Sticky particles with interaction

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Outline

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1. A one-dimensional fluid flow and the sticky particle model

2. Free motion: representation formulas by the cumulative distribution function

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1 A one-dimensional fluid flow and the sticky particle model

2 Free motion: representation formulas by the cumulative distribution function

3 Monotone rearrangement and Lagrangian representation

A one-dimensional compressible fluid flow

Consider a simple hyperbolic system modeling a one-dimensional compressible and pressureless fluid flow under the influence of a force field that is generated by the fluid itself.
The density-momentum couple $(\varrho, \varrho v)$ satisfy

$$\boxed{\begin{cases} \partial_t \varrho + \partial_x(\varrho v) = 0 \\ \partial_t(\varrho v) + \partial_x(\varrho v^2) = f[\varrho] \end{cases} \quad \text{in } [0,\infty) \times \mathbb{R},}$$

for suitable initial data $(\varrho, \varrho v)(t = 0, \cdot) =: (\varrho_0, \varrho_0 v_0)$.

- The continuity equation for the density ϱ (a nonnegative measure in time and space describing the distribution of mass or electric charge) and the real-valued Eulerian velocity field v.
- The momentum equation for ϱv

The force field $f[\varrho]$ is described by a signed measure depending on the mass distribution.

The structure of the force field f

We will assume that $f[\varrho]$ is absolutely continuous with respect to ϱ: the typical simplest form of f is

$$f[\varrho] = -\varrho\,\partial_x q_\varrho, \qquad \boxed{q_\varrho(x) = V(x) + \int_{\mathbb{R}} W(x-y)\,\mathrm{d}\varrho(y)} \qquad (*)$$

for suitable C^1 potential functions V, W with linearly growing derivatives.

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$$\boxed{f[\varrho] = -\varrho\,\partial_x q_\varrho} \quad \text{with } q_\varrho \text{ solution of } \quad \boxed{-\partial^2_{xx} q_\varrho = \lambda(\varrho - \sigma),}$$

q_ϱ admits the representation similar to (*)

$$W(x) := \frac{\lambda}{2}|x|, \quad V(x) := -\frac{\lambda}{2}\int_{\mathbb{R}} |x-y|\,\mathrm{d}\sigma(y),$$

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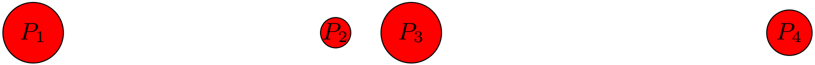
The Euler-Poisson equations in the **attractive regime (with $\lambda > 0$ and positive convex potential W)** is the one-dimensional caricature of a cosmological model for the universe at an early stage, describing the formation of galaxies.

Starting point: motion of a finite number of particles.

Discrete particle model

N particles $\quad P_i := (m_i, x_i, v_i), \quad i = 1, \ldots, N,$
with positive mass m_i satisfying $\sum_{i=1}^{N} m_i = 1$
ordered positions $x_1 < x_2 < \ldots < x_{N-1} < x_N,$



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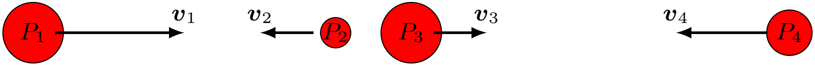

At the initial time $t = 0$ the particles are disjoint and start to move according to the system of ODE's

$$\frac{\mathrm{d}}{\mathrm{d}t} x_i(t) = v_i(t), \quad \frac{\mathrm{d}}{\mathrm{d}t} v_i(t) = a_i(x(t)).$$

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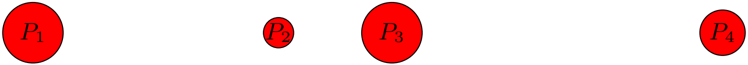

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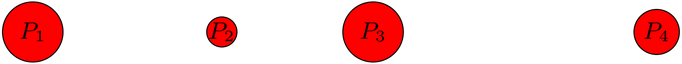

