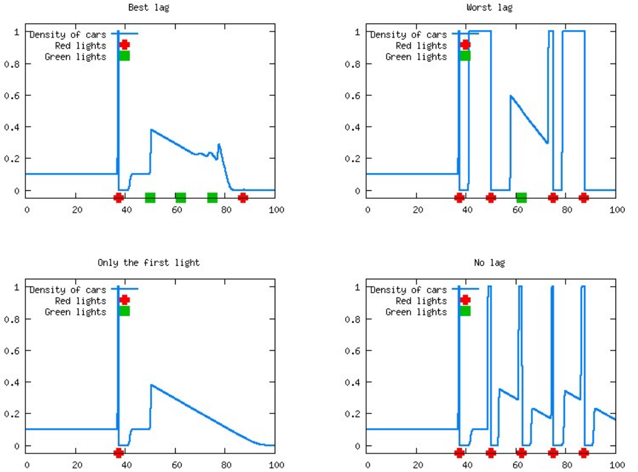

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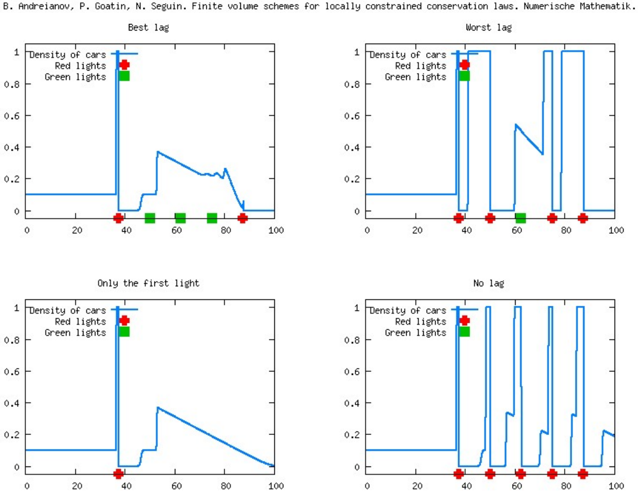
The green wave in traffic flow



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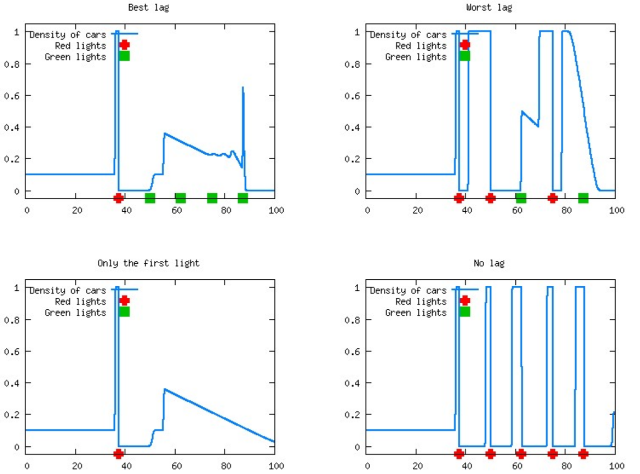

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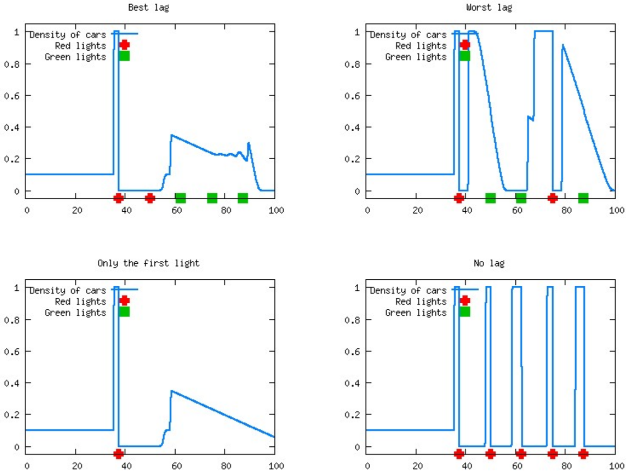

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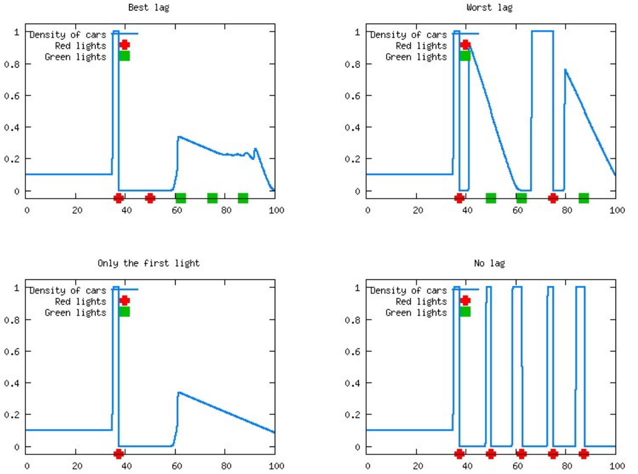

