An entropy-satisfying relaxation approximation for the isentropic Baer & Nunziato model with vanishing phases

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Introduction

Joint work with

Frédéric Coquel, Jean-Marc Hérard & Nicolas Seguin.

Framework:

- This work is financially supported by EDF.
- Two-phase flows for the study of nuclear reactors' heart.
- Modelling and development of numerical schemes.

Outline

- The Baer & Nunziato model and its properties,
- A relaxation approximation for the Baer & Nunziato model,
- The Riemann problem for the relaxation system,
- A relaxation numerical scheme,
- Conclusion and perspectives.

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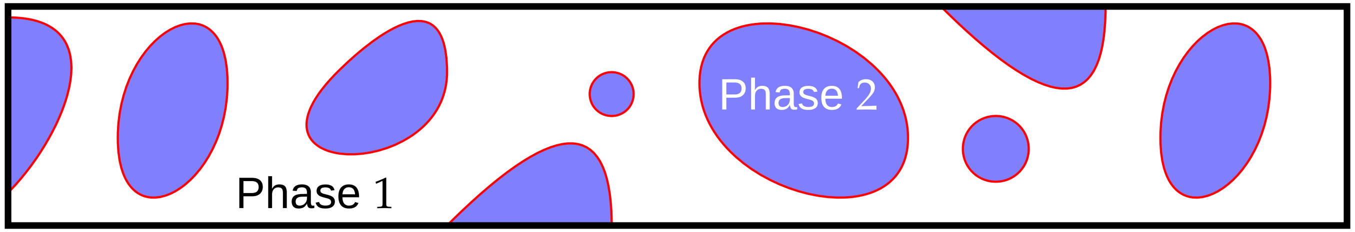
The Baer-Nunziato two phase flow model



$$
\begin{aligned}
&\partial_t \alpha_1 + V_I \partial_x \alpha_1 = 0,\\
&\partial_t(\alpha_1\rho_1) + \partial_x(\alpha_1\rho_1 u_1) = 0,\\
&\partial_t(\alpha_1\rho_1 u_1) + \partial_x(\alpha_1\rho_1 u_1^2 + \alpha_1 p_1(\rho_1)) - P_I \partial_x \alpha_1 = 0,\\
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\end{aligned}
$$

- For $k = 1, 2$, α_k, ρ_k, u_k: statistical fraction, density and velocity of phase k, with $\alpha_1 + \alpha_2 = 1$,
- $\rho_k \mapsto p_k(\rho_k)$: barotropic pressure laws,
- (V_I, P_I) interface velocity and pressure:

$$(V_I, P_I) = (u_2, p_1).$$

The Baer-Nunziato two phase flow model

Contributors

On the complete model (with energy equations)

- Abgrall & Saurel,
- Ambroso, Chalons, & Raviart,
- Andrianov & Warnecke,
- Baer & Nunziato,
- Deledicque & Papalexandris,
- Embid & Baer,
- Gallouët, Coquel, Hérard & Seguin,
- Gavrilyuk & Saurel,
- Schwendeman, Whale & Kapila,
- Stewart & Wendroff,
- Tokareva & Toro...

On the isentropic model (without energy equations)

- Ambroso, Chalons, Coquel & Galié,
- Thanh, Knöner & Nguyen Thanh Nam.

The Baer-Nunziato two phase flow model

$$(\mathscr{H}) \qquad \begin{aligned} &\partial_t \alpha_1 + u_2 \partial_x \alpha_1 = 0, \\ &\partial_t(\alpha_1\rho_1) + \partial_x(\alpha_1\rho_1 u_1) = 0, \\ &\partial_t(\alpha_1\rho_1 u_1) + \partial_x(\alpha_1\rho_1 u_1^2 + \alpha_1 p_1(\rho_1)) - p_1 \partial_x \alpha_1 = 0, \\ &\partial_t(\alpha_2\rho_2) + \partial_x(\alpha_2\rho_2 u_2) = 0, \\ &\partial_t(\alpha_2\rho_2 u_2) + \partial_x(\alpha_2\rho_2 u_2^2 + \alpha_2 p_2(\rho_2)) - p_1 \partial_x \alpha_2 = 0. \end{aligned}$$

State vector : $\mathbb{U} = (\alpha_1, \alpha_1\rho_1, \alpha_1\rho_1 u_1, \alpha_2\rho_2, \alpha_2\rho_2 u_2)$

Principal properties:

- Four GNL fields $u_k \pm c_k(\rho_k)$ with $c(\rho_k) = \sqrt{p_k'(\rho_k)}$, and one LD field : u_2,
- The system is hyperbolic iff $|u_1 - u_2| \neq c_1(\rho_1)$, otherwise: resonance.
- The entropy weak solutions of $(\mathscr{H})$ satisfy the following inequality

$$\partial_t \eta(\mathbb{U}) + \partial_x \mathcal{F}_\eta(\mathbb{U}) \leqslant 0,$$

  where $\eta(\mathbb{U}) := \sum_{k=1}^{2} \alpha_k \rho_k E_k$ and $\mathcal{F}_\eta(\mathbb{U}) := \sum_{k=1}^{2} \alpha_k(\rho_k E_k + p_k(\rho_k))u_k$. Phasic energies: $E_k := \frac{u_k^2}{2} + e_k(\tau_k)$.
- The mapping $\mathbb{U} \mapsto \eta(\mathbb{U})$ is convex.

The Baer-Nunziato two phase flow model

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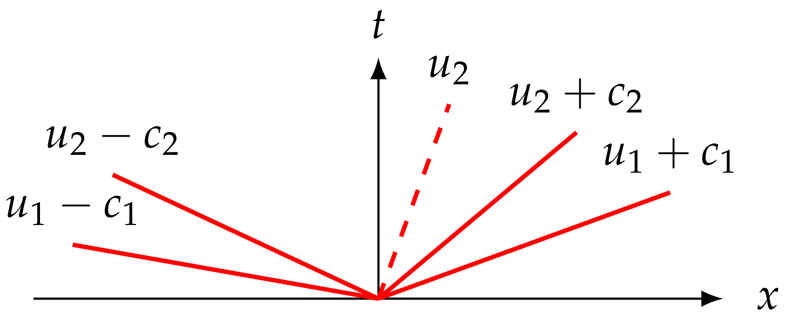
No complete resolution of the Riemann problem: $\mathbb{U}(x,t=0) = \begin{cases} \mathbb{U}_L & \text{if} \quad x < 0, \\ \mathbb{U}_R & \text{if} \quad x > 0. \end{cases}$

- Waves's ordering, resonance,



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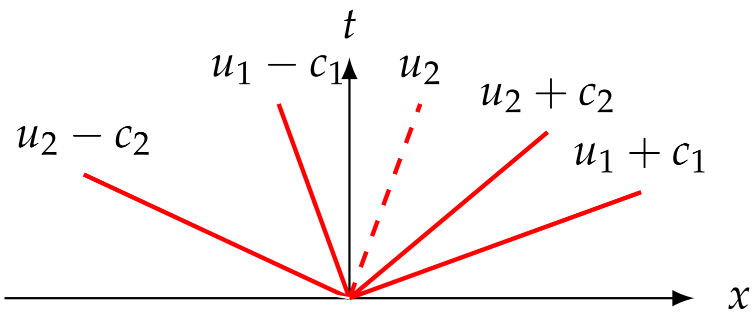

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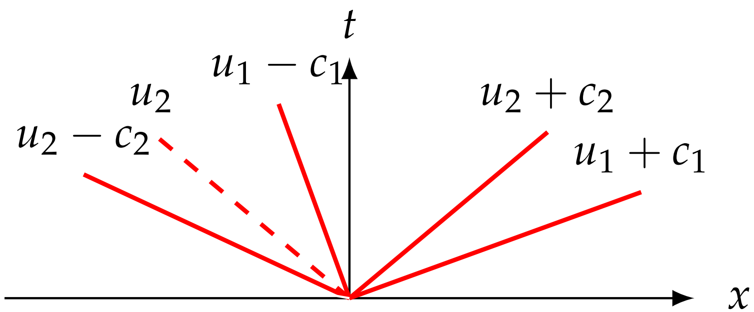