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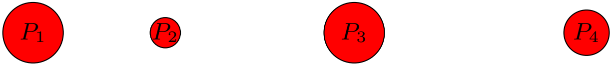

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The **first collision time** $t = t^1$ correspond to

$$x_j(t^1) = x_{j+1}(t^1) = \ldots = x_k(t^1) \quad \text{for some indices } j < k.$$

The particles $P_j, P_{j+1}, \ldots, P_k$ **collapse and stick in a new particle** P with **mass** $m := m_j + \ldots + m_k$ and

"barycentric" velocity $\quad v := \dfrac{m_j v_j(t^1) + m_{j+1} v_{j+1}(t^1) + \ldots + m_k v_k(t^1)}{m}$

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Measure-theoretic description

We thus have:
a **(finite) sequence of collision times** $0 < t^1 < t^2 < \dots$
in each interval $[t^h, t^{h+1})$ a finite number N^h of (suitably relabeled) particles $P_1(t), \cdots, P_{N^h}(t)$, $P_i(t) := (m_i, x_i(t), v_i(t))$.

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We can introduce the measures

$$\varrho_t := \sum_{i=1}^{N^h} m_i \delta_{x_i(t)} \in \mathcal{P}(\mathbb{R}), \quad (\varrho\, v)_t := \sum_{i=1}^{N^h} m_i\, v_i(t)\, \delta_{x_i(t)} \in \mathcal{M}(\mathbb{R})$$

$$f[\varrho_t] := \sum_{i=1}^{N^h} m_i a_i(t) \delta_{x_i(t)} \in \mathcal{M}(\mathbb{R})$$

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They satisfy the **1-dimensional pressureless Euler system** in the sense of distributions

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho\, v) = 0, \\ \partial_t(\varrho\, v) + \partial_x(\varrho\, v^2) = f[\varrho], \end{cases} \quad \text{in } \mathbb{R} \times (0, +\infty); \quad \varrho|_{t=0} = \varrho_0, \quad v|_{t=0} = v_0,$$

The various models

Motion driven by a potential V

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} x_i = v_i, \\ \dfrac{\mathrm{d}}{\mathrm{d}t} v_i = -\partial_x V(x_i) \end{cases} \quad \rightsquigarrow \quad \begin{cases} \partial_t \varrho + \partial_x(\varrho\, v) = 0, \\ \partial_t(\varrho\, v) + \partial_x(\varrho\, v^2) = -\varrho \partial_x V \end{cases}$$

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Non smooth interaction: the **Euler-Poisson system when $W(x) = \pm\lambda|x|$.**

Main problem: continuous limit

Consider a sequence of discrete initial data $\mu_0^n := (\varrho_0^n, \varrho_0^n v_0^n)$ converging to $\mu_0 = (\varrho_0, \varrho_0 v_0)$ in a suitable measure-theoretic sense and let $\mu_t^n = (\varrho_t^n, \varrho_t^n v_t^n)$ be the (discrete) solution of SPS.

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- *Find a suitable characterization of μ_t*
- *Show that $(\varrho_t, \varrho_t v_t)$ solves the pressureless Euler system*

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Main contributions in the case $f[\varrho] \equiv 0$

- Existence and convergence
  - Grenier '95, E-Rykov-Sinai '96: first existence and convergence result.
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$\varrho_0^n \to \varrho_0$ in the L^2-Wasserstein distance,

$v_0^n = v_0$ is given by a continuous function with (at most) linear growth.

In particular the result cover the case when ϱ_0^n, ϱ_0 have a common compact support and $\varrho_0^n \to \varrho_0$ weakly in the sense of distribution (or, equivalently, in the duality with continuous functions).

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- Pioneering ideas which lies (more or less explicitly) at the core of the papers by E-RYKOV-SINAI and BRENIER-GRENIER have been introduced by
  - SHNIRELMAN '86 and further clarified by
  - ANDRIEVWSKY-GURBATOV-SOBOELVSKIĬ '07 in a formal way.

