Stability of the free plasma-vacuum interface

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Plan

1. Plasma-vacuum interface problem
    - Formulation of the problem
    - Reduction to the fixed domain
2. Analysis of linearized stability
    - Linearized stability in H^1
    - Proof
        - Hyperbolic regularization
        - Secondary symmetrization
    - High-order energy estimate
3. Nonlinear stability
    - Nash-Moser technique
    - Main result

Ideal compressible MHD

Consider the ideal compressible MHD equations:

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho\, v) = 0\,, \\ \partial_t(\rho\, v) + \nabla\cdot(\rho\, v\otimes v - H\otimes H) + \nabla(p + \frac{1}{2}|H|^2) = 0\,, \\ \partial_t H - \nabla\times(v\times H) = 0\,, \\ \partial_t\big(\rho e + \frac{1}{2}(\rho|v|^2 + |H|^2)\big) \\ \qquad\qquad + \nabla\cdot\big(\rho v(e + \frac{1}{2}|v|^2) + vp + H\times(v\times H)\big) = 0\,, \\ \nabla\cdot H = 0\,, \end{cases} \tag{1}$$

with
ρ density, S entropy, v velocity field, H magnetic field,
$p = p(\rho, S)$ pressure (such that $p'_\rho > 0$), $e = e(\rho, S)$ internal energy.

The total pressure is $q = p + \frac{1}{2}|H|^2$.

In terms of $U = (q, v, H, S)^T$ system (1) admits the symmetrization

$$\begin{pmatrix} \rho_p/\rho & \underline{0}^T & -(\rho_p/\rho)H & 0 \\ \underline{0} & \rho I_3 & 0_3 & \underline{0} \\ -(\rho_p/\rho)H^T & 0_3 & I_3 + (\rho_p/\rho)H \otimes H & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & 1 \end{pmatrix} \partial_t \begin{pmatrix} q \\ v \\ H \\ S \end{pmatrix} +$$

$$\begin{pmatrix} (\rho_p/\rho)v \cdot \nabla & \nabla\cdot & -(\rho_p/\rho)Hv \cdot \nabla & 0 \\ \nabla & \rho v \cdot \nabla I_3 & -H \cdot \nabla I_3 & \underline{0} \\ -(\rho_p/\rho)H^T v \cdot \nabla & -H \cdot \nabla I_3 & (I_3 + (\rho_p/\rho)H \otimes H)v \cdot \nabla & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & v \cdot \nabla \end{pmatrix} \begin{pmatrix} q \\ v \\ H \\ S \end{pmatrix} = 0 \tag{2}$$

where $\underline{0} = (0, 0, 0)^T$.

We write system (2) as

$$A_0(U)\partial_t U + \sum_{j=1}^{3} A_j(U)\partial_j U = 0, \tag{3}$$

which is symmetric hyperbolic provided the hyperbolicity condition $A_0 > 0$ holds:

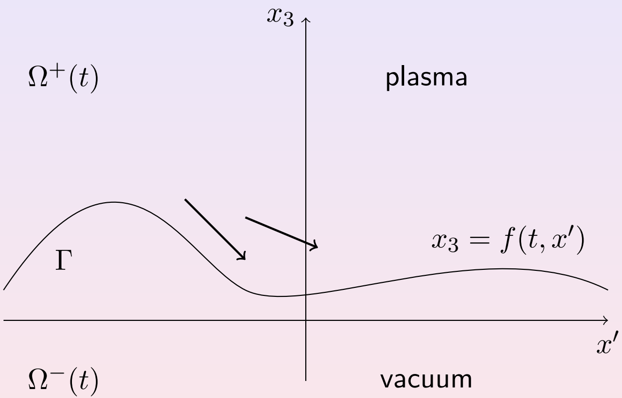
$$\rho > 0, \quad \rho_p > 0.$$

Given a smooth hypersurface

$$\Gamma(t) = \{x_3 = f(t, x')\} \qquad \text{in } [0, T] \times \mathbb{R}^3,$$

we denote $\Omega^{\pm}(t) = \mathbb{R}^3 \cap \{x_3 \gtrless f(t, x')\}$ (here $x' = (x_1, x_2)$).

The plasma is governed by equations (3) in the region $\Omega^{+}(t) = \mathbb{R}^3 \cap \{x_3 > f(t, x')\}$.


x_3
$\Omega^+(t)$
plasma
Γ
$x_3 = f(t, x')$
x'
$\Omega^-(t)$
vacuum

The vacuum region is $\Omega^-(t) = \mathbb{R}^3 \cap \{x_3 < f(t,x')\}$, where we assume the so-called *pre-Maxwell dynamics*:

$$\nabla \times \mathcal{H} = 0, \qquad \operatorname{div} \mathcal{H} = 0, \tag{4}$$

$$\nabla \times E = -\partial_t \mathcal{H}, \qquad \operatorname{div} E = 0, \tag{5}$$

$\mathcal{H}$ denotes the vacuum magnetic field and E the electric field.

As usual in nonrelativistic MHD, we neglect the displacement current $(1/c)\,\partial_t E$, where c is the speed of light.

From (5) the electric field E is a secondary variable that may be computed from the magnetic field $\mathcal{H}$. Hence, in the vacuum only one basic variable is needed, viz. $\mathcal{H}$, satisfying (4).

On the moving interface $\Gamma(t)$ the plasma and the vacuum magnetic fields are related by:

$$\partial_t f = v_N, \quad [q] = 0, \quad H_N = 0, \quad \mathcal{H}_N = 0 \quad \text{on } \Gamma(t), \tag{6}$$

where $v_N = v \cdot N$, $H_N = H \cdot N$, $\mathcal{H}_N = \mathcal{H} \cdot N$, $N = (-\partial_1 f, -\partial_2 f, 1)$, and $[q] = q|_\Gamma - \frac{1}{2}|\mathcal{H}|^2_{|\Gamma}$.

The interface $\Gamma(t)$ moves with the plasma. The total pressure is continuous across $\Gamma(t)$. The magnetic field on both sides is tangent to $\Gamma(t)$.

The function f describing the interface is one unknown of the problem, i.e. this is a free boundary problem.

System (4) for the vacuum magnetic field $\mathcal{H}$,

$$\nabla \times \mathcal{H} = 0, \qquad \operatorname{div} \mathcal{H} = 0, \tag{4}$$

is elliptic. Plasma-vacuum problem (3), (4) is a coupled hyperbolic-elliptic system.

In (4) time t plays the role of a parameter. Time dependence of $\mathcal{H}$ comes from the coupling with the plasma variables through the boundary conditions (6) at the moving front $\Gamma(t)$.

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System (3), (4), (6) is supplemented with initial conditions

$$\begin{aligned} &U(0,x) = U_0(x)\,, \quad \mathcal{H}(0,x) = \mathcal{H}_0(x)\,, \quad x \in \Omega^{\pm}(0)\,, \\ &f(0,x') = f_0(x')\,, \quad x' \in \Gamma(0), \end{aligned} \tag{7}$$

where $\operatorname{div} H_0 = 0$ in $\Omega^+(0)$, $\operatorname{div} \mathcal{H}_0 = 0$ in $\Omega^-(0)$.