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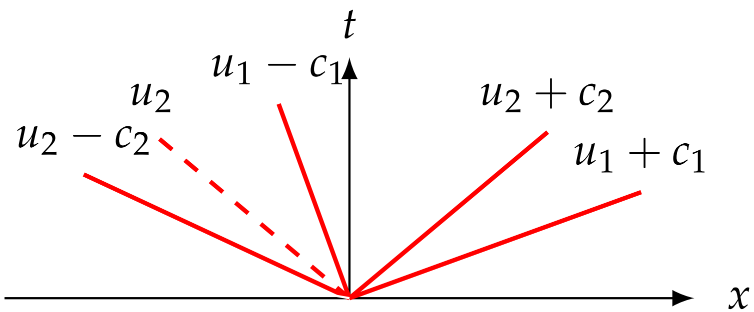

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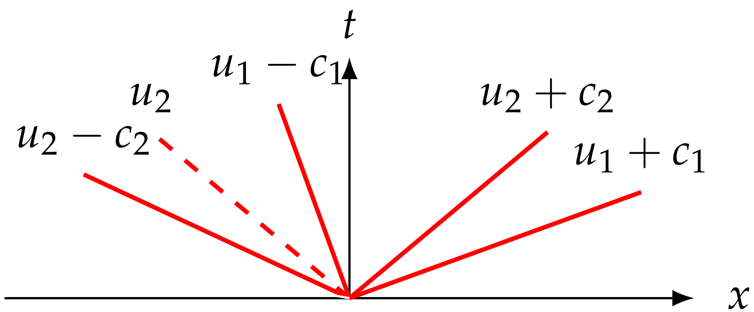

- Non-linearity of the pressure-laws, GNL fields: $\rho_k \longmapsto p_k(\rho_k)$,
- Treatment of the non-conservative terms $p_1 \partial_x \alpha_1$.

The Riemann problem for $(\mathscr{H})$

What is the definition of $p_1 \partial_x \alpha_1$?

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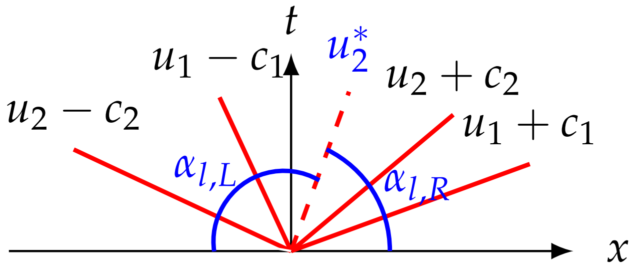
In the domain of hyperbolicity:



- ▷ $\partial_x \alpha_k = (\alpha_{k,R} - \alpha_{k,L})\delta_0(x - u_2^* t)$.

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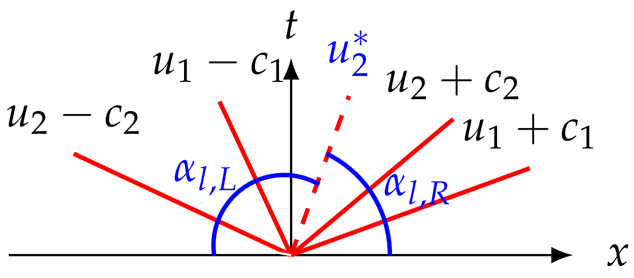

- $\partial_x \alpha_k = (\alpha_{k,R} - \alpha_{k,L}) \delta_0(x - u_2^* t)$.
- Outside the u_2-contact: $\alpha_k = cst$, classical phasic acoustic waves.

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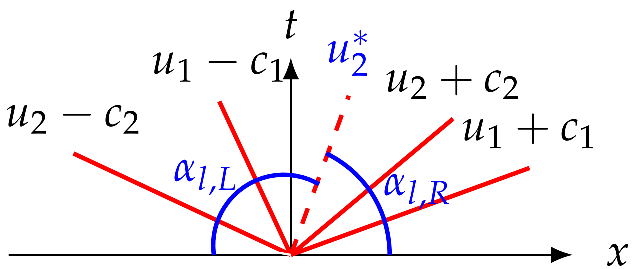

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- Outside the u_2-contact: $\alpha_k = cst$, classical phasic acoustic waves.
- Across the u_2-contact (Riemann invariants)

$$
\begin{aligned}
[u_2] &= 0, \\
-u_2^* [\alpha_1 \rho_1] + [\alpha_1 \rho_1 u_1] &= 0, \\
-u_2^* [\alpha_1 \rho_1 u_1 + \alpha_2 \rho_2 u_2] + [\alpha_1 \rho_1 u_1^2 + \alpha_1 p_1 + \alpha_2 \rho_2 u_2^2 + \alpha_2 p_2] &= 0, \\
-u_2^* [\eta] + [\mathcal{F}_\eta] &= 0.
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$$

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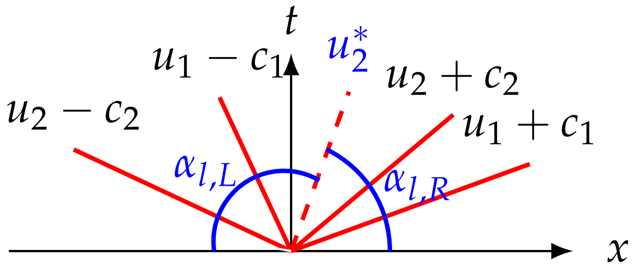

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$$[u_2] = 0,$$

$$-u_2^*[\alpha_1\rho_1] + [\alpha_1\rho_1 u_1] = 0,$$

$$-u_2^*[\alpha_1\rho_1 u_1 + \alpha_2\rho_2 u_2] + [\alpha_1\rho_1 u_1^2 + \alpha_1 p_1 + \alpha_2\rho_2 u_2^2 + \alpha_2 p_2] = 0,$$

$$-u_2^*[\eta] + [\mathcal{F}_\eta] = 0.$$

The non-conservative product $p_1 \partial_x \alpha_1$ is a Dirac measure supported by the half-line $x - u_2^* t = 0$ and whose mass is given by

$$p_1^*(\alpha_{k,R} - \alpha_{k,L}) := -u_2^*[\alpha_1\rho_1 u_1] + [\alpha_1\rho_1 u_1^2 + \alpha_1 p_1].$$

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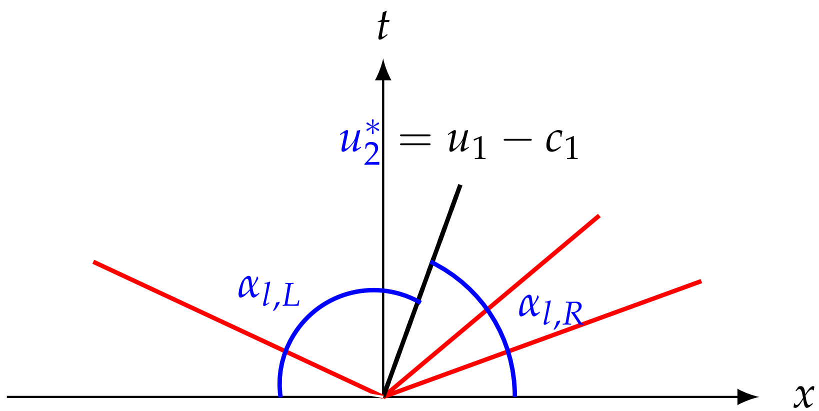
In the case of resonance:

A shock may superimpose with the u_2-contact (loss of hyperbolicity)



The total energy is dissipated through the discontinuity of α_1:

$$-u_2^*[\eta] + [\mathcal{F}_\eta] < 0.$$

Dal Maso-LeFloch-Murat, Isaacson-Temple, Goatin-LeFloch.

