HYP 2012

A positive, well-balanced and entropy-satisfying scheme for shallow water flows

Interest of the kinetic description

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Outline & Main ideas

Introduction

Kinetic description & num. scheme

general scheme

with discrete entropy

Numerical validations

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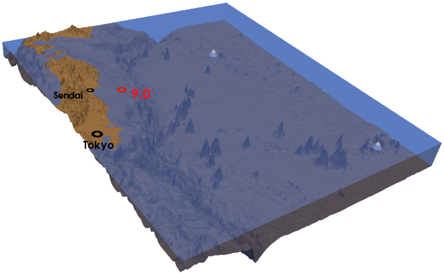
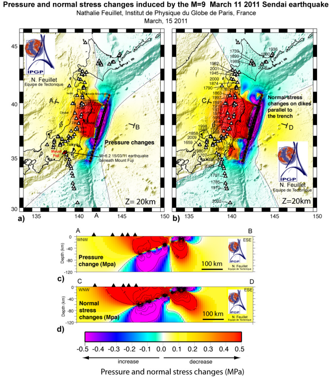
- Interest of efficient numerical methods
  - in fluid mechanics, geophysics
  - non smooth solutions, few dissipation
- Useful in practice (simple): for scientists, industrial

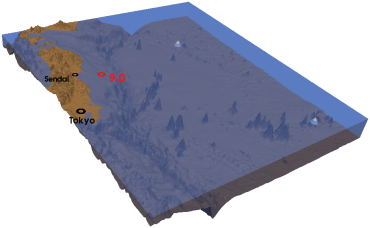
Seism : Japan, march 2011



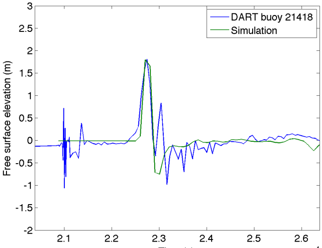


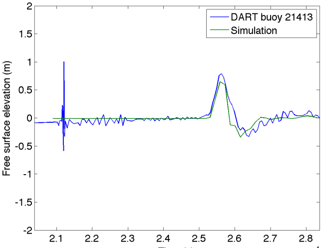
source IPGP (A. Mangeney)

Comparison with DART buoys (3d hyd. Navier-Stokes)



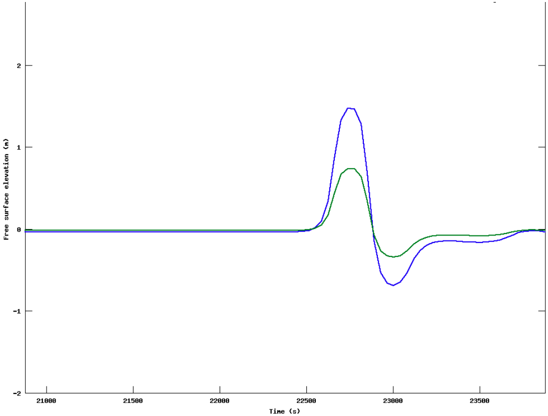
Long distance
small amplitude
⇒ accurate scheme is needed





Japan tsunami simulated with Saint-Venant

- Hydrostatic reconstruction vs. proposed scheme
  - Unstructured mesh, 2.10^6 nodes, 1^{st} order scheme (space & time)



The Saint-Venant system

$$(SV)\left\{\begin{array}{l}\frac{\partial H}{\partial t}+\frac{\partial (H\bar{u})}{\partial x}=0\\ \frac{\partial (H\bar{u})}{\partial t}+\frac{\partial}{\partial x}\left(H\bar{u}^2+\frac{g}{2}H^2\right)=-gH\frac{\partial z_b}{\partial x}\end{array}\right.$$

- The system is hyperbolic
- The water depth satisfies

$$H\geq 0,\qquad \frac{d}{dt}\int H=0$$

- Static equilibrium, "lake at rest"

$$u=0,\qquad H+z_b=Cst$$

- It admits a convex entropy (the energy)

$$\frac{\partial}{\partial t}\left(H\frac{\bar{u}^2}{2}+\frac{g}{2}H^2+gHz_b\right)+\frac{\partial}{\partial x}\bar{u}\left(H\frac{\bar{u}^2}{2}+gH^2+gHz_b\right)\leq 0$$

$\Rightarrow$ Positivity, well-balancing, consistency, discrete entropy

...without reconstruction

Num. methods for the Saint-Venant system

- Finite volume schemes [Bouchut'04]
- Various solvers (relaxation, Roe, HLL, kinetic,. . .)
- Well-balanced scheme required

$$\frac{\partial H}{\partial t} + \frac{\partial (H\bar{u})}{\partial x} = 0, \quad \Rightarrow H_i^{n+1} = H_i^n - \frac{\Delta t}{\Delta x}(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n)$$

with e.g. $\mathcal{F}_{i+1/2}^n = \frac{\max(\sqrt{gH_i}, \sqrt{gH_{i+1}})}{2}(H_i - H_{i+1}) \neq 0$
when $H_j + z_{b,j} = Cst$