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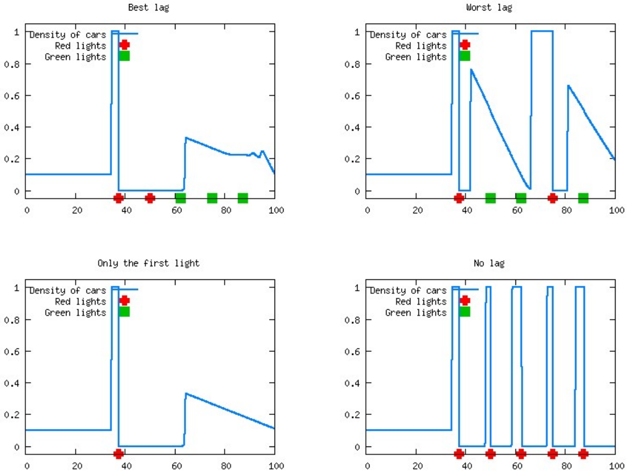

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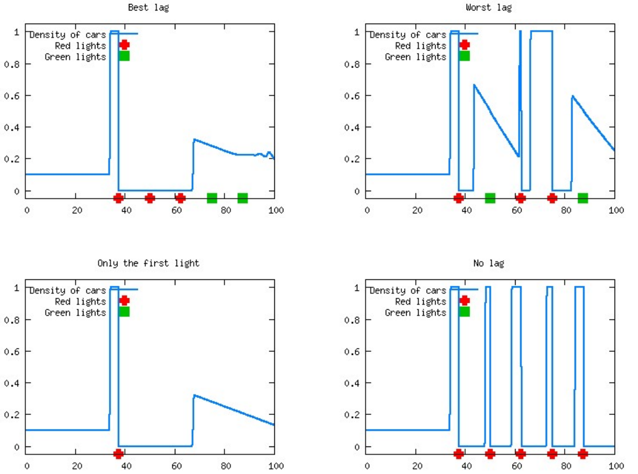

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The green wave in traffic flow

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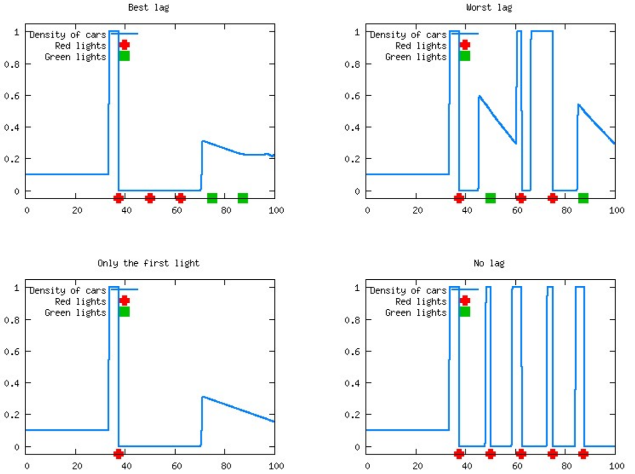

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The green wave in traffic flow

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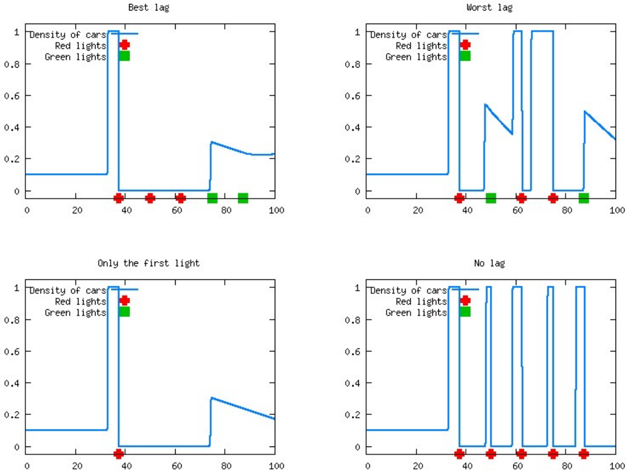

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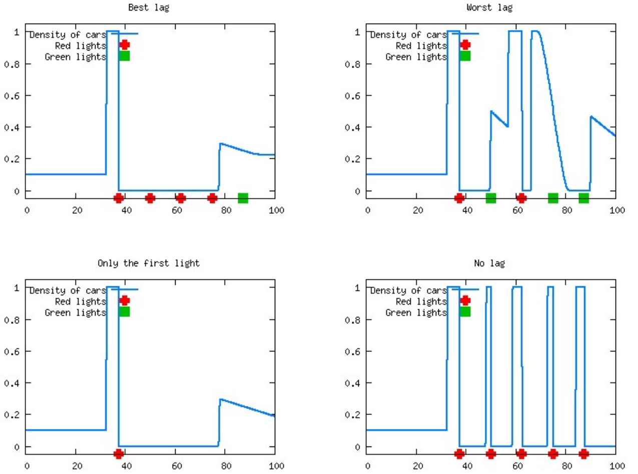

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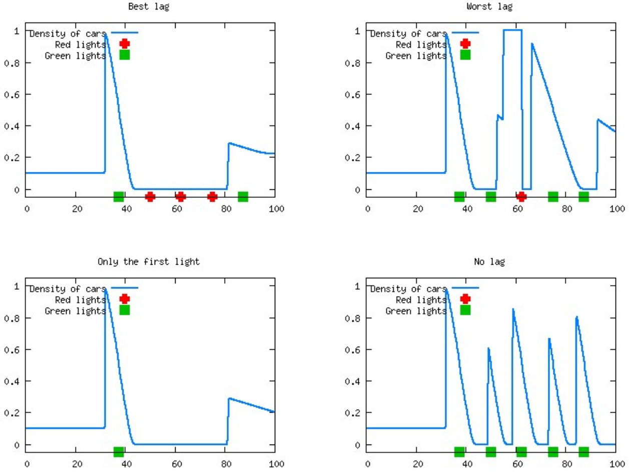