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A relaxation approximation

Main properties of the relaxation scheme:

- Conservativity for the partial masses $\alpha_k\rho_k$ and for the total momentum $\alpha_1\rho_1u_1 + \alpha_2\rho_2u_2$.
- Positivity of the fractions and densities at the discrete level:

$$0 < (\alpha_k)_j^n < 1 \quad \text{and} \quad (\alpha_k\rho_k)_j^n > 0, \qquad \forall\, j \in \mathbb{Z}, \qquad \forall\, n \in \mathbb{N},$$

- Non-linear stability: Assuming a sub-characteristic condition: a discrete entropy inequality is satisfied by the scheme:

$$\eta(\mathbb{U}_j^{n+1}) - \eta(\mathbb{U}_j^n) + \frac{\Delta t}{\Delta x}\left(H(\mathbb{U}_j^n, \mathbb{U}_{j+1}^n) - H(\mathbb{U}_{j-1}^n, \mathbb{U}_j^n)\right) \leqslant 0,$$

  where the numerical entropy flux $H(\mathbb{U}_L, \mathbb{U}_R)$ is such that $H(\mathbb{U}, \mathbb{U}) = \mathcal{F}_\eta(\mathbb{U})$.
- Good behiavour fou vanishing phase cases.
- High level of accuracy for a first-order scheme.
- Small computational cost.

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Introduce a larger but "more linear" system that *relaxes* towards $(\mathscr{H})$ in the limit $\varepsilon \to 0$: For $k \in \{1,2\}$:

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with $\pi_k(\tau_k, \mathcal{T}_k) = p_k(\mathcal{T}_k) + {a_k}^2(\mathcal{T}_k - \tau_k)$, où $\tau_k = \rho_k^{-1}$.

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Many Contributors:

Berthon, Bouchut, Bouchut-Klingenberg, Chalons-Coulombel, Coquel *et al*, Jin-Xin,...

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Good properties of $(\mathscr{R})$:

- The relaxation system has only LD fields. All the waves are contact discontinuities,
- There exists an entropy for system $(\mathscr{R})$:

$$\partial_t \eta^r(\mathbb{W}) + \partial_x \mathcal{F}_\eta^r(\mathbb{W}) = 0,$$

$$\eta^r(\mathbb{W}) := \textstyle\sum_{k=1}^2 \alpha_k \rho_k \mathcal{E}_k \text{ with } \mathcal{E}_k := \frac{u_k^2}{2} + e_k(\mathcal{T}_k) + \frac{1}{2a_k^2}(\pi_k^2(\tau_k, \mathcal{T}_k) - p_k^2(\mathcal{T}_k)).$$

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Difficulties linked with $(\mathscr{R})$:

- The eigenvalues $u_k - a_k \tau_k$, u_k and $u_k + a_k \tau_k$ are note naturally ordered.
- A possible phenomenon of resonance.
- The two phases are coupled through the non-conservative terms $\pi_1 \partial_x \alpha_k$.

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The Riemann problem for $(\mathscr{R})$

$$\begin{cases} \partial_t \mathbb{W} + \partial_x \mathbf{g}(\mathbb{W}) + \mathbf{d}(\mathbb{W})\partial_x \mathbb{W} = 0, \quad x \in \mathbb{R},\ t > 0, \\ \\ \mathbb{W}(x, t=0) = \begin{cases} \mathbb{W}_L & \text{if} \quad x < 0, \\ \mathbb{W}_R & \text{if} \quad x > 0. \end{cases} \end{cases}$$

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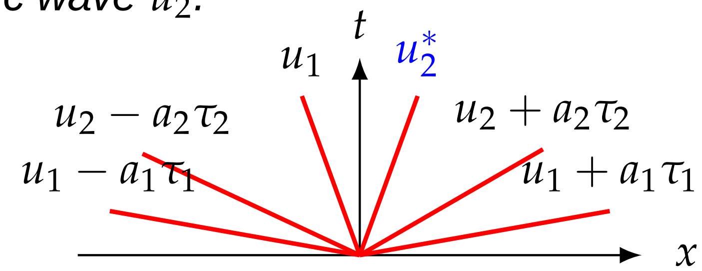
Definition. *A solution of the Riemann problem with subsonic relative speed is a self-similar weak solution* $\mathbb{R} \times \mathbb{R}_+ \ni (x, t) \mapsto \mathbb{W}(x/t; \mathbb{W}_L, \mathbb{W}_R)$ *such that*

$$|u_1 - u_2^*|(x, t) < a_1 \tau_1(x, t), \quad \text{for all } (x, t) \text{ in } \mathbb{R} \times \mathbb{R}_+.$$

where u_2^* *is the effective value, once the problem is solved, of the propagation speed associated with the kinematic wave* u_2.



The Riemann problem for $(\mathscr{R})$

$$\begin{cases} \partial_t \mathbb{W} + \partial_x \mathbf{g}(\mathbb{W}) + \mathbf{d}(\mathbb{W})\partial_x \mathbb{W} = 0, \quad x \in \mathbb{R},\ t > 0, \\ \\ \mathbb{W}(x, t = 0) = \begin{cases} \mathbb{W}_L & \text{if} \quad x < 0, \\ \mathbb{W}_R & \text{if} \quad x > 0. \end{cases} \end{cases}$$

Definition. *A solution of the Riemann problem with subsonic relative speed is a self-similar weak solution* $\mathbb{R} \times \mathbb{R}_+ \ni (x, t) \mapsto \mathbb{W}(x/t; \mathbb{W}_L, \mathbb{W}_R)$ *such that*

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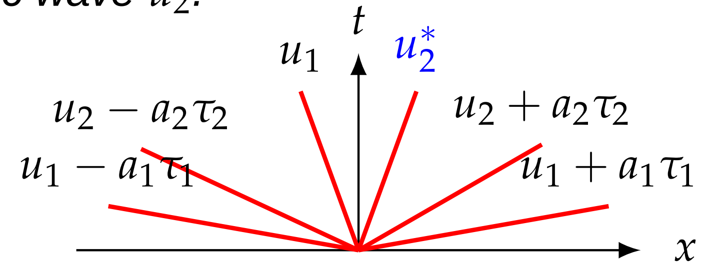

Moreover, we authorize total energy dissipation through the u_2*-wave:*

$$\partial_t \eta^r(\mathbb{W}) + \partial_x \mathcal{F}_\eta^r(\mathbb{W}) = -f\delta_0(x - u_2^* t),$$

where $f \geqslant 0$ *is a non-negative real number.*

The Riemann problem for $(\mathscr{R})$

Theorem. *System $(\mathscr{R})$ admits solutions to the Riemann problem with subsonic relative speeds if and only if the initial data $(\mathbb{W}_L, \mathbb{W}_R) \in \Omega^r \times \Omega^r$ is such that*

$$-a_1 \tau^\sharp_{1,R} < \frac{u^\sharp_1 - u^\sharp_2 - \frac{1}{a_2}\Lambda^\alpha(\pi^\sharp_1 - \pi^\sharp_2)}{1 + \frac{a_1}{a_2}|\Lambda^\alpha|} < a_1 \tau^\sharp_{1,L}$$

with

- *A term measuring the velocity and pressure difference between the phases in the initial data:*
$$\frac{u^\sharp_1 - u^\sharp_2 - \frac{1}{a_2}\Lambda^\alpha(\pi^\sharp_1 - \pi^\sharp_2)}{1 + \frac{a_1}{a_2}|\Lambda^\alpha|}$$
- $a_1 \tau^\sharp_{1,L}$ *and* $-a_1 \tau^\sharp_{1,R}$ *two "acoustic waves" built on in the initial data.*