- Hydrostatic reconstruction [ABBKP,04]
  - $z_{b,j}^*, z_{b,j+1}^* \Rightarrow H_j^* = H_{j+1}^*$ at rest
  - efficient, various situations
  - only semi discrete entropy

Kinetic representation of the Saint-Venant system

- Gibbs equilibrium $M(x,t,\xi) = \frac{H}{c}\chi\left(\frac{\xi-\bar{u}}{c}\right)$ with $c = \sqrt{gH/2}$

where $\chi(\omega) = \chi(-\omega) \geq 0$, $\text{supp}\,(\chi) \subset \Omega$, $\int_{\mathbb{R}} \chi(\omega) = \int_{\mathbb{R}} \omega^2\chi(\omega) = 1$

Proposition (Audusse, Bristeau, Perthame 04)

The functions $(H, \bar{u}, E)(t,x)$ are strong solutions of the Saint-Venant system if and only if $M(x,t,\xi)$ is solution of the kinetic equation

$$(\mathcal{B}), \qquad \frac{\partial M}{\partial t} + \xi\frac{\partial M}{\partial x} - g\frac{\partial z_b}{\partial x}\frac{\partial M}{\partial \xi} = Q(x,t,\xi)$$

where $Q(t,x,\xi)$ is a collision term.

- Macroscopic variables $(H, \bar{u}, E) = \int_{\mathbb{R}}(1, \xi, \xi^2/2)M\, d\xi$
- A linear transport equation ... easy to upwind

Discrete scheme for the Saint-Venant system (I)

- Gibbs equilibrium $M_i^n = \frac{H_i^n}{c_i^n}\chi\left(\frac{\xi-\bar{u}_i^n}{c_i^n}\right)$
- A simple upwind scheme, for a given ξ

$$
\begin{aligned}
M_i^{n+1-} \quad = \quad & M_i^n - \sigma_i^n \Bigg(\xi(M_{i+1}^n - M_i^n)\mathbb{1}_{\xi\leq 0} - g\Delta z_{b,i+1/2}\frac{\partial M_{i+1/2}^n}{\partial \xi} \\
& + \xi(M_i^n - M_{i-1}^n)\mathbb{1}_{\xi\geq 0} - g\Delta z_{b,i-1/2}\frac{\partial M_{i-1/2}^n}{\partial \xi}\Bigg)
\end{aligned}
$$

with

$$
M_{i+1/2}^n = M_{i+1/2-}^n \mathbb{1}_{\xi\leq 0} + M_{i+1/2+}^n \mathbb{1}_{\xi\geq 0}
$$
$$
M_{i+1/2-}^n, M_{i+1/2-}^n \quad \text{to be defined later}
$$

- Key point

$$
\int_{\mathbb{R}} \frac{\partial M}{\partial \xi} d\xi = 0, \quad \text{but} \quad \int_{\xi\leq 0} \frac{\partial M_{i+1/2+}}{\partial \xi} d\xi + \int_{\xi\geq 0} \frac{\partial M_{i+1/2-}}{\partial \xi} d\xi \neq 0
$$

Discrete scheme for the Saint-Venant system (II)

- Extended version of an idea in Perthame-Simeoni'01

$$\int_{\mathbb{R}} \xi^p \left(g \frac{\partial z_b}{\partial x} \frac{\partial M}{\partial \xi} - \xi \frac{\partial \widehat{M}}{\partial x} \right) d\xi = 0, \quad p = 0, 1,$$

with $\widehat{M} = \frac{\widehat{H}}{\widehat{c}} \chi \left(\frac{\xi}{\widehat{c}} \right), \quad \widehat{H} = \eta - z_b,\ \eta = Cst$

- The proposed scheme is

$$M_i^{n+1-} = M_i^n - \sigma_i^n \left(\mathcal{M}_{i+1/2}^n - \mathcal{M}_{i-1/2}^n \right)$$

with

$$\begin{aligned}
\mathcal{M}_{i+1/2}^n &= \xi M_{i+1/2}^n - \xi \widehat{M}_{i+1/2}^n \\
M_{i+1/2}^n &= M_i^n \mathbb{1}_{\xi \geq 0} + M_{i+1}^n \mathbb{1}_{\xi \leq 0} \\
\widehat{M}_{i+1/2}^n &= \widehat{M}_{i+1/2+}^n \mathbb{1}_{\xi \leq 0} + \widehat{M}_{i+1/2-}^n \mathbb{1}_{\xi \geq 0} \\
\widehat{M}_{i+1/2-}^n &= \frac{\widehat{H}_{i+1/2-}^n}{\widehat{c}_{i+1/2-}^n} \chi \left(\frac{\xi}{\widehat{c}_{i+1/2-}^n} \right), \quad \widehat{H}_{i+1/2-}^n = \eta_{i+1/2} - z_{b,i}
\end{aligned}$$