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The green wave in traffic flow

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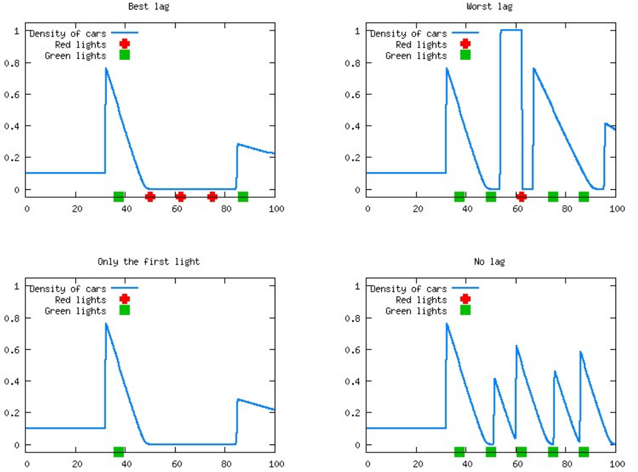

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The green wave in traffic flow

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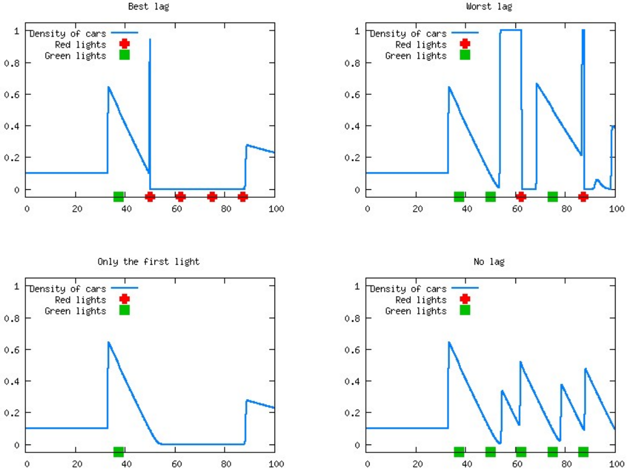

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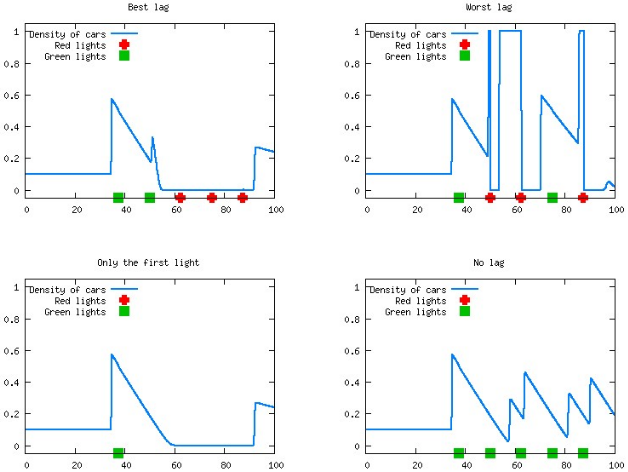

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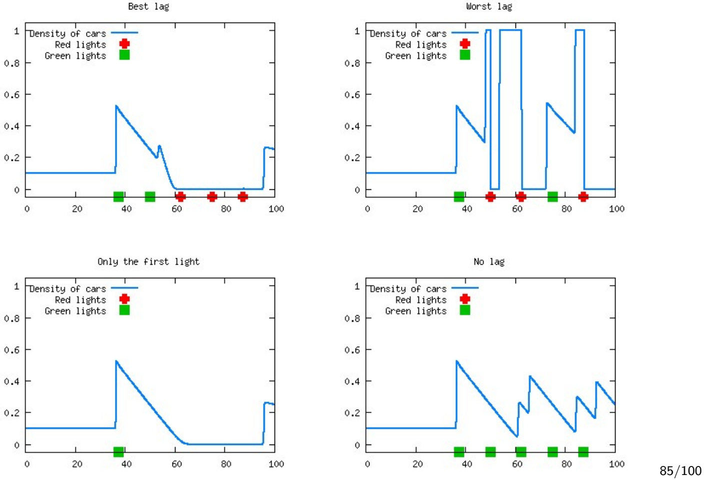

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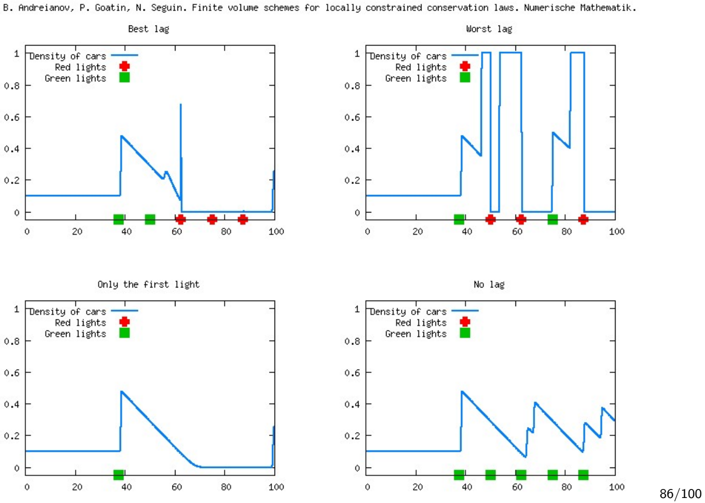