Moreover, there exists at most one energy preserving solution and in some cases, preserving subsonic solutions in relative speeds involves a dissipation of the total energy through the coupling wave u_2:

$$\partial_t \eta^r(\mathbb{W}) + \partial_x \mathcal{F}^r_\eta(\mathbb{W}) = -f\delta_0(x - u^*_2 t), \quad \text{where } f \geqslant 0.$$

The Riemann problem for $(\mathscr{R})$

Sketch of the proof:

$$(\mathscr{R}) \quad \begin{cases} \partial_t \alpha_1 + u_2 \cdot \partial_x \alpha_1 = 0, \\ \partial_t(\alpha_k \rho_k) + \partial_x(\alpha_k \rho_k u_k) = 0, \\ \partial_t(\alpha_k \rho_k u_k) + \partial_x(\alpha_k \rho_k u_k^2 + \alpha_k \pi_k(\tau_k, \mathcal{T}_k)) - \pi_1 \partial_x \alpha_k = 0, \\ \partial_t(\alpha_k \rho_k \mathcal{T}_k) + \partial_t(\alpha_k \rho_k u_k \mathcal{T}_k) = 0. \end{cases}$$

The Riemann problem for $(\mathscr{R})$

Sketch of the proof:

$$
(\mathscr{R}) \qquad
\begin{aligned}
&\partial_t \alpha_1 + u_2 \cdot \partial_x \alpha_1 = 0, \\
&\partial_t(\alpha_k \rho_k) + \partial_x(\alpha_k \rho_k u_k) = 0, \\
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&\partial_t(\alpha_k \rho_k \mathcal{T}_k) + \partial_t(\alpha_k \rho_k u_k \mathcal{T}_k) = 0.
\end{aligned}
$$

Let u_2^* be the effective propagation velocity of the u_2-wave once the Riemann problem solved.

The Riemann problem for $(\mathscr{R})$

Sketch of the proof:

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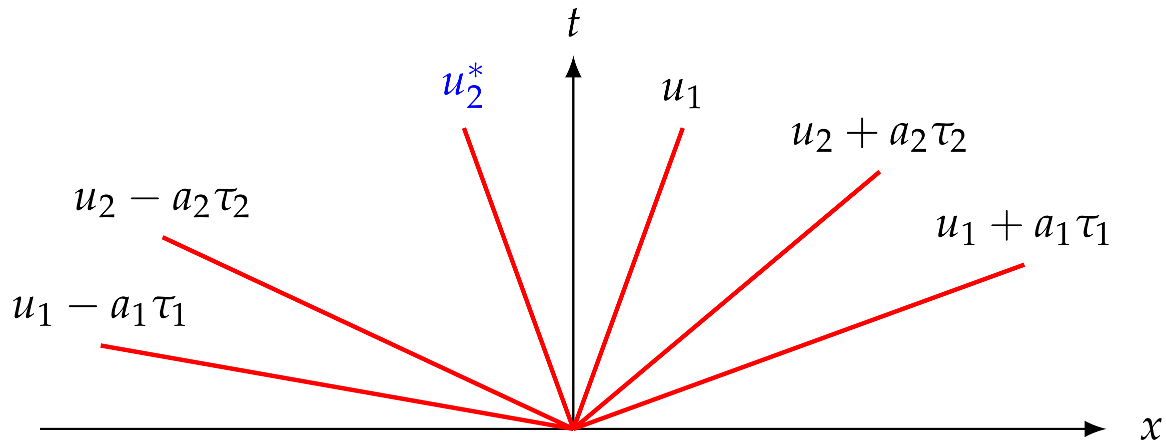
▷ We have $\partial_x \alpha_k = (\alpha_{k,R} - \alpha_{k,L})\delta_0(x - u_2^* t)$.



The Riemann problem for $(\mathscr{R})$

Sketch of the proof:

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Let u_2^* be the effective propagation velocity of the u_2-wave once the Riemann problem solved.

- ▷ We have $\partial_x \alpha_k = (\alpha_{k,R} - \alpha_{k,L})\delta_0(x - u_2^* t)$.
- ▷ Suppose we can predict the value of the Dirac's weight : π_1^*

The Riemann problem for $(\mathscr{R})$

Sketch of the proof:

$$(\mathscr{R}) \quad \begin{aligned} &\partial_t \alpha_1 + u_2^* \cdot \partial_x \alpha_1 = 0, \\ &\partial_t(\alpha_k \rho_k) + \partial_x(\alpha_k \rho_k u_k) = 0, \\ &\partial_t(\alpha_k \rho_k u_k) + \partial_x(\alpha_k \rho_k u_k^2 + \alpha_k \pi_k(\tau_k, \mathcal{T}_k)) - \pi_1 \partial_x \alpha_k = 0, \\ &\partial_t(\alpha_k \rho_k \mathcal{T}_k) + \partial_t(\alpha_k \rho_k u_k \mathcal{T}_k) = 0. \end{aligned}$$

Let u_2^* be the effective propagation velocity of the u_2-wave once the Riemann problem solved.

- ▷ We have $\partial_x \alpha_k = (\alpha_{k,R} - \alpha_{k,L})\delta_0(x - u_2^* t)$.
- ▷ Suppose we can predict the value of the Dirac's weight : π_1^*

$$(\mathscr{R}) \quad \begin{aligned} &\partial_t \alpha_1 + u_2^* \cdot \partial_x \alpha_1 = 0, \\ &\partial_t(\alpha_k \rho_k) + \partial_x(\alpha_k \rho_k u_k) = 0, \\ &\partial_t(\alpha_k \rho_k u_k) + \partial_x(\alpha_k \rho_k u_k^2 + \alpha_k(\pi_k(\tau_k, \mathcal{T}_k) - \pi_1^*)) = 0, \\ &\partial_t(\alpha_k \rho_k \mathcal{T}_k) + \partial_t(\alpha_k \rho_k u_k \mathcal{T}_k) = 0. \end{aligned}$$

The Riemann problem for $(\mathscr{R})$

Sketch of the proof:

The two phase are decoupled

The Riemann problem for $(\mathscr{R})$

Sketch of the proof:
The two phase are decoupled
Phase 2

$$\partial_t \alpha_2 + u_2^* \partial_x \alpha_2 = 0,$$
$$\partial_t (\alpha_2 \rho_2) + \partial_x (\alpha_2 \rho_2 u_2) = 0,$$
$$\partial_t (\alpha_2 \rho_2 u_2) + \partial_x (\alpha_2 \rho_2 u_2^2 + \alpha_2 (\pi_2(\tau_2, \mathcal{T}_2) - \pi_1^*)) = 0,$$
$$\partial_t (\alpha_2 \rho_2 \mathcal{T}_2) + \partial_x (\alpha_2 \rho_2 u_2 \mathcal{T}_2) = 0$$

Knowing π_1^*, we can calculate u_2^* : $\qquad u_2^* = \Phi(\pi_1^*)$.

The Riemann problem for $(\mathscr{R})$

Sketch of the proof:
The two phase are decoupled
Phase 2

$$\begin{aligned}
&\partial_t\alpha_2 + u_2^*\partial_x\alpha_2 = 0,\\
&\partial_t(\alpha_2\rho_2) + \partial_x(\alpha_2\rho_2 u_2) = 0,\\
&\partial_t(\alpha_2\rho_2 u_2) + \partial_x(\alpha_2\rho_2 u_2^2 + \alpha_2(\pi_2(\tau_2,\mathcal{T}_2) - \pi_1^*)) = 0,\\
&\partial_t(\alpha_2\rho_2\mathcal{T}_2) + \partial_x(\alpha_2\rho_2 u_2\mathcal{T}_2) = 0
\end{aligned}$$

Knowing π_1^*, we can calculate u_2^* : $\quad u_2^* = \Phi(\pi_1^*)$.