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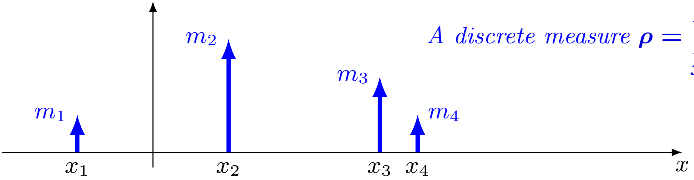
$v_0^n = v_0$ is given by a continuous function with (at most) linear growth.

In particular the result cover the case when ϱ_0^n, ϱ_0 have a common compact support and $\varrho_0^n \to \varrho_0$ weakly in the sense of distribution (or, equivalently, in the duality with continuous functions).

- Pioneering ideas which lies (more or less explicitly) at the core of the papers by E-Rykov-Sinai and Brenier-Grenier have been introduced by
  - Shnirelman '86 and further clarified by
  - Andrievwsky-Gurbatov-Soboelvskiĭ '07 in a formal way.
- Different approaches and models:
  - Bouchut-James '95, Poupaud-Rascle '97
  - Sobolevskiĭ '97, Boudin '00: viscous regularization.
  - Wolansky '07: particles with finite size.

The cumulative distribution function

For every probability measure $\varrho \in \mathcal{P}(\mathbb{R})$



A discrete measure $\boldsymbol{\rho} = \sum_{j=1}^{4} \boldsymbol{m_j \delta_{x_j}}$

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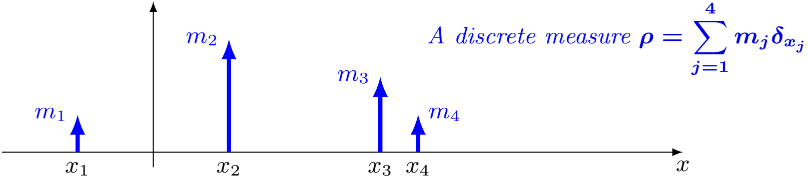
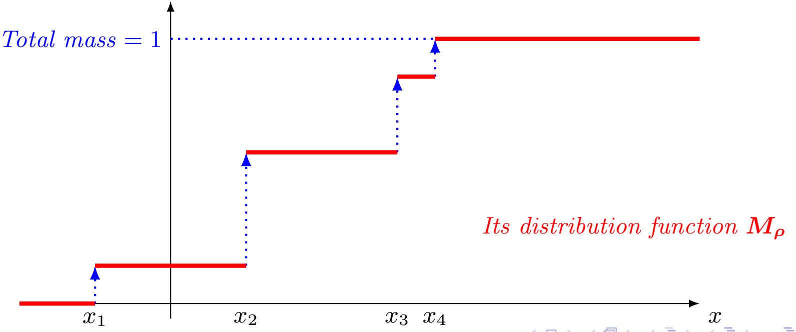


we introduce the **distribution function $M_\rho(x)$** $:= \boldsymbol{\rho}\big((-\infty, x]\big)$,



The Brenier-Grenier formulation in the absence of force

$$M_\varrho(x) := \varrho\big((-\infty, x]\big), \quad x \in \mathbb{R}, \quad \text{so that } \varrho = \partial_x M_\varrho \quad \text{in } \mathscr{D}'(\mathbb{R}).$$

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Theorem (Brenier-Grenier '96)

M is the unique entropy solution of the scalar conservation law

$$\partial_t M + \partial_x A(M) = 0 \quad \textit{in } \mathbb{R} \times (0, +\infty)$$

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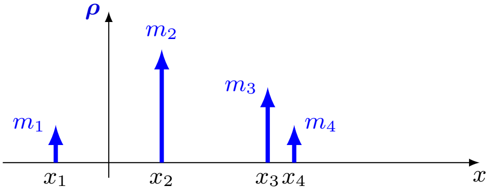
Outline