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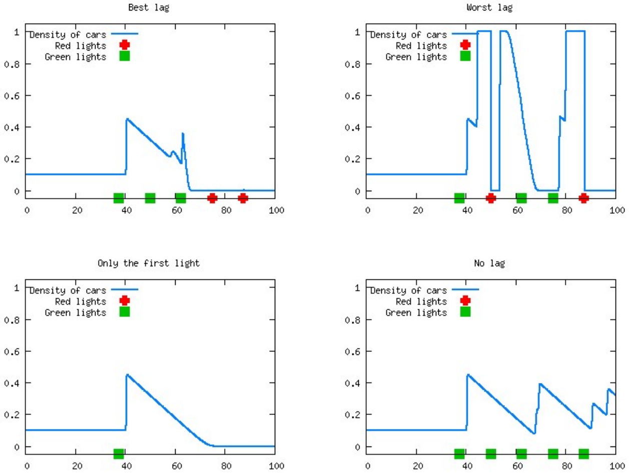

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The green wave in traffic flow

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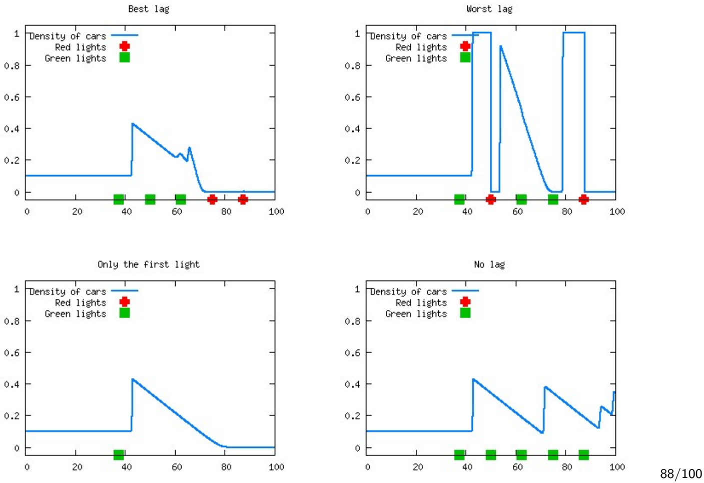

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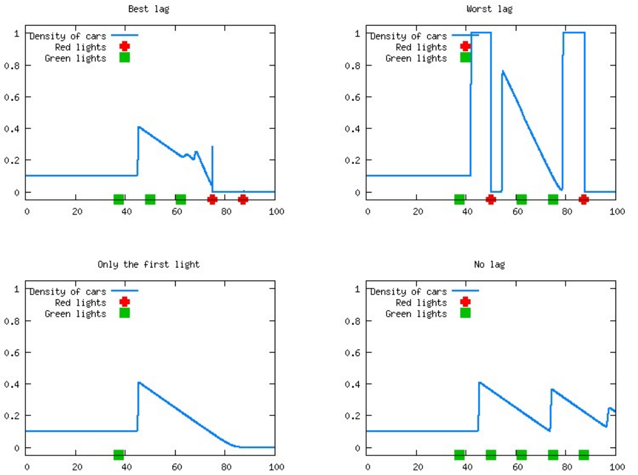

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The green wave in traffic flow

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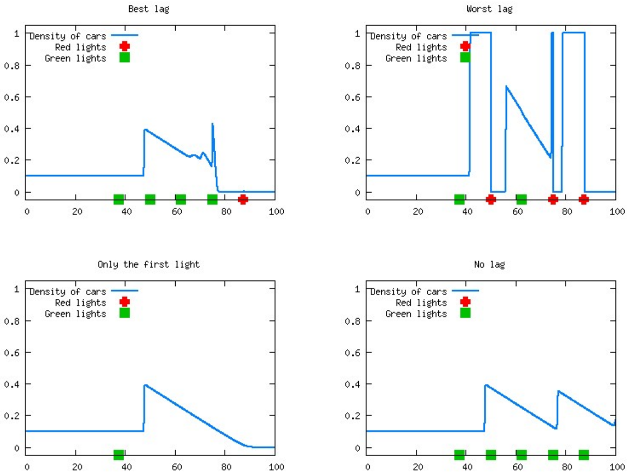

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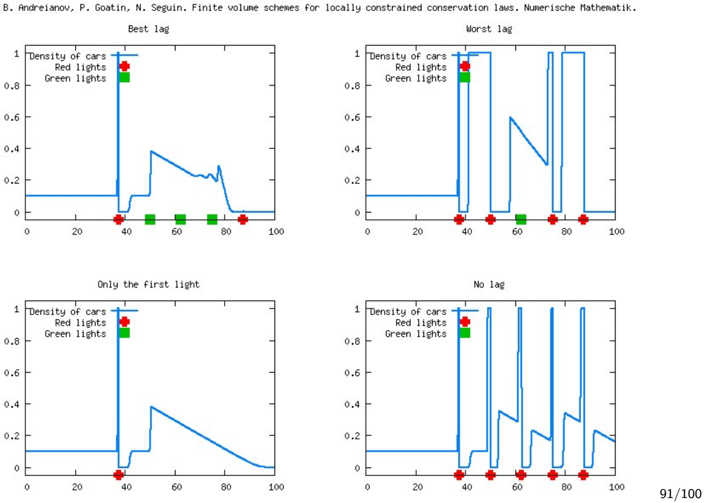