Motivation from astrophysics:

the study of stars or the solar corona

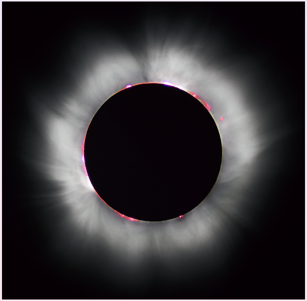
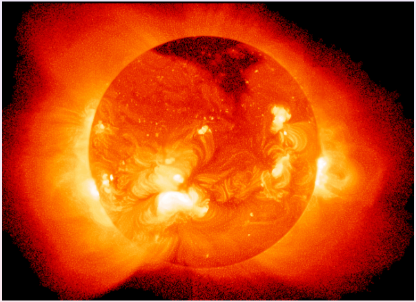
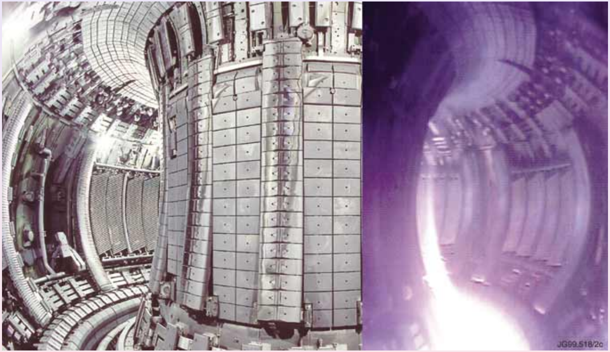


Image by Luc Viatour.



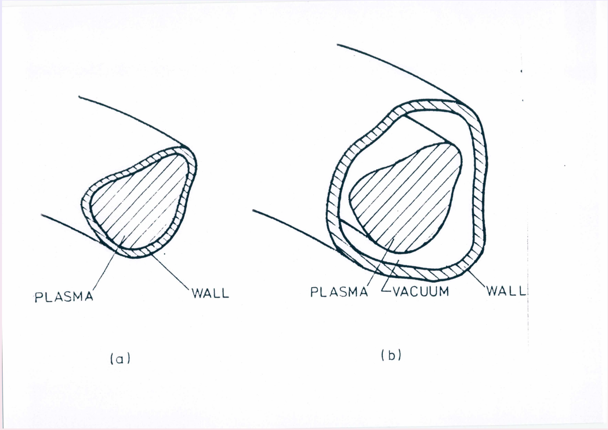
From Yohkoh satellite (Courtesy by JAXA)

Other motivation: the study of magnetic confinement



USSR stamp 1987

Tunnel at Monte Carlo (Courtesy by JET)



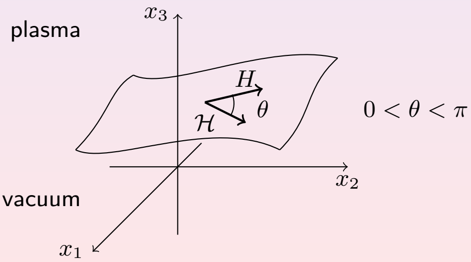
A toroidal plasma configuration: (a) surrounded by a perfectly conducting wall; (b) isolated from a wall by a vacuum region.

The stability condition

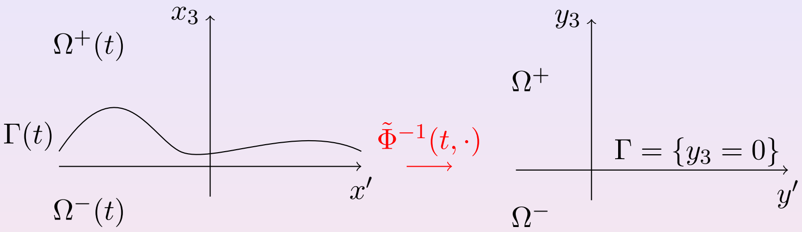
Our goal is to prove the solvability of (3), (4), (6), (7) under the stability condition

$$|H \times \mathcal{H}| > 0 \qquad \text{on } [0,T] \times \Gamma, \tag{8}$$

i.e. the magnetic fields on the two sides of the free-boundary are not collinear.



Reduction to the fixed domain



Change of variables

$$\tilde{\Phi}(t,\cdot) : y = (y', y_3) \to x = (x', x_3)$$

such that

$$x' = y', \qquad x_3 = \tilde{\Phi}(t,y),$$

$$\tilde{\Phi}(t,y',0) = f(t,x'), \quad \partial_{y_3}\tilde{\Phi}(t,y) > 0.$$

We write again x instead of y.

Possible choice:

$$\tilde{\Phi}(t, x', x_3) = x_3 + f(t, x')$$

[Majda, Proc. AMS 1983], [Métivier, 2003] for uniformly stable shocks.

We consider a different change of variables, inspired from [Lannes, JAMS 2005].

Lemma

Let $m \geq 3$ *be an integer. For all* $T > 0$, *and for all* $f \in \cap_{j=0}^{m-1} \mathcal{C}^j([0,T]; H^{m-j-0.5}(\mathbb{R}^2))$, *satisfying without loss of generality* $\|f\|_{\mathcal{C}([0,T];H^2(\mathbb{R}^2))} \leq 1$, *there exists a function* $\Psi \in \cap_{j=0}^{m-1} \mathcal{C}^j([0,T]; H^{m-j}(\mathbb{R}^3))$ *such that the function*

$$\Phi(t,x) := \big(x', x_3 + \Psi(t,x)\big), \qquad (t,x) \in [0,T] \times \mathbb{R}^3, \tag{9}$$

defines an H^m*-diffeomorphism of* $\mathbb{R}^3$ *for all* $t \in [0,T]$. *Moreover, there holds* $\partial_t^j \Phi \in \mathcal{C}([0,T]; H^{m-j}(\mathbb{R}^3))$ *for* $j = 0, \ldots, m-1$, $\Phi(t,x',0) = (x', f(t,x'))$, $\partial_3 \Phi(t,x',0) = (0,0,1)$.

We set

$$\Omega^{\pm} := \mathbb{R}^3 \cap \{x_3 \gtrless 0\} \quad , \Gamma := \mathbb{R}^3 \cap \{x_3 = 0\},$$

and introduce the change of independent variables defined by (9)

$$\widetilde{U}(t,x) := U(t,\Phi(t,x)), \quad \widetilde{\mathcal{H}}(t,x) := \mathcal{H}(t,\Phi(t,x)).$$

Dropping for convenience tildes in $\widetilde{U}$ and $\widetilde{\mathcal{H}}$, problem (3), (4), (6), (7) can be reformulated on the fixed reference domains $\Omega^{\pm}$ as

$$\mathbb{P}(U,\Psi) = 0 \quad \text{in } [0,T]\times\Omega^+, \quad \mathbb{V}(\mathcal{H},\Psi) = 0 \quad \text{in } [0,T]\times\Omega^-, \tag{10}$$

$$\mathbb{B}(U,\mathcal{H},f) = 0 \quad \text{on } [0,T]\times\Gamma, \tag{11}$$

$$(U,\mathcal{H})|_{t=0} = (U_0,\mathcal{H}_0) \quad \text{in } \Omega^+\times\Omega^-, \qquad f|_{t=0} = f_0 \quad \text{on } \Gamma, \tag{12}$$

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where $\mathbb{P}(U,\Psi) = P(U,\Psi)U$,

$$P(U,\Psi) = A_0(U)\partial_t + A_1(U)\partial_1 + A_2(U)\partial_2 + \widetilde{A}_3(U,\Psi)\partial_3,$$

$$\widetilde{A}_3(U,\Psi) = \frac{1}{\partial_3\Phi_3}\Big(A_3(U) - A_0(U)\partial_t\Psi - \sum_{k=1}^{2} A_k(U)\partial_k\Psi\Big),$$

$$\mathbb{V}(\mathcal{H},\Psi) = \begin{pmatrix} \nabla\times\mathfrak{H} \\ \operatorname{div}\mathfrak{h} \end{pmatrix},$$

$$\mathfrak{H} = (\mathcal{H}_{\tau_1}, \mathcal{H}_{\tau_2}, \mathcal{H}_3\partial_3\Phi), \quad \mathfrak{h} = (\mathcal{H}_1\partial_3\Phi_3, \mathcal{H}_2\partial_3\Phi_3, \mathcal{H}_N),$$