1. A one-dimensional fluid flow and the sticky particle model
2. Free motion: representation formulas by the cumulative distribution function
3. Monotone rearrangement and Lagrangian representation

Monotone rearrangement



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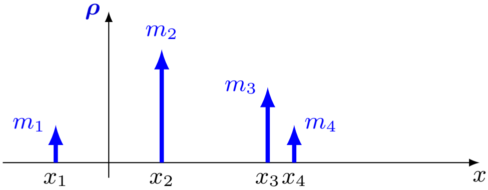
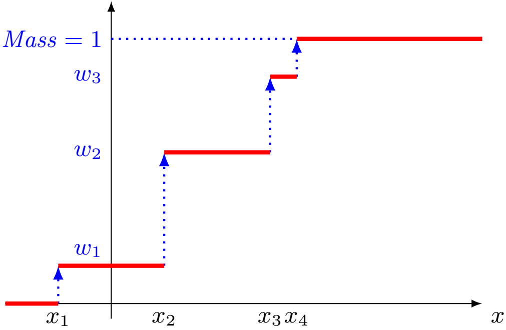


The distribution function $\boldsymbol{M_\rho}$



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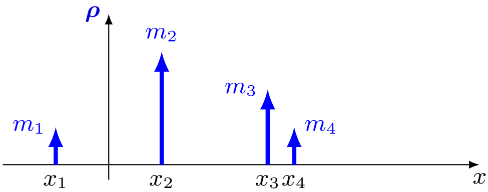

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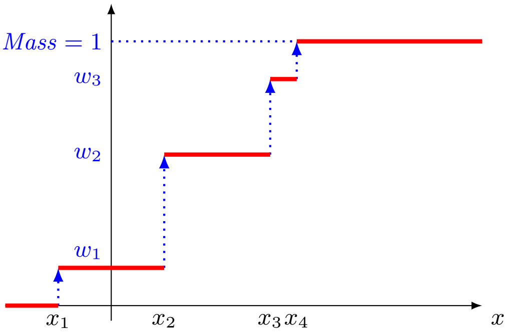

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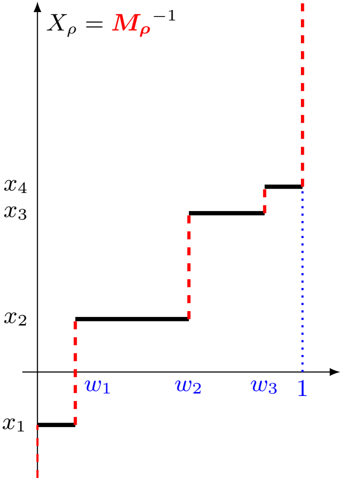

$$w_1 = m_1$$
$$w_2 = m_1 + m_2$$
$$w_3 = m_1 + m_2 + m_3$$

Optimal transport and monotone rearrangement

Point of view of 1-dimensional optimal transport: instead of using the cumulative distribution function $M_\varrho(x) = \varrho\big((-\infty, x]\big)$, we

represent each probability measure ϱ by its monotone rearrangement $X_\varrho : (0,1) \to \mathbb{R}$: $X_\varrho(w)$ is the position x such that $\varrho((-\infty, x]) = w$.

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Wasserstein distance and the L^2 isometry

The map $\varrho \mapsto X_\varrho$ is a **one-to-one correspondence** between

the space $\mathcal{P}_2(\mathbb{R})$ of probability measures with finite quadratic moment $\mathfrak{m}_2(\varrho) = \int_{\mathbb{R}} |x|^2 \, d\varrho(x) < +\infty$

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In this way $\varrho \leftrightarrow X_\varrho$ is an **isometry** between $(\mathcal{P}_2(\mathbb{R}), W_2)$ and $(\mathcal{K}, \| \cdot \|_{L^2(0,1)})$.