Discrete scheme for the Saint-Venant system (III)

- The proposed scheme is

$$M_i^{n+1-} = M_i^n - \sigma_i^n \left(\mathcal{M}_{i+1/2}^n - \mathcal{M}_{i-1/2}^n\right)$$

with

$$\begin{aligned}
\mathcal{M}_{i+1/2}^n &= \xi M_{i+1/2}^n - \xi \widehat{M}_{i+1/2}^n \\
M_{i+1/2}^n &= M_i^n \mathbb{1}_{\xi \geq 0} + M_{i+1}^n \mathbb{1}_{\xi \leq 0} \\
\widehat{M}_{i+1/2}^n &= \widehat{M}_{i+1/2+}^n \mathbb{1}_{\xi \leq 0} + \widehat{M}_{i+1/2-}^n \mathbb{1}_{\xi \geq 0} \\
\widehat{M}_{i+1/2-}^n &= \frac{\widehat{H}_{i+1/2-}^n}{\widehat{c}_{i+1/2-}^n} \chi\left(\frac{\xi}{\widehat{c}_{i+1/2-}^n}\right), \quad \widehat{H}_{i+1/2-}^n = \eta_{i+1/2} - z_{b,i}
\end{aligned}$$

- Macroscopic scheme
  $H_i^{n+1} = \int_{\mathbb{R}} M_i^{n+1-} d\xi, \quad H_i^{n+1} \overline{u}_i^{n+1} = \int_{\mathbb{R}} \xi M_i^{n+1-} d\xi$
- only analytic quadrature formula

Properties of the scheme

- Key point

$$\int_{\mathbb{R}} \frac{\partial M}{\partial \xi} d\xi = 0, \quad \text{but} \quad \int_{\xi \leq 0} \frac{\partial M_{i+1/2+}}{\partial \xi} d\xi + \int_{\xi \geq 0} \frac{\partial M_{i+1/2-}}{\partial \xi} d\xi \neq 0$$

- Well-balanced
  - trivial
- Positive
  - the CFL does not depend on $\frac{\partial z_b}{\partial x}$
  - well behaves when $H \to 0$
- Consistency
- 2^{nd} order in time (Modified Heun) and space (centered term)
- Convergence rate : $C\Delta x$ vs. $c\Delta x$ with $c < C$
- With modified $\hat{c}^n_{i+1/2\pm}$: can be used with other FV solvers (HLL, Rusanov)
- No discrete entropy

Scheme for $H \geq |\Delta z_b|$ (discrete entropy) - I

- Gibbs equilibrium
  - $M(x,t,\xi) = \frac{H}{c}\chi_0\left(\frac{\xi-\bar{u}}{c}\right), \quad \overline{M}(x,t,\xi) = \frac{H}{c}\phi_{\chi_0}\left(\frac{\xi-\bar{u}}{c}\right)$
  - $\chi_0(z) = \frac{1}{\pi}\sqrt{1-\frac{z^2}{4}}, \quad \phi_{\chi_0}(z) = \int_z^{+\infty} z_1\chi_0(z_1)dz_1$
- χ_0 is the minimum of the set (energy), see [Perthame-Simeoni 01]

$$\mathcal{E}(f) = \int_{\mathbb{R}} \left(\frac{\xi^2}{2}f(\xi) + \frac{g^2}{8}f^3(\xi) + gz_bf(\xi)\right)d\xi$$

- Modified Boltzmann equation

$$\frac{\partial M}{\partial t} + \xi\frac{\partial M}{\partial x} - g\frac{\partial z_b}{\partial x}\frac{\partial \overline{M}}{\partial \xi} = Q$$

$$\Leftrightarrow \quad \frac{\partial M}{\partial t} + \xi\frac{\partial M}{\partial x} + g\frac{\xi - u}{c^2}M\frac{\partial z_b}{\partial x} = Q$$

∂_ξ eliminated in the Boltzmann equation...