$$\mathcal{H}_N = \mathcal{H}_3 - \mathcal{H}_1\partial_1\Psi - \mathcal{H}_2\partial_2\Psi, \quad \mathcal{H}_{\tau_i} = \mathcal{H}_3\partial_i\Psi + \mathcal{H}_i, \quad i = 1,2,$$

$$\mathbb{B}(U,\mathcal{H},\varphi) = \begin{pmatrix} \partial_t f - v_{N|x_3=0} \\ [q] \\ \mathcal{H}_{N|x_3=0} \end{pmatrix}, \quad [q] = q_{|x_3=0} - \frac{1}{2}|\mathcal{H}|^2_{x_3=0},$$

$$v_N = v_3 - v_1\partial_1\Psi - v_2\partial_2\Psi.$$

where $\mathbb{P}(U,\Psi)=P(U,\Psi)U$,

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$$\mathbb{B}(U,\mathcal{H},\varphi)=\begin{pmatrix}\partial_t f-v_{N|x_3=0}\\ [q]\\ \mathcal{H}_{N|x_3=0}\end{pmatrix},\quad [q]=q_{|x_3=0}-\frac{1}{2}|\mathcal{H}|^2_{x_3=0},$$

$$v_N=v_3-v_1\partial_1\Psi-v_2\partial_2\Psi.$$

where $\mathbb{P}(U,\Psi) = P(U,\Psi)U$,

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$$v_N = v_3 - v_1\partial_1\Psi - v_2\partial_2\Psi.$$

In the previous system we don't include the equation

$$\operatorname{div} h = 0 \quad \text{in } [0,T] \times \Omega^+,$$

and the boundary condition

$$H_N = 0 \quad \text{on } [0,T] \times \Gamma,$$

where $h = (H_1 \partial_3 \Phi_3, H_2 \partial_3 \Phi_3, H_N)$, $H_N = H_3 - H_1 \partial_1 \Psi - H_2 \partial_2 \Psi$, because they are just restrictions on the initial data.

Linearization. The basic state

Let us denote

$$Q_T^\pm :=]-\infty, T] \times \Omega^\pm, \quad \omega_T :=]-\infty, T] \times \Gamma.$$

Let

$$(\widehat{U}(t,x), \widehat{\mathcal{H}}(t,x), \hat{f}(t,x')) \tag{13}$$

be a given sufficiently smooth vector-function, respectively defined on Q_T^+, Q_T^-, ω_T, with $\widehat{U} = (\hat{q}, \hat{v}, \widehat{H}, \widehat{S})$.
Corresponding to the given $\hat{f}$ we construct $\hat{\Psi}$, $\hat{\Phi}$ as in Lemma 1 such that

$$\partial_3 \widehat{\Phi}_3 \geq 1/2$$

($\widehat{\Phi}$ is a diffeomorphism).

Assume the basic state (13) satisfies

$$\rho(\hat{p},\widehat{S})>0,\quad \rho_p(\hat{p},\widehat{S})>0 \qquad \text{in } \overline{Q}_T^+,$$

$$\partial_t\widehat{H}+\frac{1}{\partial_3\widehat{\Phi}_3}\left\{(\hat{w}\cdot\nabla)\widehat{H}-(\hat{h}\cdot\nabla)\hat{v}+\widehat{H}\operatorname{div}\hat{u}\right\}=0 \quad \text{in } Q_T^+,$$

$$\operatorname{div}\hat{\mathfrak{h}}=0 \qquad \text{in } Q_T^-,$$

$$\partial_t\hat{\varphi}-\hat{v}_N=0,\quad \widehat{\mathcal{H}}_N=0 \qquad \text{on } \omega_T,$$

(all the "hat" functions are determined like corresponding ones for $(U,\mathcal{H},\varphi)$), where

$$\hat{u}=(\hat{v}_1\partial_3\widehat{\Phi}_3,\hat{v}_2\partial_3\widehat{\Phi}_3,\hat{v}_N),\quad \hat{w}=\hat{u}-(0,0,\partial_t\widehat{\Psi}).$$

Linearizing about the basic state (13) leads to the hyperbolic-elliptic boundary value problem

$$
\begin{aligned}
&\widehat{A}_0\partial_t U + \textstyle\sum_{j=1}^3 \widehat{A}_j\partial_j U + \widehat{\mathcal{C}}U = F && \text{in } Q_T^+,\\
&\nabla\times\mathfrak{H} = 0, \quad \operatorname{div}\mathfrak{h} = 0 && \text{in } Q_T^-,\\
&\left.\begin{aligned}
&\partial_t f = v_N - \hat{v}_1\partial_1 f - \hat{v}_2\partial_2 f + f\,\partial_3\hat{v}_N + g_1,\\
&q = \widehat{\mathcal{H}}\cdot\mathcal{H} - [\partial_3\hat{q}]f + g_2,\\
&\mathcal{H}_N = \partial_1\big(\widehat{\mathcal{H}}_1 f\big) + \partial_2\big(\widehat{\mathcal{H}}_2 f\big)
\end{aligned}\right. && \text{on } \omega_T,\\
&(U,\mathcal{H},f) = 0 && \text{for } t<0,
\end{aligned}
\tag{14}
$$

for data F and $g=(g_1,g_2)$ vanishing in the past, where ...

$$\widehat{A}_\alpha =: A_\alpha(\widehat{U}), \quad \alpha = 0, 1, 2,$$
$$\widehat{A}_3 =: \widetilde{A}_3(\widehat{U}, \widehat{\Psi}), \quad \widehat{\mathcal{C}} := \mathcal{C}(\widehat{U}, \widehat{\Psi}),$$

$$\mathfrak{H} = (\mathcal{H}_{\tau_1}, \mathcal{H}_{\tau_2}, \mathcal{H}_3 \partial_3 \widehat{\Phi}_3), \quad \mathfrak{h} = (\mathcal{H}_1 \partial_3 \widehat{\Phi}_3, \mathcal{H}_2 \partial_3 \widehat{\Phi}_3, \mathcal{H}_N),$$

$$\mathcal{H}_N = \mathcal{H}_3 - \mathcal{H}_1 \partial_1 \widehat{\Psi} - \mathcal{H}_2 \partial_2 \widehat{\Psi}, \quad \mathcal{H}_{\tau_i} = \mathcal{H}_3 \partial_i \widehat{\Psi} + \mathcal{H}_i, \quad i = 1, 2.$$

Stability result in H^1

Theorem (S. & Trakhinin, 2011)

Let $T > 0$. *Assume the basic state* (13) *satisfies*

$$|\widehat{H} \times \widehat{\mathcal{H}}| \geq \delta > 0 \qquad \text{on } \omega_T, \tag{15}$$

where δ *is a fixed constant. For all* $(F, g) \in H^1_{tan}(Q^+_T) \times H^{1.5}(\omega_T)$ *vanishing in the past, problem* (14) *has a unique solution* $(U, \mathcal{H}, f) \in H^1_{tan}(Q^+_T) \times H^1(Q^-_T) \times H^{1.5}(\omega_T)$ *such that*

$$\|U\|_{H^1_{tan}(Q^+_T)} + \|\mathcal{H}\|_{H^1(Q^-_T)} + \|(q, v_N, H_N)|_{\omega_T}\|_{H^{0.5}(\omega_T)} + \|f\|_{H^{1.5}(\omega_T)} \leq C\big(\|F\|_{H^1_{tan}(Q^+_T)} + \|g\|_{H^{1.5}(\omega_T)}\big)$$

where $C = C(\delta, T) > 0$ *is a constant independent of the data* (F, g).

Similar a priori estimate in [Trakhinin, JDE 2010].