Lagrangian parametrizations

To the (discrete) data $\mu_t = (\varrho_t, \varrho_t v_t)$ we associate the functions $(X_t, V_t) \in \mathcal{K} \times L^2(0,1)$ by

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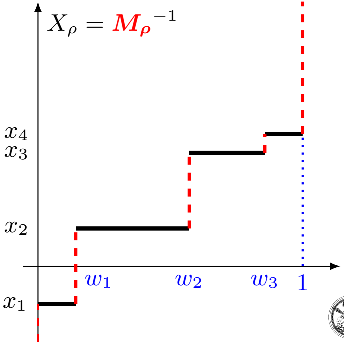
The reduced cone of the discrete particle system with masses $\boldsymbol{m} = (m_1, m_2, \cdots, m_N)$

$$\mathcal{K}^{\boldsymbol{m}} := \Big\{ X \in \mathcal{K} : X|_{[w_{i-1}, w_i)} \equiv x_i \text{ is constant} \Big\}$$

The monotone rearrangement



$w_1 = m_1, \; w_2 = m_1 + m_2, \; \cdots$

The normal cone $N_X\mathcal{K}$

$\Xi \in N_X\mathcal{K}$ if and only if $X \in \mathcal{K}$ and one of the following equivalent conditions holds

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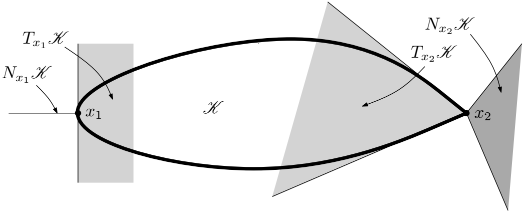

Figure: Normal and tangent cones.

Second order differential equations and jump conditions

Force field: $f[\varrho] \rightsquigarrow F[X_\varrho](w)$ such that

$$\int_{\mathbb{R}} \zeta(x)\,\mathrm{d}f[\varrho] = \int_0^1 \zeta(X_\varrho)F\,\mathrm{d}w \quad \text{for every test function } \zeta \in C_b^0(\mathbb{R}).$$

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After a collision at a time $\bar{t}$:

$$V_{\bar{t}+} = \mathsf{P}_{T_{X(\bar{t})}\mathcal{K}}(V_{\bar{t}-}) \quad \text{i.e. } V_{\bar{t}+} + N_{X(\bar{t})}\mathcal{K} \ni V_{\bar{t}-}$$

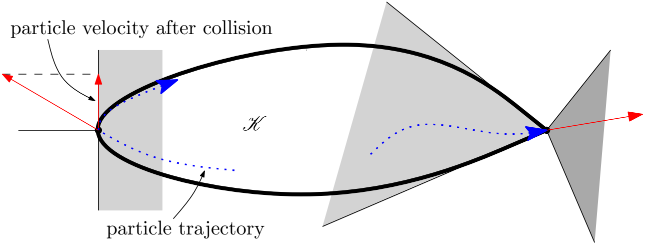

Figure: Projection of velocities onto the tangent cone.

Second order differential inclusions: the role of the sticky condition

"Formal" differential inclusion for the SPS:

$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}t^2}X_t + \partial \mathrm{I}_{\mathcal{K}}(X_t) \ni F[X_t]} \qquad (\star)$$

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By the monotonicity property of $\partial \mathrm{I}_{\mathcal{K}}$ we have

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Reduction to first-order differential inclusions

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We can then integrate $(\star)$ with respect to time from 0 to a final time t: since

$$\int_0^t \partial \mathrm{I}_{\mathcal{K}}(X_s)\,\mathrm{d}s \in \partial \mathrm{I}_{\mathcal{K}}(X_t).$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}X_t - V_0 + \partial \mathrm{I}_{\mathcal{K}}(X_t) \ni \int_0^t F[X_s]\,\mathrm{d}s} \qquad (\star\star)$$