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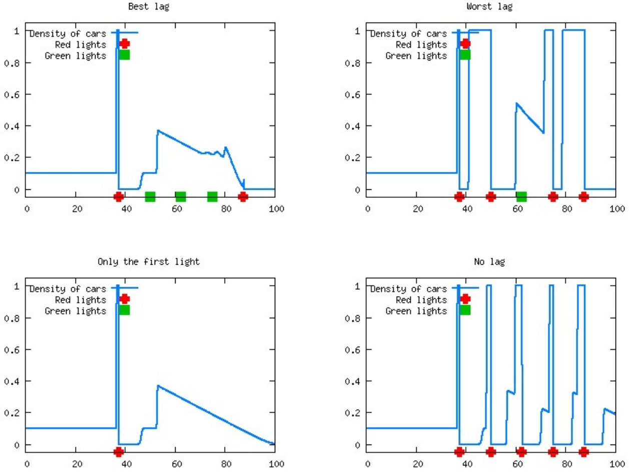

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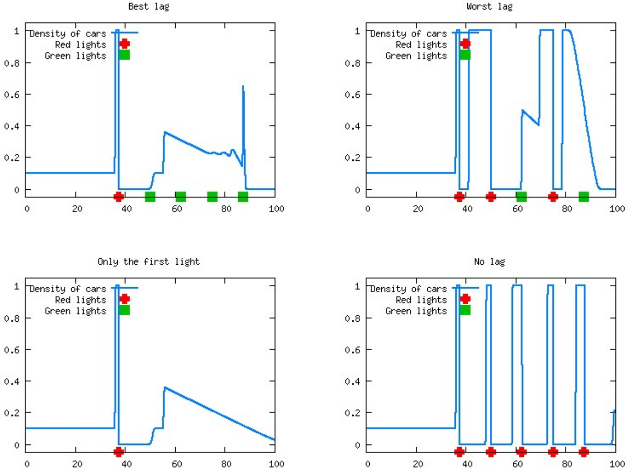

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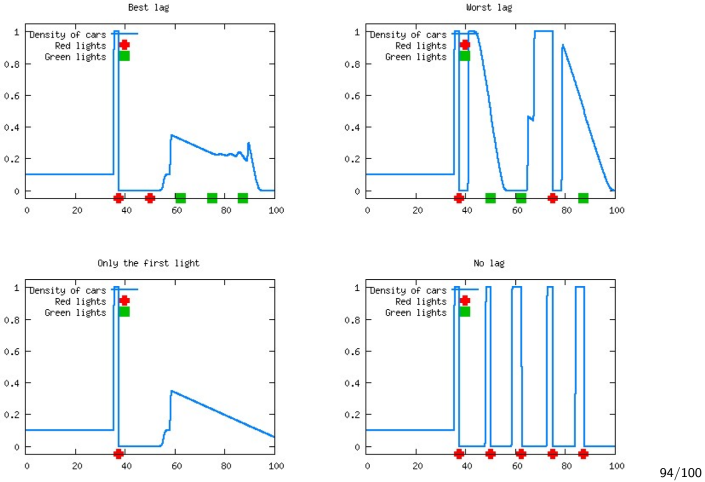

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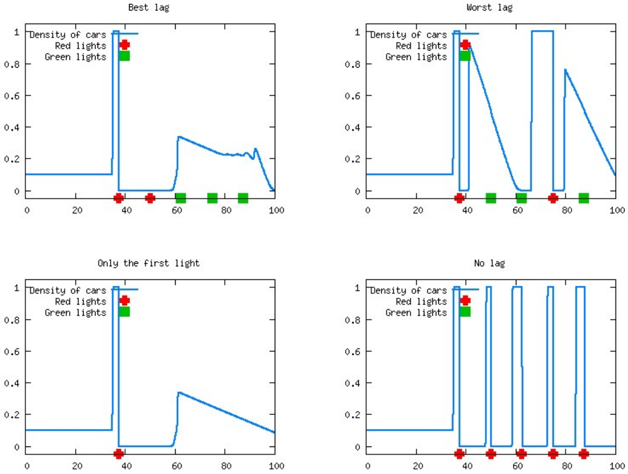

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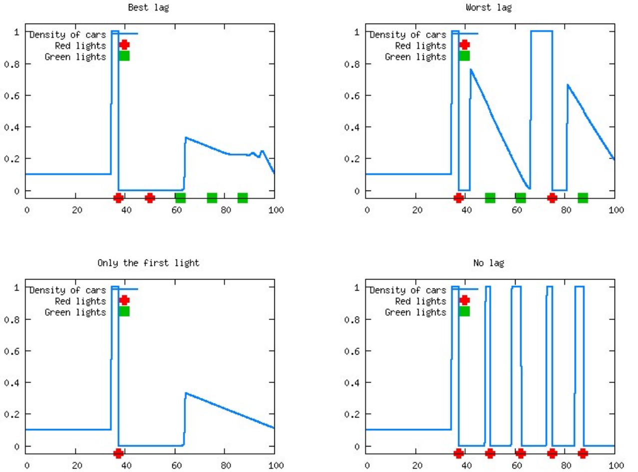