Two main ideas for the proof:

1. Hyperbolic regularization
2. Secondary symmetrization

1. Hyperbolic regularization

We re-introduce the displacement current $\partial_t E$ and accordingly modify the boundary conditions:

$$\widehat{A}_0\partial_t U^\varepsilon + \sum_{j=1}^3 \widehat{A}_j\partial_j U^\varepsilon + \widehat{\mathcal{C}}U^\varepsilon = F \qquad \text{in } Q_T^+,$$

$$\varepsilon\,\partial_t\mathfrak{e}^\varepsilon - \nabla\times\mathfrak{H}^\varepsilon = 0, \qquad \varepsilon\,\partial_t\mathfrak{h}^\varepsilon + \nabla\times\mathfrak{E}^\varepsilon = 0 \qquad \text{in } Q_T^-,$$

$$\begin{aligned} &\partial_t f^\varepsilon + \hat{v}_1\partial_1 f^\varepsilon + \hat{v}_2\partial_2 f^\varepsilon - f^\varepsilon\partial_3\hat{v}_N - v_N^\varepsilon = g_1, \\ &q^\varepsilon + [\partial_3\hat{q}]f^\varepsilon - \hat{\mathfrak{h}}\cdot\mathfrak{H}^\varepsilon + \varepsilon\,\hat{\mathfrak{e}}\cdot\mathfrak{E}^\varepsilon = g_2, \\ &\mathfrak{E}_1^\varepsilon - \varepsilon\,\partial_t(\widehat{\mathcal{H}}_2 f^\varepsilon) + \varepsilon\,\partial_1(\widehat{E}_3 f^\varepsilon) = 0, \\ &\mathfrak{E}_2^\varepsilon + \varepsilon\,\partial_t(\widehat{\mathcal{H}}_1 f^\varepsilon) + \varepsilon\,\partial_2(\widehat{E}_3 f^\varepsilon) = 0 \qquad \text{on } \omega_T, \end{aligned} \tag{16}$$

$$(U^\varepsilon, \mathfrak{H}^\varepsilon, \mathfrak{E}^\varepsilon, f^\varepsilon) = 0 \qquad \text{for } t < 0,$$

where $\epsilon > 0$ is a parameter that will converge to zero and where . . .

$$E^\varepsilon = (E_1^\varepsilon, E_2^\varepsilon, E_3^\varepsilon), \quad \widehat{E} = (\widehat{E}_1, \widehat{E}_2, \widehat{E}_3),$$

$$\mathfrak{E}^\varepsilon = (E_{\tau_1}^\varepsilon, E_{\tau_2}^\varepsilon, E_3^\varepsilon \partial_3 \widehat{\Phi}_3), \qquad \mathfrak{e}^\varepsilon = (E_1^\varepsilon \partial_3 \widehat{\Phi}_3, E_2^\varepsilon \partial_3 \widehat{\Phi}_3, E_N^\varepsilon),$$

$$E_{\tau_k}^\varepsilon = E_3^\varepsilon \partial_k \widehat{\Psi} + E_k^\varepsilon, \ k = 1, 2, \quad E_N^\varepsilon = E_3^\varepsilon - E_1^\varepsilon \partial_1 \widehat{\Psi} - E_2^\varepsilon \partial_2 \widehat{\Psi}.$$

The coefficients $\widehat{E}_j$ will be chosen later on.

All the other notations for U^ε and $\mathcal{H}^\varepsilon$ (e.g., v_N^ε, $\mathfrak{h}^\varepsilon$, $\hat{\mathfrak{h}}$, etc.) are analogous to those for U and $\mathcal{H}$.

Solutions to problem (16) satisfy

$$\begin{aligned}
&\operatorname{div} h^\varepsilon = 0 && \text{in } Q_T^+,\\
&\operatorname{div} \mathfrak{h}^\varepsilon = 0, \quad \operatorname{div} \mathfrak{e}^\varepsilon = 0 && \text{in } Q_T^-,\\
&H_N^\varepsilon = \widehat{H}_1 \partial_1 f^\varepsilon + \widehat{H}_2 \partial_2 f^\varepsilon - f^\varepsilon \partial_3 \widehat{H}_N, &&\\
&\mathcal{H}_N^\varepsilon = \partial_1\big(\widehat{\mathcal{H}}_1 f^\varepsilon\big) + \partial_2\big(\widehat{\mathcal{H}}_2 f^\varepsilon\big) && \text{on } \omega_T,
\end{aligned} \tag{17}$$

(as restrictions on the initial data).

If $\Psi = 0, \Phi_3 = x_3$, then $\mathfrak{h}^\varepsilon = \mathfrak{H}^\varepsilon = \mathcal{H}^\varepsilon$, $\mathfrak{e}^\varepsilon = \mathfrak{E}^\varepsilon = E^\varepsilon$; when $\varepsilon = 1$ $(16)_2$ is nothing else than the usual Maxwell equations.

Under the boundary conditions in (16), (17), the boundary ω_T is

- characteristic for the plasma equations (size $N = 8$, rank$(\widehat{A}_3) = 2$); we expect a loss of regularity in the normal direction to the boundary. We are forced to study the system in weighted anisotropic Sobolev spaces H^m_*;
- characteristic for the vacuum equations (size $N = 6$, rank=4). Full regularity in standard Sobolev spaces H^m is expected thanks to the constraints (17).

If we look for a standard L^2 energy estimate we get the boundary integral

$$\int_{\omega_T} \left(-q^\varepsilon v_N^\varepsilon + \frac{1}{\epsilon} (\mathfrak{H}_1^\varepsilon \mathfrak{E}_2^\varepsilon - \mathfrak{H}_2^\varepsilon \mathfrak{E}_1^\varepsilon) \right) dx'\, dt.$$

We don't know how to control it.

As regards existence of solutions, main difficulties are:

- the coupling with the front f^ε (UKL doesn't hold)
- the so-called non-reflexivity [Ohkubo, Hokkaido MJ 1981]:

$$\underbrace{ker \begin{pmatrix} -\widehat{A}_{3|x_3=0} & 0 \\ 0 & B_3^\varepsilon \end{pmatrix}}_{\substack{\uparrow \\ \text{boundary matrix}}} \not\subset \underset{\substack{\uparrow \\ \text{boundary space (for } f^\varepsilon = 0)}}{\mathcal{N}}$$

(B_3^ε denotes the boundary matrix in the Maxwell equations)

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In fact, at $\{x_3 = 0\}$

$$\left| \begin{pmatrix} -\widehat{A}_3 & 0 \\ 0 & B_3^\varepsilon \end{pmatrix} \begin{pmatrix} U^\varepsilon \\ V^\varepsilon \end{pmatrix} \right| \not\geq C(|\mathcal{H}_3^\varepsilon| + |E_3^\varepsilon|)$$

(where $V^\varepsilon = (\mathcal{H}^\varepsilon, E^\varepsilon)$),
so that the boundary conditions (involving $\mathcal{H}_3^\varepsilon, E_3^\varepsilon$) do not have (weak $H^{-1/2}$) sense in a weak formulation.

Thus we consider the following secondary symmetrization for the modified Maxwell equations obtained from a linear combination of $(16)_2$ and the restrictions $(17)_2$.

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Thus we consider the following secondary symmetrization for the modified Maxwell equations obtained from a linear combination of $(16)_2$ and the restrictions $(17)_2$.