Reduction to first-order differential inclusions

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} X_t + \partial \mathrm{I}_{\mathcal{K}}(X_t) \ni F[X_t], \qquad s < t \quad \Rightarrow \quad \partial \mathrm{I}_{\mathcal{K}}(X_s) \subset \partial \mathrm{I}_{\mathcal{K}}(X_t) \qquad (\star)$$

We can then integrate $(\star)$ with respect to time from 0 to a final time t: since

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Integrating again we get

$$X_t - (X_0 + tV_0) + \partial \mathrm{I}_{\mathcal{K}}(X_t) \ni \int_0^t (t-s) F[X_s]\,\mathrm{d}s$$

i.e.

$$\boxed{X_t = \mathsf{P}_{\mathcal{K}}\Big(X_0 + tV_0 + \int_0^t (t-s) F[X_s]\,\mathrm{d}s\Big)} \qquad (\star\star\star)$$

A description of the evolution by a differential inclusion

Recall that to the (discrete) data $\mu_t = (\varrho_t, \varrho_t v_t)$ we associate the functions $(X_t, V_t) \in \mathcal{K} \times L^2(0,1)$ by

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Theorem (Lagrangian representation)

A family $\mu_t = (\varrho_t, \varrho_t v_t)$ is a solution of the (discrete) SPS if and only if X is the unique strong solution of the differential inclusion

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Equivalently, setting $Y_t := V_0 + \int_0^t F[X_s]\,\mathrm{d}s$ we get the **well posed** system

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X_t + \partial \mathrm{I}_{\mathcal{K}}(X_t) \ni Y_t \\ \qquad\qquad \dfrac{\mathrm{d}}{\mathrm{d}t} Y_t = F[X_t] \end{cases}$$

Stability properties

Suppose that F is Lipschitz in $L^2(0,1)$. By general results on solution of differential inclusion of the type $Z_t' + \partial\phi(Z_t) \ni G_t$, ϕ convex

Theorem

If X_t^1, X_t^2 are the Lagrangian representation of two (discrete) solutions ϱ_t^1, ϱ_t^2 of the SPS we have

$$\sup_{t\in[0,T]} \|X_t^1 - X_t^2\|_{L^2} \le C_T\Big(\|X_0^1 - X_0^2\|_{L^2} + \|V_0^1 - V_0^2\|_{L^2}\Big).$$

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X is right-differentiable in each point and the velocity field v_t can be recovered by the formula

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One gets [S. '96] *the following integral estimate for the velocity component:*

$$\int_0^T \|V^1_r - V^2_r\|^2_{L^2}\,\mathrm{d}r \le C_T\Big(\sum_\ell \|X^\ell_0\| + \|V^\ell_0\|\Big)\Big(\|X^1_0 - X^2_0\| + \|V^1_0 - V^2_0\|\Big).$$

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and $(\rho_t, \rho_t v_t)$ is a solution of

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho\, v) = 0, \\ \partial_t(\varrho\, v) + \partial_x(\varrho\, v^2) = f[\varrho], \end{cases} \qquad \varrho|_{t=0} = \varrho_0, \quad v|_{t=0} = v_0,$$

An explicit representation formula for the Euler-Poisson system

[Tadmor-Wei]

$$f[\varrho] = -\varrho\,\partial_x q_\varrho \quad \text{with } q_\varrho \text{ solution of} \quad -\partial^2_{xx} q_\varrho = \varrho$$

$$F[X] = 1 - 2w$$

$$X_t = \mathsf{P}_{\mathcal{K}}\Big(X_0 + tV_0 + (1-2w)t^2\Big).$$

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The L^2-projection on $\mathcal{K}$

Theorem

If $X \in L^2(0,1)$ *and* $\mathscr{X}(w) = \int_0^w X(s)\,\mathrm{d}s$ *is its primitive then*

$$\mathsf{P}_{\mathcal{K}}(X) = \frac{\mathrm{d}}{\mathrm{d}w}\mathscr{X}^{**} \qquad \textit{where } \mathscr{X}^{**} \textit{ is the convex envelope of } \mathscr{X}.$$