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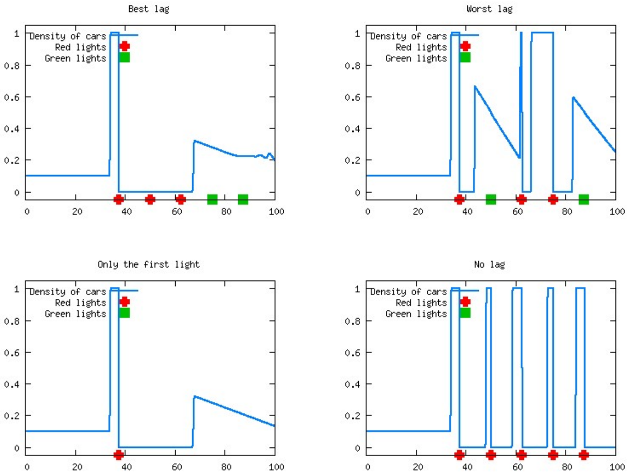

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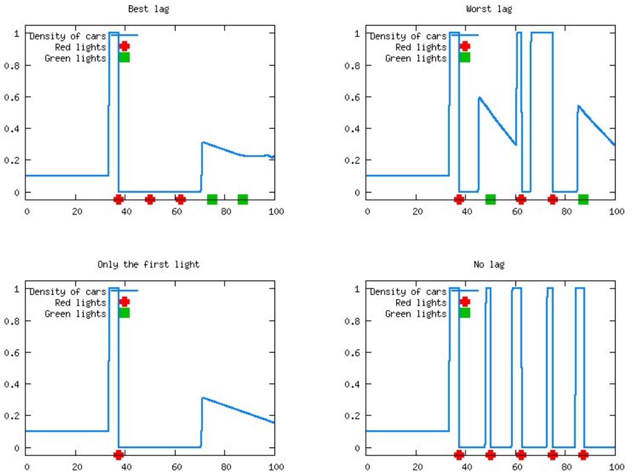

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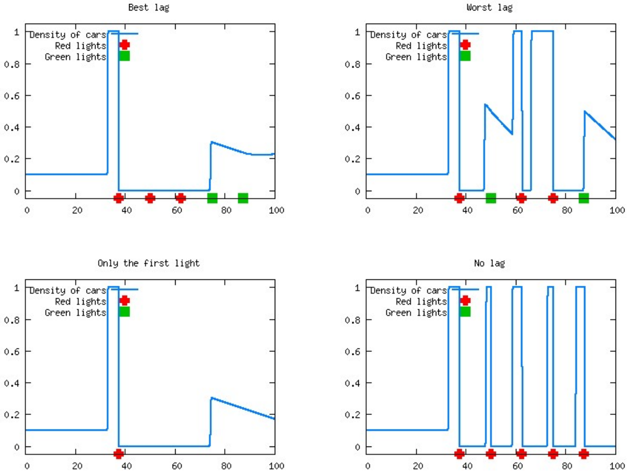

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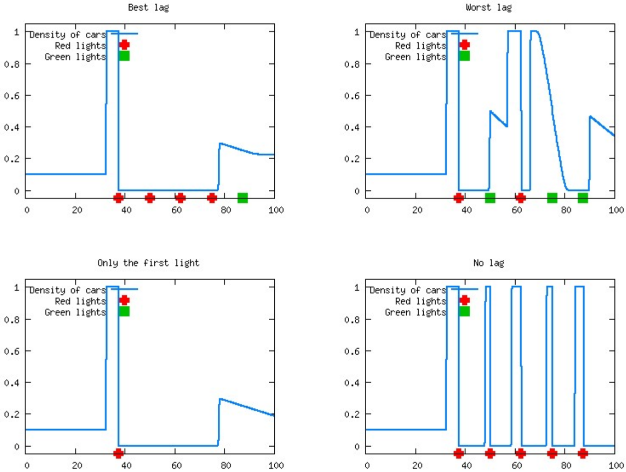
The green wave in traffic flow

B. Andreianov, P. Goatin, N. Seguin. Finite volume schemes for locally constrained conservation laws. Numerische Mathematik.



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Burgers equation with pointwise friction

The formal problem

$$\partial_t u + \partial_x(u^2/2) + \lambda u \partial_x \alpha = 0$$
$$\alpha(t,x) = \mathbf{H}(x)$$

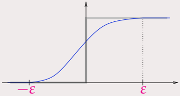
where $\lambda > 0$ is the drag coefficient

Regularization of the interface

$$\partial_t u + \partial_x(u^2/2) + \lambda u \partial_x \alpha = 0$$
$$\alpha(t,x) = \mathbf{H}_\varepsilon(x)$$

where $\mathbf{H}$ is replaced by $\mathbf{H}_\varepsilon \in \mathscr{C}^1(\mathbb{R})$, a nondecreasing function such that

$$\forall |x| \geqslant \varepsilon \quad \mathbf{H}_\varepsilon(x) = \mathbf{H}(x)$$