2. Secondary symmetrization

Let us define the matrix

$$\hat\eta = \begin{pmatrix} \partial_3\widehat\Phi_3 & 0 & 0 \\ 0 & \partial_3\widehat\Phi_3 & 0 \\ -\partial_1\widehat\Psi & -\partial_2\widehat\Psi & 1 \end{pmatrix}.$$

For every choice of vector functions $\vec\nu \neq 0$, consider the system, obtained from $(16)_2$, $(17)_2$,

$$\Big(\partial_t\mathfrak{h}^\varepsilon + \frac{1}{\varepsilon}\nabla\times\mathfrak{E}^\varepsilon\Big) - \hat\eta\,\Big(\vec\nu\times\hat\eta^{-1}(\varepsilon\partial_t\mathfrak{e}^\varepsilon - \nabla\times\mathfrak{H}^\varepsilon)\Big) + \frac{\hat\eta\,\vec\nu}{\partial_3\widehat\Phi_3}\operatorname{div}\mathfrak{h}^\varepsilon = 0,$$

$$\Big(\partial_t\mathfrak{e}^\varepsilon - \frac{1}{\varepsilon}\nabla\times\mathfrak{H}^\varepsilon\Big) + \hat\eta\,\Big(\vec\nu\times\hat\eta^{-1}(\varepsilon\partial_t\mathfrak{h}^\varepsilon + \nabla\times\mathfrak{E}^\varepsilon)\Big) + \frac{\hat\eta\,\vec\nu}{\partial_3\widehat\Phi_3}\operatorname{div}\mathfrak{e}^\varepsilon = 0. \quad (18)$$

(18) is symmetric hyperbolic provided

$$\varepsilon|\vec{\nu}| < 1,$$

and equivalent to $(16)_2$ on solutions with initial data satisfying the constraints

$$\operatorname{div} \mathfrak{h}^\varepsilon = 0, \quad \operatorname{div} \mathfrak{e}^\varepsilon = 0 \qquad \text{for } t = 0.$$

Thus we may deal with (18) instead of $(16)_2$.

Lemma

Let $T > 0$. Assume the basic state (13) satisfies $|\widehat{H} \times \widehat{\mathcal{H}}| \geq \delta > 0$ on ω_T, where δ is a fixed constant.
Then for all $\epsilon > 0$ sufficiently small and all $F \in H^1_{tan}(Q_T^+)$, $g \in H^{1.5}(\omega_T)$, vanishing in the past, problem (16) has a unique solution $(U^\varepsilon, \mathfrak{H}^\varepsilon, \mathfrak{E}^\varepsilon, f^\varepsilon) \in H^1_{tan}(Q_T^+) \times H^1(Q_T^-) \times H^1(Q_T^-) \times H^{1.5}(\omega_T)$ such that

$$\|U^\varepsilon\|_{H^1_{tan}(Q_T^+)} + \|\mathfrak{H}^\varepsilon, \mathfrak{E}^\varepsilon\|_{H^1(Q_T^-)} + \|(q^\varepsilon, v_N^\varepsilon, H_N^\varepsilon)|_{\omega_T}\|_{H^{0.5}(\omega_T)} + \|f^\varepsilon\|_{H^{1.5}(\omega_T)} \leq C\big(\|F\|_{H^1_{tan}(Q_T^+)} + \|g\|_{H^{1.5}(\omega_T)}\big) \quad (19)$$

where $C = C(\delta, T) > 0$ is a constant independent of ϵ and the data (F, g).

Proof of the Lemma. Control of the front

The boundary condition $(16)_3$

$$\partial_t f^\varepsilon + \hat{v}_1 \partial_1 f^\varepsilon + \hat{v}_2 \partial_2 f^\varepsilon = g_1 + f^\varepsilon \partial_3 \hat{v}_N + v_N^\varepsilon \tag{20}$$

is a linear transport equation. Solving it, f^ε gets the regularity of v_N^ε.
On the other hand, the boundary constraints (17) yield

$$\begin{cases} \widehat{H}_1 \partial_1 f^\varepsilon + \widehat{H}_2 \partial_2 f^\varepsilon = H_N^\varepsilon + f^\varepsilon \partial_3 \widehat{H}_N, \\ \widehat{\mathcal{H}}_1 \partial_1 f^\varepsilon + \widehat{\mathcal{H}}_2 \partial_2 f^\varepsilon = \mathcal{H}_N^\varepsilon - \left(\partial_1 \widehat{\mathcal{H}}_1 + \partial_2 \widehat{\mathcal{H}}_2\right) f^\varepsilon \quad \text{on } \omega_T, \end{cases} \tag{21}$$

Under the stability condition (15) we have

$$\widehat{H}_1 \widehat{\mathcal{H}}_2 - \widehat{H}_2 \widehat{\mathcal{H}}_1 \neq 0,$$

and we may solve the above linear system (21) and (20) for $\nabla_{t,x'} f^\varepsilon$.
Thus $\nabla_{t,x'} f^\varepsilon$ has the regularity of $v_N^\varepsilon, H_N^\varepsilon, \mathcal{H}_N^\varepsilon$ at Γ, i.e. f^ε gains one derivative.

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The boundary condition $(16)_3$

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▸ N − M

Proof of the Lemma. Analysis of boundary terms

Write the secondary symmetrization (18) as

$$M_0^\varepsilon \partial_t W^\varepsilon + \sum_{j=1}^{3} M_j^\varepsilon \partial_j W^\varepsilon + M_4^\varepsilon W^\varepsilon = 0, \tag{22}$$

where $W^\varepsilon = (\mathfrak{H}^\varepsilon, \mathfrak{E}^\varepsilon)$.
Look for a L^2 energy estimate for system $(16)_1$, (22), where we choose

$$\nu_1 = \hat{v}_1, \quad \nu_2 = \hat{v}_2, \quad \nu_3 = \hat{v}_1\partial_1\hat{f} + \hat{v}_2\partial_2\hat{f}.$$

Under this choice the boundary is characteristic for (22).

We get the boundary integral

$$\mathcal{A} := -\frac{1}{2}\int_{\omega_T} (\widehat{A}_1 U^\varepsilon, U^\varepsilon) - (M_1^\varepsilon W^\varepsilon, W^\varepsilon) dx'\, dt =$$
$$= \int_{\omega_T} \Big(- q^\varepsilon v_N^\varepsilon + \frac{1}{\epsilon}(\mathfrak{H}_1^\varepsilon \mathfrak{E}_2^\varepsilon - \mathfrak{H}_2^\varepsilon \mathfrak{E}_1^\varepsilon)$$
$$+ (\hat{v}_1 \mathfrak{H}_1^\varepsilon + \hat{v}_2 \mathfrak{H}_2^\varepsilon)\mathcal{H}_N^\varepsilon + (\hat{v}_1 \mathfrak{E}_1^\varepsilon + \hat{v}_2 \mathfrak{E}_2^\varepsilon) E_N^\varepsilon \Big) dx'\, dt.$$

Inserting the boundary conditions of (16) (where $\mathfrak{E}_1^\varepsilon, \mathfrak{E}_2^\varepsilon$ are chosen proportional to ε) gives

$$\mathcal{A} := \int_{\omega_T} \big(\widehat{E}_3 + \hat{v}_1 \widehat{\mathcal{H}}_2 - \hat{v}_2 \widehat{\mathcal{H}}_1\big)\big(\varepsilon E_N^\varepsilon \partial_t f^\varepsilon + \mathfrak{H}_1^\varepsilon \partial_2 f^\varepsilon - \mathfrak{H}_2^\varepsilon \partial_1 f^\varepsilon\big)$$
$$+ \varepsilon\big(\widehat{E}_{\tau_1} E_1^\varepsilon + \widehat{E}_{\tau_2} E_2^\varepsilon\big)\big(\partial_t f^\varepsilon + \hat{v}_1 \partial_1 f^\varepsilon + \hat{v}_2 \partial_2 f^\varepsilon\big)$$

$$+ f^\varepsilon\big\{[\partial_3 \hat{q}]\, v_N^\varepsilon - \partial_3 \hat{v}_N (q^\varepsilon + [\partial_3 \hat{q}] f^\varepsilon) + (\partial_t \hat{\mathcal{H}}_2 - \partial_1 \hat{E}_3)(\mathfrak{H}_2^\varepsilon + \varepsilon \hat{v}_1 E_N^\varepsilon)$$
$$+ (\partial_t \hat{\mathcal{H}}_1 + \partial_2 \hat{E}_3)(\mathfrak{H}_1^\varepsilon - \varepsilon \hat{v}_2 E_N^\varepsilon) + (\partial_1 \hat{\mathcal{H}}_1 + \partial_2 \hat{\mathcal{H}}_2)(\hat{v}_1 \mathfrak{H}_1^\varepsilon + \hat{v}_2 \mathfrak{H}_2^\varepsilon)\big\}.$$