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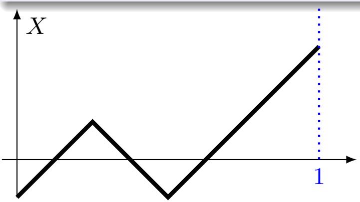

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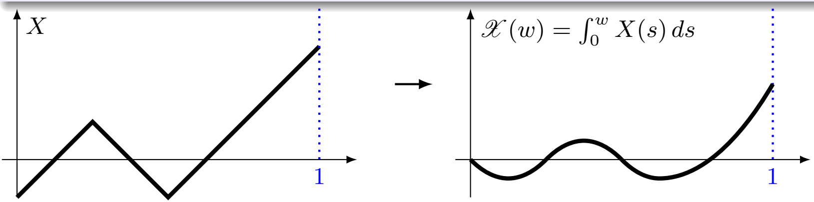

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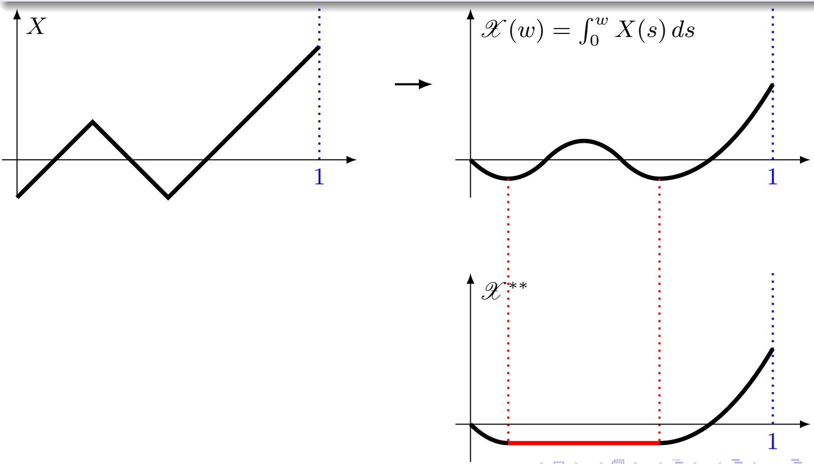

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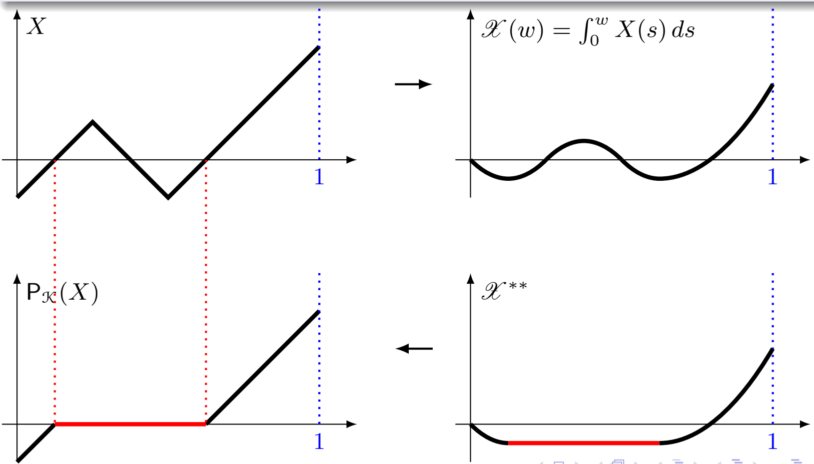

The subdifferential of $\mathrm{I}_{\mathcal{K}}$

If $X \in \mathcal{K}$ we consider the open set Ω_X where X is (essentially) constant:

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Theorem

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Notice that if $Z = f(X) \in \mathcal{H}_X$ depends on X then

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Corollary (Monotonicity property of $\partial \mathrm{I}_{\mathcal{K}}$)

If $Z = f(X) \in \mathcal{H}_X$ depends on X then

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