Phase 1

$$\begin{aligned}
&\partial_t\alpha_1 + u_2^*\partial_x\alpha_1 = 0,\\
&\partial_t(\alpha_1\rho_1) + \partial_x(\alpha_1\rho_1 u_1) = 0,\\
&\partial_t(\alpha_1\rho_1 u_1) + \partial_x(\alpha_1\rho_1 u_1^2 + \alpha_1\pi_1(\tau_1,\mathcal{T}_1)) = \pi_1^*\partial_x\alpha_1,\\
&\partial_t(\alpha_1\rho_1\mathcal{T}_1) + \partial_x(\alpha_1\rho_1 u_1\mathcal{T}_1) = 0
\end{aligned}$$

Knowing u_2^*, we can calculate π_1^*: $\quad \pi_1^* = \Psi(u_2^*)$.

The Riemann problem for $(\mathscr{R})$

Sketch of the proof:

The two phase are decoupled

Phase 2

$$\partial_t \alpha_2 + u_2^* \partial_x \alpha_2 = 0,$$

$$\partial_t(\alpha_2 \rho_2) + \partial_x(\alpha_2 \rho_2 u_2) = 0,$$

$$\partial_t(\alpha_2 \rho_2 u_2) + \partial_x(\alpha_2 \rho_2 u_2^2 + \alpha_2(\pi_2(\tau_2, \mathcal{T}_2) - \pi_1^*)) = 0,$$

$$\partial_t(\alpha_2 \rho_2 \mathcal{T}_2) + \partial_x(\alpha_2 \rho_2 u_2 \mathcal{T}_2) = 0$$

Knowing π_1^*, we can calculate u_2^* : $\quad u_2^* = \Phi(\pi_1^*)$.

Phase 1

$$\partial_t \alpha_1 + u_2^* \partial_x \alpha_1 = 0,$$

$$\partial_t(\alpha_1 \rho_1) + \partial_x(\alpha_1 \rho_1 u_1) = 0,$$

$$\partial_t(\alpha_1 \rho_1 u_1) + \partial_x(\alpha_1 \rho_1 u_1^2 + \alpha_1 \pi_1(\tau_1, \mathcal{T}_1)) = \pi_1^*(\alpha_{1,R} - \alpha_{1,L})\delta_0(x - u_2^* t),$$

$$\partial_t(\alpha_1 \rho_1 \mathcal{T}_1) + \partial_x(\alpha_1 \rho_1 u_1 \mathcal{T}_1) = 0$$

Knowing u_2^*, we can calculate π_1^*: $\quad \pi_1^* = \Psi(u_2^*)$.

The Riemann problem for $(\mathscr{R})$

Sketch of the proof:

The two phase are decoupled

Phase 2

$$\partial_t \alpha_2 + u_2^* \partial_x \alpha_2 = 0,$$
$$\partial_t(\alpha_2\rho_2) + \partial_x(\alpha_2\rho_2 u_2) = 0,$$
$$\partial_t(\alpha_2\rho_2 u_2) + \partial_x(\alpha_2\rho_2 u_2^2 + \alpha_2(\pi_2(\tau_2,\mathcal{T}_2) - \pi_1^*)) = 0,$$
$$\partial_t(\alpha_2\rho_2\mathcal{T}_2) + \partial_x(\alpha_2\rho_2 u_2\mathcal{T}_2) = 0$$

Knowing π_1^*, we can calculate u_2^* : $\quad u_2^* = \Phi(\pi_1^*)$.

Phase 1

$$\partial_t \alpha_1 + u_2^* \partial_x \alpha_1 = 0,$$
$$\partial_t(\alpha_1\rho_1) + \partial_x(\alpha_1\rho_1 u_1) = 0,$$
$$\partial_t(\alpha_1\rho_1 u_1) + \partial_x(\alpha_1\rho_1 u_1^2 + \alpha_1\pi_1(\tau_1,\mathcal{T}_1)) = \pi_1^*(\alpha_{1,R} - \alpha_{1,L})\delta_0(x - u_2^* t),$$
$$\partial_t(\alpha_1\rho_1\mathcal{T}_1) + \partial_x(\alpha_1\rho_1 u_1\mathcal{T}_1) = 0$$

Knowing u_2^*, we can calculate π_1^*: $\quad \pi_1^* = \Psi(u_2^*)$.

Fixed Point: $\quad u_2^* = (\Phi \circ \Psi)(u_2^*)$

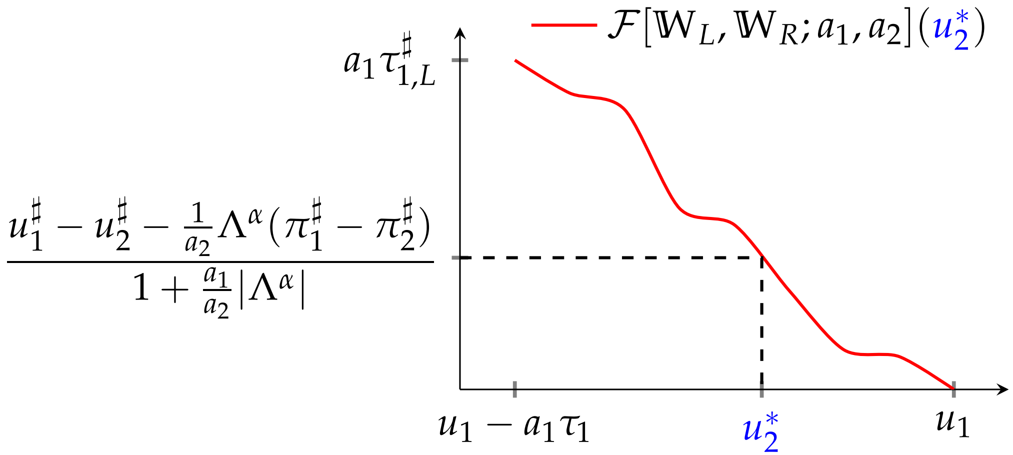
The Riemann problem for $(\mathscr{R})$

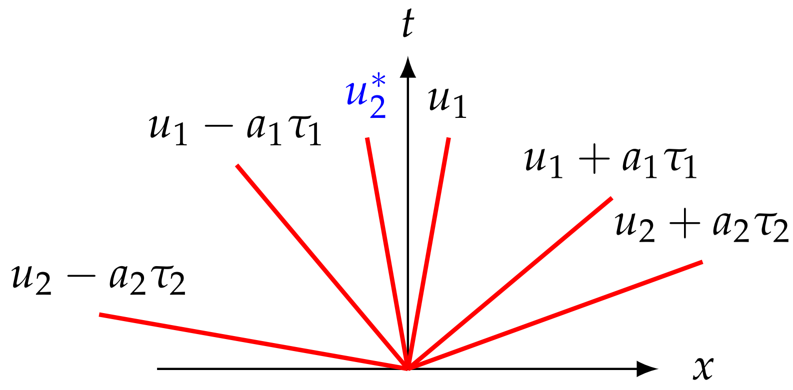
Interpretation of the inequalities:

$$0 < \frac{u_1^\sharp - u_2^\sharp - \frac{1}{a_2}\Lambda^\alpha(\pi_1^\sharp - \pi_2^\sharp)}{1 + \frac{a_1}{a_2}|\Lambda^\alpha|} < a_1 \tau_{1,L}^\sharp$$

Necessary and sufficient condition for the existence of a solution with a subsonic wave ordering, *i.e.* for the existence of a solution to the fixed point problem:

$$u_2^* = \Big(\Phi \circ \Psi\Big)(u_2^*) \quad \Longleftrightarrow \quad \mathcal{F}[\mathbb{W}_L, \mathbb{W}_R; a_1, a_2](u_2^*) = \frac{u_1^\sharp - u_2^\sharp - \frac{1}{a_2}\Lambda^\alpha(\pi_1^\sharp - \pi_2^\sharp)}{1 + \frac{a_1}{a_2}|\Lambda^\alpha|}.$$





Outline

- The Baer & Nunziato model and its properties,
- A relaxation approximation for the Baer & Nunziato model,
- The Riemann problem for the relaxation system,
- A relaxation numerical scheme,
- Conclusion and perspectives.