We choose

$$\hat{E} = -\vec{\nu} \times \hat{\mathcal{H}},$$

so that

$$\widehat{E}_3 + \hat{v}_1 \widehat{\mathcal{H}}_2 - \hat{v}_2 \widehat{\mathcal{H}}_1 = 0, \quad \widehat{E}_{\tau_1} = 0, \quad \widehat{E}_{\tau_2} = 0.$$

The choice is related to Ohm's law.

$$\mathcal{A} := \int_{\omega_T} \big(\widehat{E}_3 + \hat{v}_1\widehat{\mathcal{H}}_2 - \hat{v}_2\widehat{\mathcal{H}}_1\big)\big(\varepsilon E^\varepsilon_N \partial_t f^\varepsilon + \mathfrak{H}^\varepsilon_1 \partial_2 f^\varepsilon - \mathfrak{H}^\varepsilon_2 \partial_1 f^\varepsilon\big) + \varepsilon\big(\widehat{E}_{\tau_1} E^\varepsilon_1 + \widehat{E}_{\tau_2} E^\varepsilon_2\big)\big(\partial_t f^\varepsilon + \hat{v}_1\partial_1 f^\varepsilon + \hat{v}_2\partial_2 f^\varepsilon\big)$$

$$+ f^\varepsilon\big\{[\partial_3\hat{q}]\, v^\varepsilon_N - \partial_3\hat{v}_N(q^\varepsilon + [\partial_3\hat{q}] f^\varepsilon) + (\partial_t\hat{\mathcal{H}}_2 - \partial_1\hat{E}_3)(\mathfrak{H}^\varepsilon_2 + \varepsilon\hat{v}_1 E^\varepsilon_N) + (\partial_t\hat{\mathcal{H}}_1 + \partial_2\hat{E}_3)(\mathfrak{H}^\varepsilon_1 - \varepsilon\hat{v}_2 E^\varepsilon_N) + (\partial_1\hat{\mathcal{H}}_1 + \partial_2\hat{\mathcal{H}}_2)(\hat{v}_1\mathfrak{H}^\varepsilon_1 + \hat{v}_2\mathfrak{H}^\varepsilon_2)\big\}.$$

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We are left with no derivatives of f^ε:

$$
\begin{aligned}
\mathcal{A} := \int_{\omega_T} f^\varepsilon \Big\{ [\partial_3 \hat{q}]\, v_N^\varepsilon - \partial_3 \hat{v}_N (q^\varepsilon + [\partial_3 \hat{q}] f^\varepsilon) \\
+ (\partial_t \hat{\mathcal{H}}_2 - \partial_1 \hat{E}_3)(\mathfrak{H}_2^\varepsilon + \varepsilon \hat{v}_1 E_N^\varepsilon) \\
+ (\partial_t \hat{\mathcal{H}}_1 + \partial_2 \hat{E}_3)(\mathfrak{H}_1^\varepsilon - \varepsilon \hat{v}_2 E_N^\varepsilon) \\
+ (\partial_1 \hat{\mathcal{H}}_1 + \partial_2 \hat{\mathcal{H}}_2)(\hat{v}_1 \mathfrak{H}_1^\varepsilon + \hat{v}_2 \mathfrak{H}_2^\varepsilon) \Big\}.
\end{aligned}
$$

To exploit the diminished order we pass to an energy estimate in H^1_{tan} (instead of L^2), take tangential derivatives, perform some integration by parts, use the higher regularity at the boundary of the noncharacteristic part of the vector solution, etc etc ...
In the end we get the (uniform in ϵ) a priori estimate (19).

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Proof of Theorem 1

Given the uniform (in ϵ) a priori estimate (19), we may pass to the limit in the hyperbolic regularizing system (16) as $\epsilon \to 0$ and find the solution $(U, \mathcal{H}, f) \in H^1_{tan}(Q_T^+) \times H^1(Q_T^-) \times H^1(\omega_T)$ of the linearized problem (14):

$$\widehat{A}_0 \partial_t U + \sum_{j=1}^3 \widehat{A}_j \partial_j U + \widehat{\mathcal{C}} U = F \qquad \text{in } Q_T^+,$$

$$\nabla \times \mathfrak{H} = 0, \quad \operatorname{div} \mathfrak{h} = 0 \qquad \text{in } Q_T^-,$$

$$\begin{aligned} &\partial_t f = v_N - \hat{v}_1 \partial_1 f - \hat{v}_2 \partial_2 f + f\, \partial_3 \hat{v}_N + g_1, \\ &q = \widehat{\mathcal{H}} \cdot \mathcal{H} - [\partial_3 \hat{q}] f + g_2, \\ &\mathcal{H}_N = \partial_1 \big(\widehat{\mathcal{H}}_1 f\big) + \partial_2 \big(\widehat{\mathcal{H}}_2 f\big) \end{aligned} \qquad \text{on } \omega_T,$$

$$(U, \mathcal{H}, f) = 0 \qquad \text{for } t < 0.$$

High-order energy estimate

As we want to work with functions in Sobolev spaces (vanishing at infinity), in contradiction with a uniform stability condition

$$|H \times \mathcal{H}| \geq \delta > 0 \qquad \text{on } [0,T] \times \Gamma,$$

we make a shift by a constant solution.

Let us consider constant solutions $\bar{U}$ and $\bar{\mathcal{H}}$ (with $f = 0$), where

$$\bar{U} = (\bar{q}, 0, 0, 0, \bar{H}, 0), \quad \bar{H} = (\bar{H}_1, \bar{H}_2, 0), \quad \bar{\mathcal{H}} = (\bar{\mathcal{H}}_1, \bar{\mathcal{H}}_2, 0), \tag{23}$$

$$\begin{aligned} &\bar{q} = \bar{p} + \tfrac{\bar{H}_1^2 + \bar{H}_2^2}{2} = \tfrac{\bar{\mathcal{H}}_1^2 + \bar{\mathcal{H}}_2^2}{2}, \quad \bar{p} > 0, \\ &\rho(\bar{p}, 0) > 0, \quad \rho_p(\bar{p}, 0) > 0 \qquad \text{(hyperbolicity condition)}, \\ &\bar{H}_1 \bar{\mathcal{H}}_2 - \bar{H}_2 \bar{\mathcal{H}}_1 \neq 0 \qquad \text{(stability condition)}. \end{aligned} \tag{24}$$

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$$\begin{array}{ll} \bar{q} = \bar{p} + \frac{\bar{H}_1^2 + \bar{H}_2^2}{2} = \frac{\bar{\mathcal{H}}_1^2 + \bar{\mathcal{H}}_2^2}{2}, & \bar{p} > 0, \\ \rho(\bar{p}, 0) > 0, \quad \rho_p(\bar{p}, 0) > 0 & \text{(hyperbolicity condition)}, \\ \bar{H}_1 \bar{\mathcal{H}}_2 - \bar{H}_2 \bar{\mathcal{H}}_1 \neq 0 & \text{(stability condition)}. \end{array} \tag{24}$$

Make a shift by the change of unknowns

$$\breve{U} = U - \bar{U}, \qquad \breve{\mathcal{H}} = \mathcal{H} - \bar{\mathcal{H}}, \tag{25}$$

then write again $U, \mathcal{H}$ instead of $\breve{U}, \breve{\mathcal{H}}$.