We first seek stationary solutions $U(x)$ for $|x| \geqslant \varepsilon$

Burgers equation with pointwise friction

Find all the pairs $(\kappa_-, \kappa_+) \in \mathbb{R}^2$ such that it exists U verifying

$$\begin{cases} (U^2/2)'(x) + \lambda\, U(x)\mathbf{H}'_\varepsilon(x) = 0, \quad x \in (-\varepsilon, \varepsilon) \\ U(-\varepsilon) = \kappa_- \\ U(\varepsilon) = \kappa_+ \end{cases} \tag{$\star$}$$

in the entropy weak sense

- Smooth parts: $(\star)$ becomes $U(x)(U + \lambda \mathbf{H}_\varepsilon)'(x) = 0$

Either $U(x) = 0$
Or $U(x) + \lambda \mathbf{H}_\varepsilon(x) = \mathsf{Cst}$

- Shock waves at $x_0 \in (-\varepsilon, \varepsilon)$: $(U(x_0^-) + U(x_0^+))/2 = 0$ and $U(x_0^-) > U(x_0^+)$

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in the entropy weak sense

The set $\mathcal{G}_\lambda \subset \mathbb{R}^2$ of admissible pairs is the union $\mathcal{G}_\lambda = \mathcal{G}_\lambda^1 \cup \mathcal{G}_\lambda^2 \cup \mathcal{G}_\lambda^3$, where

- $\mathcal{G}_\lambda^1 = \{(a, a-\lambda), a \in \mathbb{R}\}$. (classical Riemann invariant)
- $\mathcal{G}_\lambda^2 = [0, \lambda] \times [-\lambda, 0]$ (with shock wave)
- $\mathcal{G}_\lambda^3 = \{(a, b) \in (\mathbb{R}^+ \times \mathbb{R}^-) \setminus \mathcal{G}_\lambda^2,\ -\lambda \leqslant a + b \leqslant \lambda\}$ (with shock wave)

Burgers equation with pointwise friction

Proposition (Stability of the definition of the "nonconservative product")

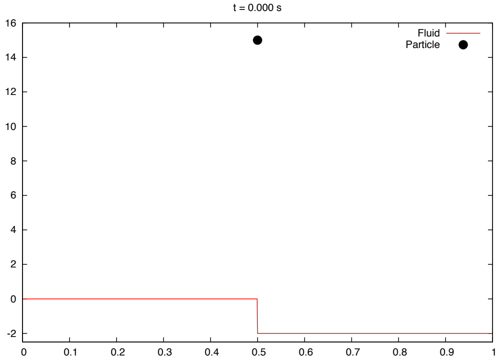
For all nondecreasing $\mathbf{H}_\varepsilon \in \mathscr{C}^1(\mathbb{R})$ such that $\mathbf{H}_\varepsilon(x) = \mathbf{H}(x)$ if $|x| > \varepsilon$,
it exists a weak entropy U to

$$\begin{cases} (U^2/2)'(x) + \lambda\, U(x)\mathbf{H}_\varepsilon'(x) = 0 \\ U(-\varepsilon) = \kappa_- \\ U(\varepsilon) = \kappa_+ \end{cases}$$

if and only if $(\kappa_-, \kappa_+) \in \mathcal{G}_\lambda$.

- The germ $\mathcal{G}_\lambda$ is dissipative
  $\Longrightarrow$ Uniqueness + Lipshitz-continuous dependence
- Construction of monotone numerical schemes preserving $\mathcal{G}_\lambda^1$
  $\Longrightarrow$ Strong convergence + Existence

Interaction between a particle and a Burgers fluid



$$\begin{cases} \partial_t u + \partial_x (u^2/2) = \lambda \ (h'(t) - u) \ \delta_0(x - h(t)) \\ m \ h''(t) = \lambda \ (u(t, h(t)) - h'(t)) \end{cases}$$

[Lagoutière, Seguin, Takahashi '08], [Andreianov, Lagoutière, Seguin, Takahashi '10]
See also [Borsche, Colombo, Gravello '10 & '12]

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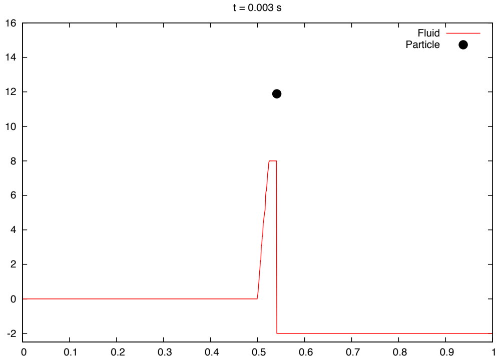

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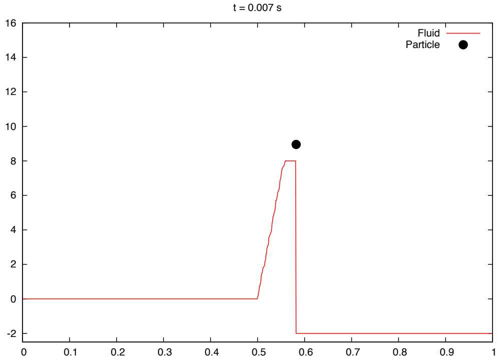