A relaxation scheme for the Baer-Nunziato model

We may design a numerical scheme relying on a relaxation approximate Riemann solver.

$$\partial_t \mathbb{W} + \partial_x \mathbf{f}(\mathbb{W}) + \mathbf{d}(\mathbb{W})\partial_x \alpha_1 = 0.$$

Integrate over $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [t^n, t^{n+1}]$:

$$\mathbb{W}_j^{n+1} - \mathbb{W}_j^n + \frac{\Delta t}{\Delta x}(\mathbf{F}^n_{j+\frac{1}{2}} - \mathbf{F}^n_{j+\frac{1}{2}}) + \frac{1}{\Delta x}\int_{t^n}^{t^{n+1}} \int_{\Omega_j} \mathbf{d}(\mathbb{W})\partial_x \alpha_1 dx dt = 0,$$

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where

$$\mathbf{F}^n_{j\pm\frac{1}{2}} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathbf{f}\left(\mathbb{W}(x_{j\pm\frac{1}{2}}, t)\right) dt,$$

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$$\mathbb{W}_j^{n+1} - \mathbb{W}_j^n + \frac{\Delta t}{\Delta x}\left(\mathbf{F}^n_{j+\frac{1}{2}} - \mathbf{F}^n_{j+\frac{1}{2}}\right) + \frac{1}{\Delta x}\int_{t^n}^{t^{n+1}} \int_{\Omega_j} \mathbf{d}(\mathbb{W})\partial_x \alpha_1 dx dt = 0,$$

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$$\mathbf{F}^n_{j\pm\frac{1}{2}} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathbf{f}\left(\mathbb{W}(x_{j\pm\frac{1}{2}}, t)\right) dt,$$

and

$$\int_{t^n}^{t^{n+1}} \int_{\Omega_j} \mathbf{d}(\mathbb{W})\partial_x \alpha_1 dx dt = \Delta t \begin{bmatrix} u_2^* \\ 0 \\ -\pi_1^* \\ 0 \\ 0 \\ +\pi_1^* \\ 0 \end{bmatrix}_{j-\frac{1}{2}} \left(\alpha^n_{1,j} - \alpha^n_{1,j-1}\right) \mathbb{1}_{u^*_{2,j-\frac{1}{2}} > 0} + \Delta t \begin{bmatrix} u_2^* \\ 0 \\ -\pi_1^* \\ 0 \\ 0 \\ +\pi_1^* \\ 0 \end{bmatrix}_{j+\frac{1}{2}} \left(\alpha^n_{1,j+1} - \alpha^n_{1,j}\right) \mathbb{1}_{u^*_{2,j+\frac{1}{2}} < 0}$$

A relaxation scheme for the Baer-Nunziato model

We may design a numerical scheme relying on a relaxation approximate Riemann solver.

$$\partial_t \mathbb{W} + \partial_x \mathbf{f}(\mathbb{W}) + \mathbf{d}(\mathbb{W})\partial_x \alpha_1 = 0.$$

Integrate over $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [t^n, t^{n+1}]$:

$$\mathbb{W}_j^{n+1} - \mathbb{W}_j^n + \frac{\Delta t}{\Delta x}(\mathbf{F}_{j+\frac{1}{2}}^n - \mathbf{F}_{j+\frac{1}{2}}^n) + \frac{1}{\Delta x}\int_{t^n}^{t^{n+1}} \int_{\Omega_j} \mathbf{d}(\mathbb{W})\partial_x \alpha_1 dx dt = 0,$$

where

$$\mathbf{F}_{j\pm\frac{1}{2}}^n = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathbf{f}\left(\mathbb{W}(x_{j\pm\frac{1}{2}}, t)\right) dt,$$

and

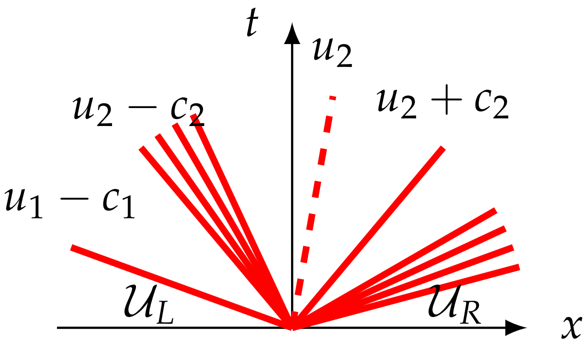
$$\int_{t^n}^{t^{n+1}} \int_{\Omega_j} \mathbf{d}(\mathbb{W})\partial_x \alpha_1 dx dt = \Delta t \begin{bmatrix} u_2^* \\ 0 \\ -\pi_1^* \\ 0 \\ 0 \\ +\pi_1^* \\ 0 \end{bmatrix}_{j-\frac{1}{2}} \left(\alpha_{1,j}^n - \alpha_{1,j-1}^n\right) \mathbb{1}_{u^*_{2,j-\frac{1}{2}}>0} + \Delta t \begin{bmatrix} u_2^* \\ 0 \\ -\pi_1^* \\ 0 \\ 0 \\ +\pi_1^* \\ 0 \end{bmatrix}_{j+\frac{1}{2}} \left(\alpha_{1,j+1}^n - \alpha_{1,j}^n\right) \mathbb{1}_{u^*_{2,j+\frac{1}{2}}<0}$$

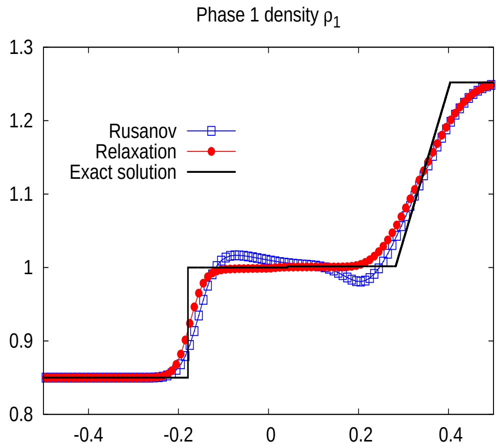
Then drop the additional variable $\mathcal{T}$:

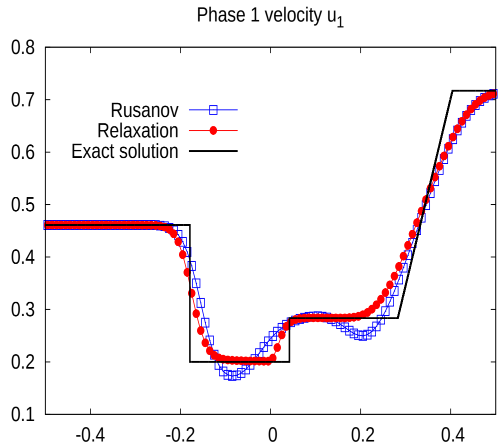
$$\mathbb{U}_j^{n+1} = \mathscr{P}\mathbb{W}_j^{n+1}$$