We reformulate the problem in terms of the new unknowns as:

$$\mathbb{P}(U, \Psi) = 0 \quad \text{in } [0,T] \times \Omega^+, \tag{26}$$

$$\mathbb{V}(\mathcal{H}, \Psi) = 0 \quad \text{in } [0,T] \times \Omega^-, \tag{27}$$

$$\mathbb{B}(U, \mathcal{H}, \varphi) = 0 \quad \text{on } [0,T] \times \Gamma, \tag{28}$$

$$\lim_{|x| \to \infty} (U, \mathcal{H}, \varphi) = 0, \tag{29}$$

$$(U, \mathcal{H})|_{t=0} = (U_0, \mathcal{H}_0) \quad \text{in } \Omega^+ \times \Omega^-, \qquad \varphi|_{t=0} = \varphi_0 \quad \text{on } \Gamma, \tag{30}$$

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where now $\mathbb{P}(U,\Psi)=P(U,\Psi)U$,

$$P(U,\Psi)=A_0(U+\bar U)\partial_t+A_1(U+\bar U)\partial_1+A_2(U+\bar U)\partial_2+\widetilde A_3(U+\bar U,\Psi)\partial_3,$$

$$\mathbb{B}(U,\mathcal{H},\varphi)=\begin{pmatrix}\partial_t\varphi-v_{N|x_3=0}\\ [q]\\ \mathcal{H}_{N|x_3=0}-\partial_1(\bar{\mathcal{H}}_1\varphi)-\partial_2(\bar{\mathcal{H}}_2\varphi)\end{pmatrix},$$

$$[q]=q_{|x_3=0}-\frac12|\mathcal{H}|^2_{x_3=0}-\bar{\mathcal{H}}\cdot\mathcal{H}_{|x_3=0}.$$

As for the constraints, we have the new one

$$H_N=\partial_1(\bar H_1\varphi)+\partial_2(\bar H_2\varphi)\quad\text{on } [0,T]\times\Gamma,$$

instead of $H_N=0$.

Now linearize about the basic state

$$(\widehat{U}(t,x) + \bar{U}, \widehat{\mathcal{H}}(t,x) + \bar{\mathcal{H}}, \hat{f}(t,x')). \tag{31}$$

Assume the basic state (31) satisfies

$$\rho(\hat{p} + \bar{p}, \widehat{S}) > 0, \quad \rho_p(\hat{p} + \bar{p}, \widehat{S}) > 0 \qquad \text{in } \overline{Q}_T^+,$$

$$\partial_t \widehat{H} + \frac{1}{\partial_3 \widehat{\Phi}_3} \left\{ (\hat{w} \cdot \nabla)\widehat{H} - ((\hat{h} + \bar{h}) \cdot \nabla)\hat{v} + (\widehat{H} + \bar{H}) \operatorname{div} \hat{u} \right\} = 0 \quad \text{in } Q_T^+,$$

$$\operatorname{div} \hat{\mathfrak{h}} = 0 \qquad \text{in } Q_T^-,$$

$$\partial_t \hat{\varphi} - \hat{v}_N = 0, \quad \widehat{\mathcal{H}}_N = \partial_1(\bar{\mathcal{H}}_1 \varphi) + \partial_2(\bar{\mathcal{H}}_2 \varphi) \qquad \text{on } \omega_T,$$

(all the "hat" and "bar" functions are determined like corresponding ones for $(U, \mathcal{H}, \varphi)$).

Linearization leads to the nonhomogeneous hyperbolic-elliptic boundary value problem

$$\begin{aligned}
&\widehat{A}_0\partial_t U + \textstyle\sum_{j=1}^3 \widehat{A}_j\partial_j U + \widehat{\mathcal{C}}U = F && \text{in } Q_T^+,\\
&\nabla\times\mathfrak{H} = \chi, \quad \operatorname{div}\mathfrak{h} = \Xi && \text{in } Q_T^-,\\
&\partial_t f = v_N - \hat{v}_1\partial_1 f - \hat{v}_2\partial_2 f + f\,\partial_3\hat{v}_N + g_1, \\
&q = (\widehat{\mathcal{H}} + \bar{\mathcal{H}})\cdot\mathcal{H} - [\partial_3\hat{q}]f + g_2, \\
&\mathcal{H}_N = \partial_1\big((\widehat{\mathcal{H}}_1 + \bar{\mathcal{H}}_1)f\big) + \partial_2\big((\widehat{\mathcal{H}}_2 + \bar{\mathcal{H}}_2)f\big) + g_3 && \text{on } \omega_T,\\
&(U,\mathcal{H},f) = 0 && \text{for } t<0,
\end{aligned} \tag{32}$$

where

$$\widehat{A}_\alpha =: A_\alpha(\widehat{U} + \bar{U}), \quad \alpha = 0, 1, 2,$$

$$\widehat{A}_3 =: \widetilde{A}_3(\widehat{U} + \bar{U}, \widehat{\Psi}), \quad \widehat{\mathcal{C}} := \mathcal{C}(\widehat{U}, \widehat{\Psi}),$$

for data (F, χ, Ξ) and $g = (g_1, g_2, g_3)$ vanishing in the past, and satisfying the compatibility conditions

$$\operatorname{div} \chi = 0, \qquad \int_{\Omega^-} \Xi \, dx = \int_\Gamma g_3 \, dx',$$

Assumptions:

Let $T>0$, $m\in\mathbb{N}, m\geq 1, s=\max\{m,7\}$. Assume the basic state (31) satisfies the stability condition

$$|(\widehat{H}+\bar{H})\times(\widehat{\mathcal{H}}+\bar{\mathcal{H}})|\geq\delta>0 \qquad \text{on } \omega_T,$$

where δ is a fixed constant.

$\widehat{U}\in H^{s+1}_*(Q_T^+)$, $\widehat{\mathcal{H}}\in H^s(Q_T^-)$, $\nabla\widehat{\Psi}\in H^{s+1}(Q_T)$,
$F\in H^{m+1}_*(Q_T^+)$, $(\chi,\Xi)\in H^{m-1}(Q_T^-)\cap L^{6/5}(Q_T^-)$,
$g\in H^{m+1/2}(\omega_T)$ with $g_3\in L^{4/3}(\omega_T)$, all functions vanishing in the past.

H^m_*

Theorem (S. & Trakhinin, 2012)

Under the previous assumptions, problem (32) *has a unique solution* $(U, \mathcal{H}, f) \in H^m_*(Q^+_T) \times H^m(Q^-_T) \times H^{m+1/2}(\omega_T)$.
For $m \geq 7$ *the solution obeys the tame estimate*

$$\begin{aligned}
&\|U\|^2_{H^m_*(Q^+_T)} + \|\mathcal{H}\|^2_{H^m(Q^-_T)} + \|(q, v_N, H_N)|_{\omega_T}\|_{H^{m-1/2}(\omega_T)} \\
&+ \|f\|^2_{H^{m+1/2}(\omega_T)} \leq C\Big\{ \Big(\|\tilde{f}\|^2_{H^8_*(Q_T)} + \|\chi, \Xi\|^2_{H^7(Q^-_T)} + \|g\|^2_{H^{7.5}(\omega_T)} \\
&\qquad + \|\chi, \Xi\|^2_{L^2(0,T;L^{6/5}(\Omega^-))} + \|g_3\|^2_{L^2(0,T;L^{4/3}(\Gamma))} \Big) \times \\
&\qquad \times \Big(\|\widehat{U}\|^2_{H^{m+1}_*(Q^+_T)} + \|\widehat{\mathcal{H}}\|^2_{H^m(Q^-_T)} + \|\nabla\widehat{\Psi}\|^2_{H^{m+1}(Q_T)} \Big) \\
&\qquad + \|F\|^2_{H^{m+1}_*(Q_T)} + \|\chi, \Xi\|^2_{H^{m-1}(Q^-_T)} + \|g\|^2_{H^{m+1/2}(\omega_T)} \\
&\qquad + \|\chi, \Xi\|^2_{L^2(0,T;L^{6/5}(\Omega^-))} + \|g_3\|^2_{L^2(0,T;L^{4/3}(\Gamma))} \Big\}. \qquad (33)
\end{aligned}$$

Proof of Theorem 2

It follows from Theorem 1 and estimates of commutators.