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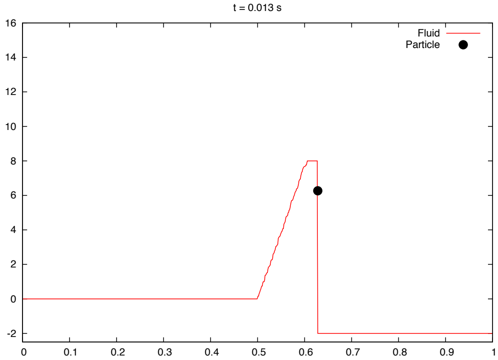

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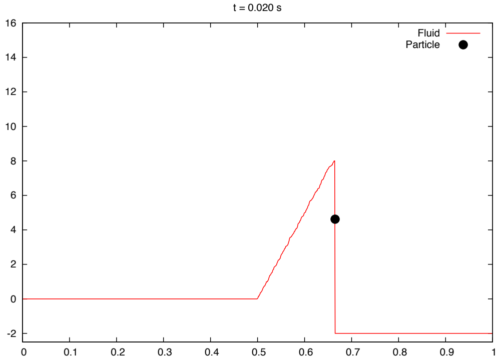

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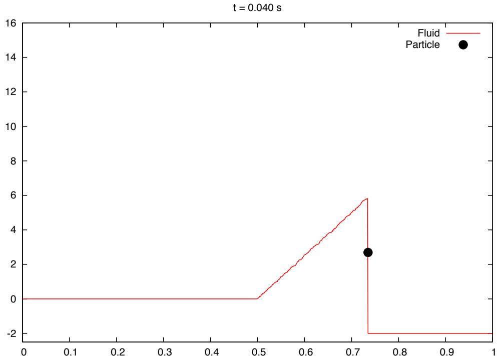

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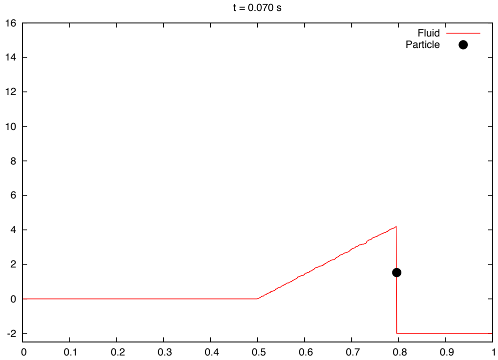

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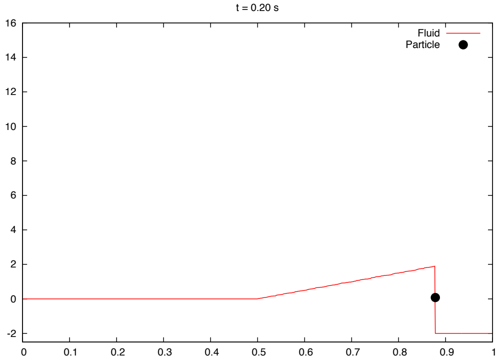

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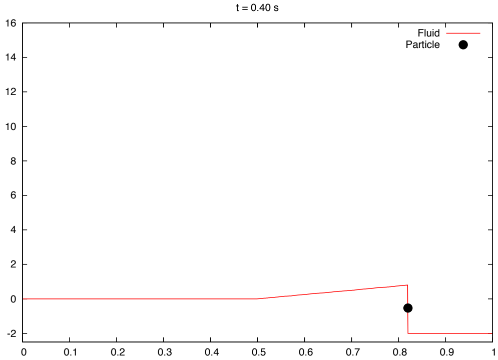

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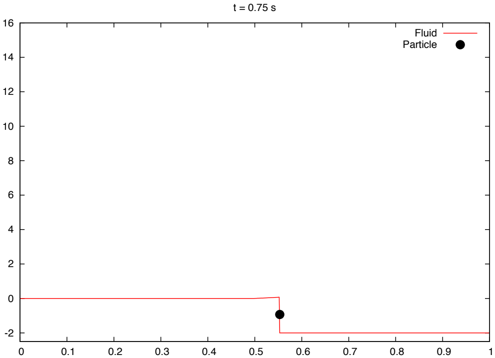

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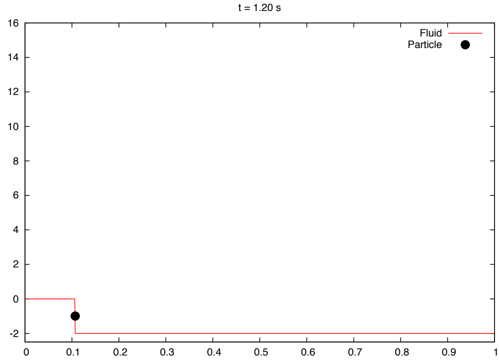

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with $\beta \in [0,1]$

- Same theory (existence, uniqueness, convergence of adapted schemes)

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- Same theory (existence, uniqueness, convergence of adapted schemes)
- BUT, if $\lambda < 0$
  - the case $\beta = 0$ is ill-posed (non-uniqueness)
  - the case $\beta = 1$ is well-posed ($\mathbf{L}^1$-contraction becomes Lipschitz-continuous semi-group)

Conclusion

- Classical theory of Kruzhkov can be adapted in some scalar cases
  - conservative equations
  - nonconvervative equations in some cases
  - if not, no chance to select a "good" solution (continuous dependence fails)
- Systems of singular balance laws
  - Mainly, available results only on the Riemann problem
  - Well-posedness in the conservative case
  - Extension of the notion of dissipative germs to systems?