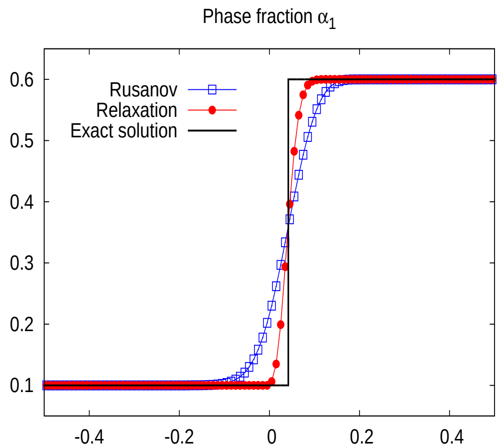
Numerical tests

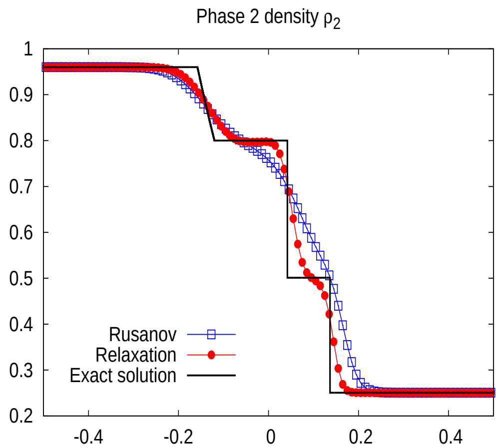
Subsonic shock-tube:

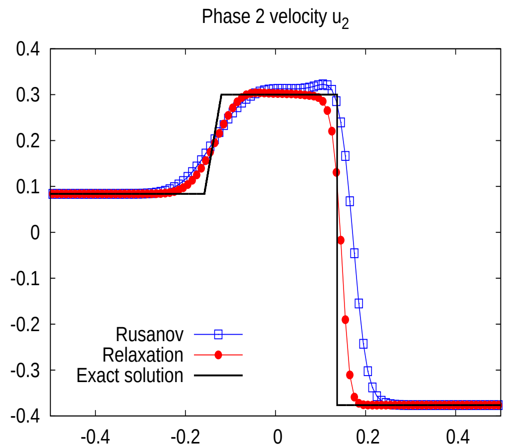






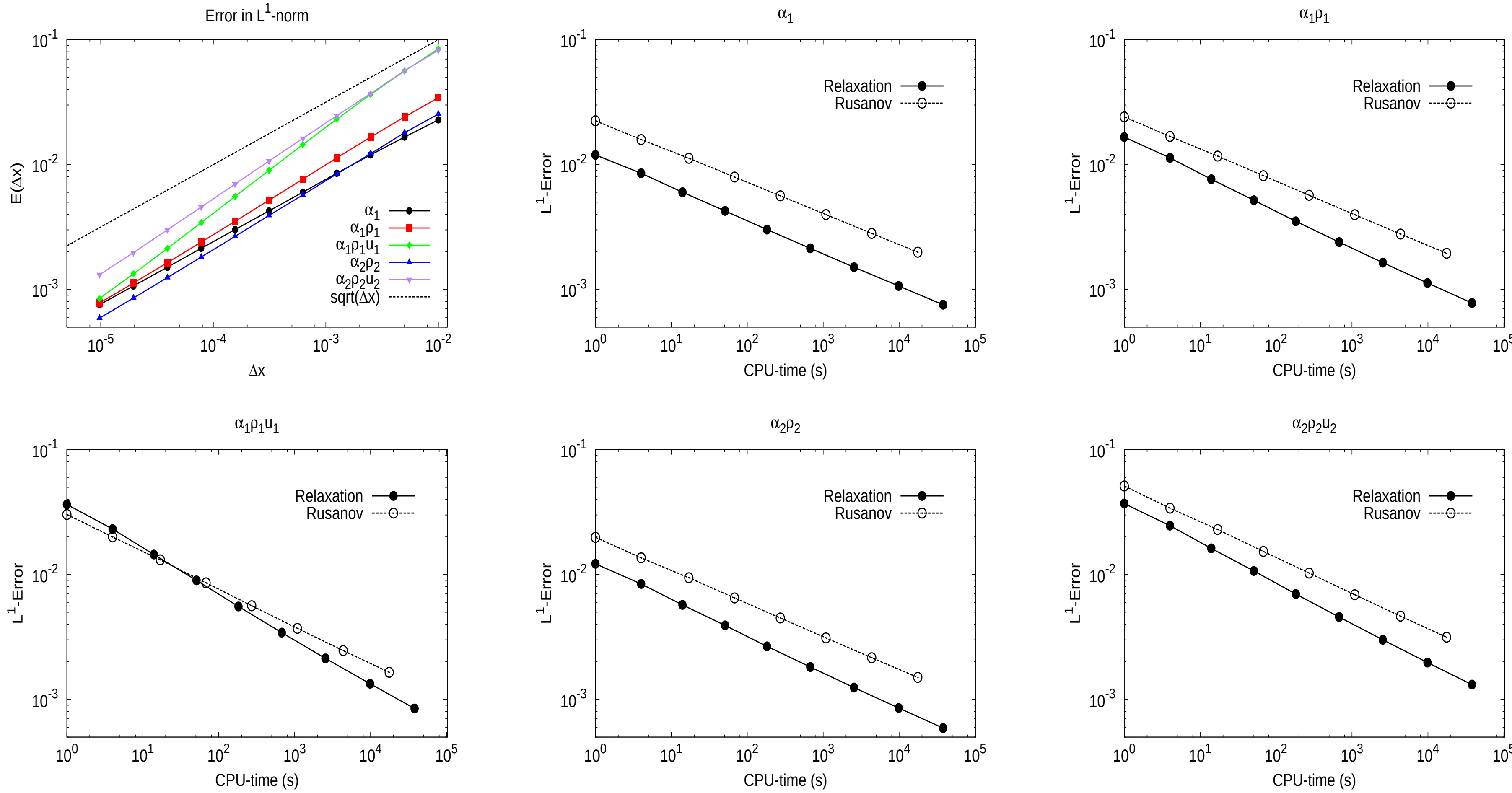






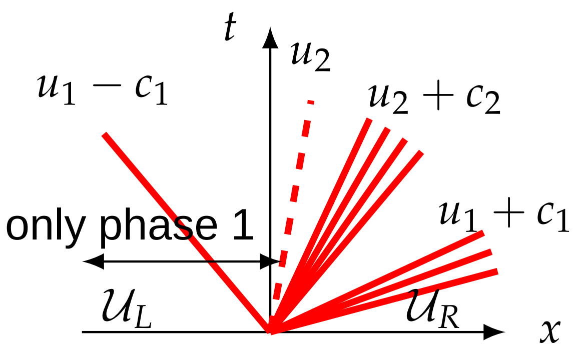
Numerical tests

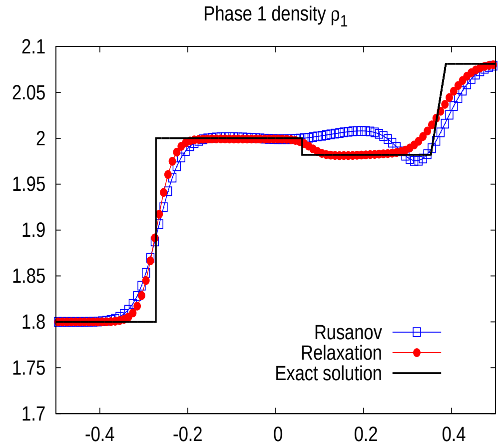
Subsonic shock-tube:

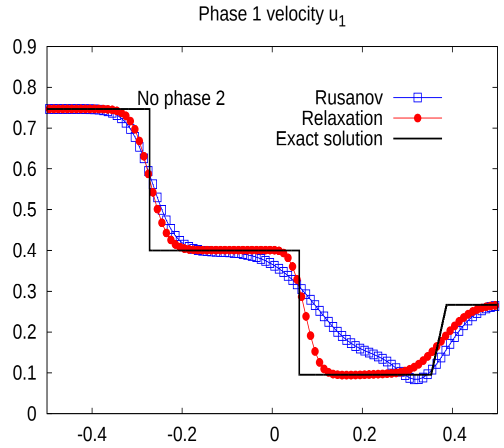


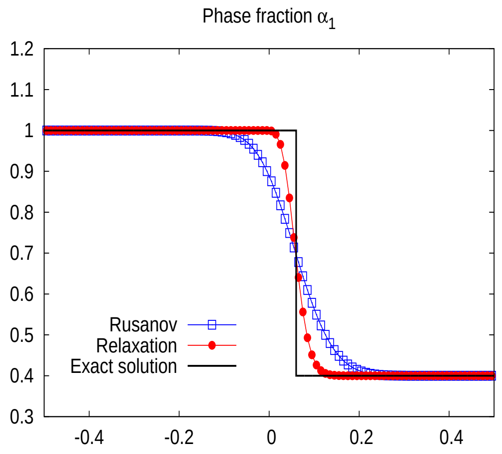
Numerical tests

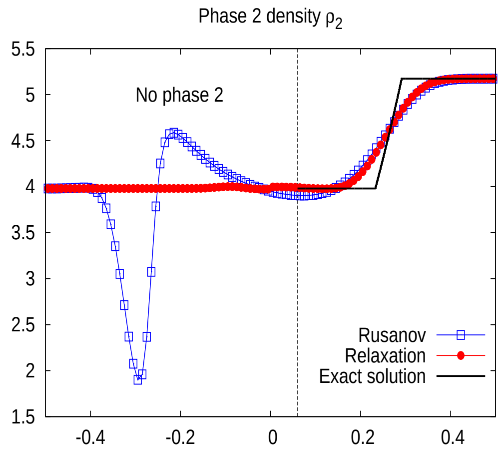
A vanishing phase shock-tube:

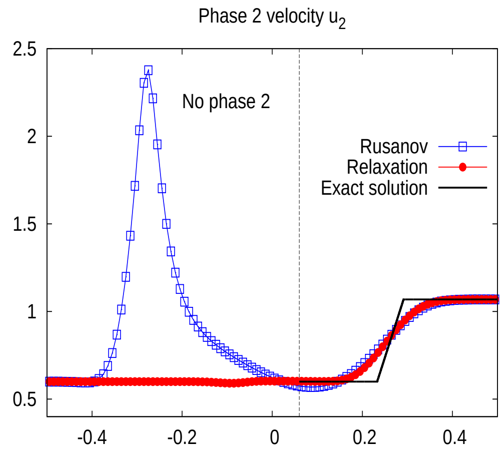






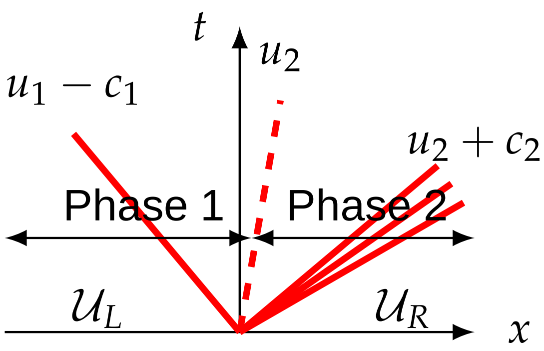


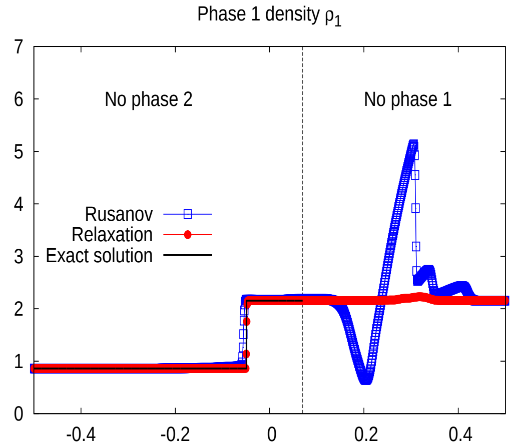


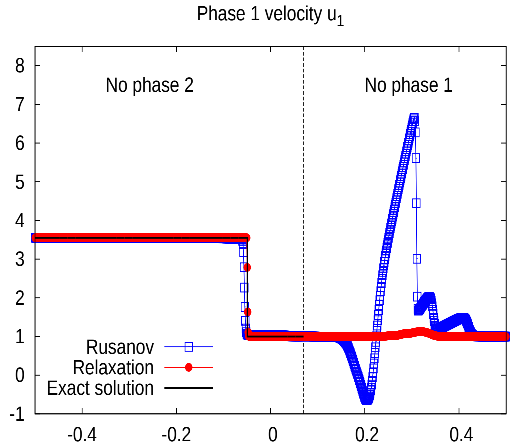


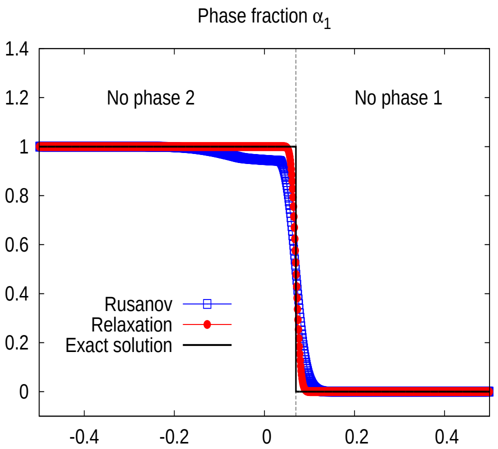
Numerical tests

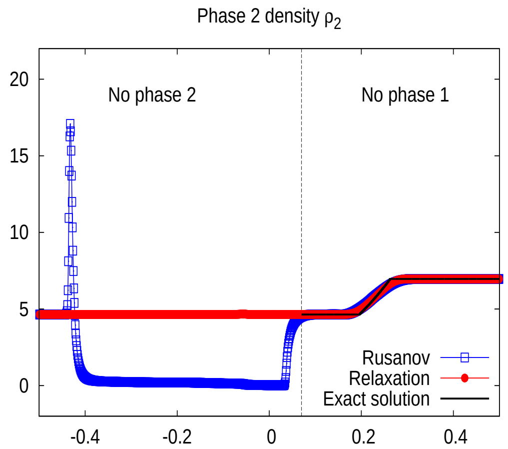
Coupling between two pure phases:

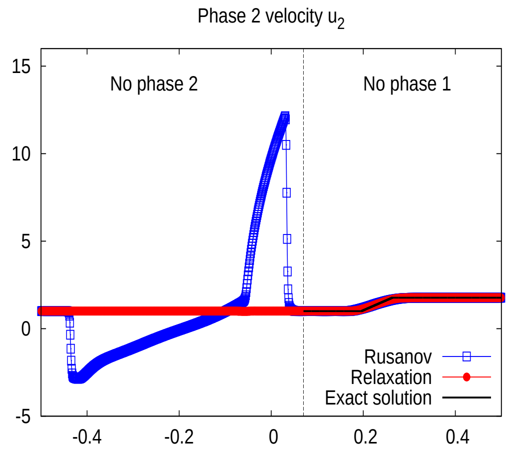












Outline

- The Baer & Nunziato model and its properties,
- A relaxation approximation for the Baer & Nunziato model,
- The Riemann problem for the relaxation system,
- A relaxation numerical scheme,
- Conclusion and perspectives.

Conclusion and perspectives

Conclusion:

- ▷ A relaxation system for the isentropic Baer-Nunziato model,
- ▷ Solutions of the relaxation Riemann problem for the regimes of subsonic flows in relative speed: *ab initio* conditions for the waves' ordering,
- ▷ Much more accurate than Rusanov's scheme for a given mesh,
- ▷ Much smaller computational cost for the same level of accuracy,

Perspectives:

- ▷ Solutions for all the regimes (supersonic and resonant flows),
- ▷ What information can we get on the Baer & Nuziato system ?
- ▷ 2D extension: ongoing work,
- ▷ Extension to the complete model (with energies) using the energy/entropy duality: ongoing work.

Thank you for your attention.