Summarizing:

1^{st} step:
Linearized stability in H^1.

2^{nd} step:
Higher-order tame estimate.

3^{rd} step:
Solve the original nonlinear problem (10), (11), (12) by a Nash-Moser iteration.

Nash-Moser technique

Given $\mathcal{F} : X \mapsto X$, with X a Banach space (the same space for the sake of simplicity), we want to solve the nonlinear equation

$$\mathcal{F}(u) = w, \tag{34}$$

where we may assume $\mathcal{F}(0) = 0$.

1) Assume $\mathcal{F}$ is continuously differentiable and the linear application $\mathcal{F}'(\cdot)$ is invertible in a neighborhood of $u = 0$. Then $\mathcal{F}$ is locally invertible.

By Newton's method we may solve (34) by the approximating sequence

$$\begin{aligned} &u_0 = 0, \\ &u_{k+1} = u_k + (\mathcal{F}'(u_k))^{-1}(w - \mathcal{F}(u_k)), \quad k \geq 1. \end{aligned} \tag{35}$$

Newton's method has a fast convergence rate:

$$\|u_{k+1} - u_k\|_X \leq C\|u_k - u_{k-1}\|_X^2.$$

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2) Instead of one single space X, we are given a scale of Banach spaces $X_0 \supset X_1 \supset \cdots \supset X_m \supset \dots$ with norms $\|\cdot\|_m$, $m \geq 0$, and $\cap_{m\geq 0} X_m = C^\infty$.
For instance $X_m = H^m$ (Sobolev spaces), $X_s = C^s$ (Hölder spaces).

It may happen that $\mathcal{F} : X_m \mapsto X_m$, but $\mathcal{F}'(\cdot)$ is only invertible between X_m and X_{m-r}, with a loss of regularity of order r.

Trying to solve (34) again by Newton's method (35) we get

$$\|u_{k+1} - u_k\|_{m-r} \leq C\|u_k - u_{k-1}\|_m^2,$$

with a finite loss of regularity at each step. Iteration is impossible!

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The idea is to compensate the loss of regularity with the fast convergence rate.
To do so we introduce a family of smoothing operators $\{S_\theta\}_{\theta\geq 1}$

$$S_\theta : \cup_{m\geq 0} X_m \mapsto \cap_{m\geq 0} X_m$$

with the following properties (α and β in a bounded interval):

$$\begin{array}{lll} i) & \|S_\theta u\|_\alpha \leq C\ \|u\|_\beta & \alpha\leq\beta, \\ ii) & \|S_\theta u\|_\alpha \leq C\ \theta^{\alpha-\beta}\|u\|_\beta & \beta\leq\alpha, \\ iii) & \|S_\theta u - u\|_\alpha \leq C\ \theta^{\alpha-\beta}\|u\|_\beta & \alpha\leq\beta, \\ iv) & \|\frac{d}{d\theta} S_\theta u\|_\alpha \leq C\ \theta^{\alpha-\beta-1}\|u\|_\beta & \forall\alpha,\beta. \end{array}$$

We modify (35) by considering the approximating sequence

$$\begin{aligned} u_0 &= 0, \\ u_{k+1} &= u_k + (\mathcal{F}'(S_{\theta_k} u_k))^{-1}(w - \mathcal{F}(u_k)), \end{aligned} \tag{36}$$

where $\theta_k \to \infty$ as $k \to \infty$.

Balancing in appropriate way the fast convergence rate of Newton's scheme and loss of regularity gives the convergence of the approximating sequence.
Since formally $S_{\theta_k} \to I$ as $k \to \infty$ (in low norm), the sequence $\{u_k\}$ is expected to converge to a solution u of (34).

By adapting the Nash-Moser technique to our problem, we get our main result:

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Theorem (S. & Trakhinin, 2012)

Let $m \geq 13$. *Consider the constant solution* (23) $(\bar{U}, \bar{\mathcal{H}}, 0)$, *satisfying* (24). *Consider initial data* $(U_0, \mathcal{H}_0, f_0)$ *that are compactly supported perturbations in* $H^{m+9.5}(\Omega^+) \times H^{m+9.5}(\Omega^-) \times H^{m+10}(\Gamma)$ *of the constant solution* (23), *and that satisfy the hyperbolicity condition together with suitable compatibility conditions. The initial magnetic fields satisfy the necessary initial constraints and the stability condition*

$$|H_0 \times \mathcal{H}_0| \geq \delta > 0 \qquad \text{on } \Gamma,$$

where δ *is a fixed constant.*

If $T > 0$ *is sufficiently small, then there exists a unique solution* $(U, \mathcal{H}, f)$ *on* $[0, T]$ *of* (26)–(30) *with initial data* $(U_0, \mathcal{H}_0, f_0)$. *The solution is such that* $(U - \bar{U}, \mathcal{H} - \bar{\mathcal{H}}, f) \in H^m_*(]0, T[\times\Omega^+)\times H^m(]0, T[\times\Omega^-)\times H^{m+0.5}(]0, T[\times\Gamma)$.

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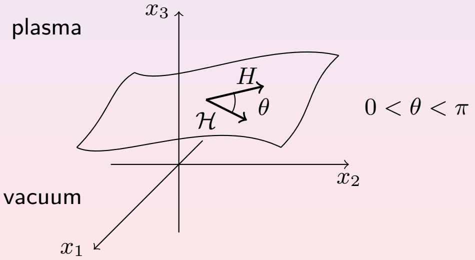
Conclusion

Under the stability condition

$$|H \times \mathcal{H}| \geq \delta > 0 \qquad \text{on } [0,T] \times \Gamma,$$

we have shown the well-posedness of the nonlinear plasma-vacuum interface problem (10), (11), (12).



Thank you for your attention!

The plasma variables $U = (q, v, H, S)$ solve an IBVP with characteristic boundary. The natural function space is the anisotropic weighted Sobolev space $H^m_*(\Omega)$ where the trace operator

$$\gamma_0 : U \mapsto U_{|\Gamma}, \qquad \gamma_0 : H^m_*(\Omega^+) \mapsto H^{m-1}(\Gamma).$$

Then for every fixed t

$$\begin{aligned}
& U \in H^m_*(\Omega^+), \mathcal{H} \in H^m(\Omega^-) \\
& \Rightarrow (v, H, \mathcal{H})_{|\Gamma} \in H^{m-1}(\Gamma) \\
& \Rightarrow \nabla_{t,x'} f \in H^{m-1}(\Gamma) \Rightarrow f \in H^m(\Gamma) \Rightarrow \Phi \in H^{m+0.5}(\mathbb{R}^3) \\
& \Rightarrow U \in H^{m-1}_*(\Omega^+), \mathcal{H} \in H^{m-1}(\Omega^-).
\end{aligned}$$

We lose one derivative!
The loss of regularity forces the use of a Nash-Moser iteration.
This fact justifies the study of the linearized problem. back

For characteristic boundaries, the natural function space is the weighted anisotropic Sobolev space

$$H^m_*(\Omega) := \{u \in L^2(\Omega) : \ Z^\alpha \partial^k_{x_n} u \in L^2(\Omega), \ |\alpha| + 2k \leq m\},$$

where

$$Z^\alpha := Z_1^{\alpha_1} \dots Z_n^{\alpha_n} \,, \quad \alpha = (\alpha_1, \dots, \alpha_n)\,,$$
$$Z_j = \partial_{x_j} \quad \text{for } j = 1, \dots, n-1 \quad \text{and} \quad Z_n = x_n \partial_{x_n},$$

if $\Omega = \{x_n > 0\}$.

Generally speaking, one normal derivative (w.r.t. $\partial\Omega$) is controlled by two tangential derivatives.

back2