Scheme for $H \geq H_0 > 0$ (discrete entropy) - II

- Goal : M_i^{n+1-} as a convex combination of M_{i-1}^n, M_i^n and M_{i+1}^n
- A simple upwind scheme, for a given ξ

$$M_i^{n+1-} = M_i^n - \sigma_i^n \left(\mathcal{M}_{i+1/2}^n - \mathcal{M}_{i-1/2}^n\right)$$

with

$$\mathcal{M}_{i+1/2}^n = \mathcal{M}_{i+1/2+}^n + \mathcal{M}_{i+1/2-}^n$$

$$\mathcal{M}_{i+1/2+}^n = \left(\xi + 2\frac{\Delta z_{b,i+1/2}}{H_{i+1}^n} \frac{\xi - u_{i+1}^n}{c_{i+1}^n} \overline{c}_{i+1/2+}^n\right) \mathbb{1}_{\xi \leq \xi_{i+1/2+}} M_{i+1}^n$$

$$\mathcal{M}_{i+1/2-}^n = \left(\xi + 2\frac{\Delta z_{b,i+1/2}}{H_i^n} \frac{\xi - u_i^n}{c_i^n} \overline{c}_{i+1/2-}^n\right) \mathbb{1}_{\xi \geq \xi_{i+1/2-}} M_i^n$$

- So

$$M_i^{n+1-} = (1 - \mathcal{A}_i^n)\, M_i^n + \mathcal{A}_{i-1/2+}^n M_{i-1}^n + \mathcal{A}_{i+1/2-}^n M_{i+1}^n$$

with $\mathcal{A}_j^n \geq 0$, $1 - \mathcal{A}_i^n \geq 0$

Scheme for $H \geq H_0 > 0$ (discrete entropy) - III

- The scheme is well-balanced, consistent and positive

Proposition

Let us consider a real convex function $e(.)$ defined over $\mathbb{R}_+$. Under the CFL condition, the scheme satisfies the in-cell entropy inequality

$$E_i^{n+1} \leq E_i^n + \sigma_i \left(\Lambda_{i+1/2}^n - \Lambda_{i-1/2}^n \right)$$

with

$$\begin{aligned} E_i^n &= \int_{\mathbb{R}} e(M_i^n) d\xi \\ \Lambda_{i+1/2}^n &= \sigma_i^n \int_{\mathbb{R}} \left(\mathcal{A}_{i+1/2-}^n e(M_{i+1}^n) - \mathcal{A}_i^n e(M_i^n) \right) d\xi \end{aligned}$$

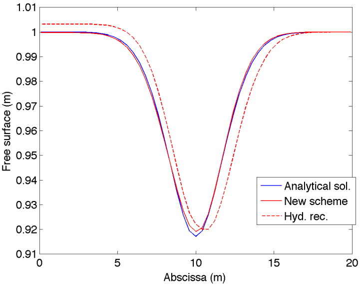
In particular the choice $e(f) = \frac{\xi^2}{2} f + \frac{g^2}{8} f^3 + g z_b f$ gives a discrete version of the energy balance.

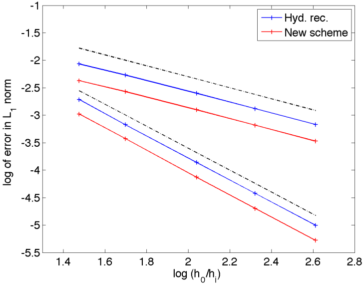
Numerical validations

- Only analytical solutions
- Stationary/transient, continuous/discontinuous solutions
- 1^{st} and 2^{nd} order schemes

$\Rightarrow$ not exhaustive validations

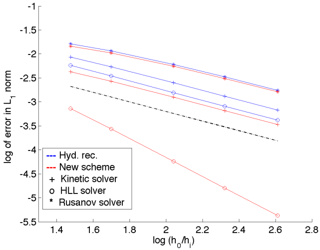
- Two main ideas
  - Systematic biais & accuracy
  - fluvial regime over a bump (anim)

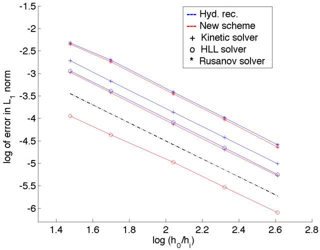




Other solvers

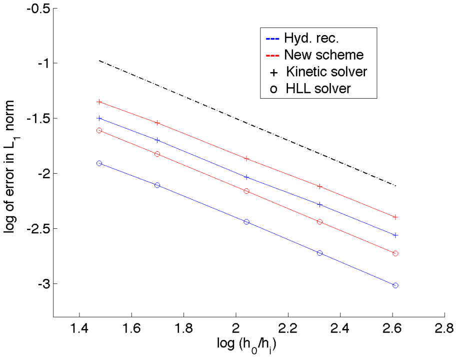
- HLL, kinetic & Rusanov fluxes
- 1^{st} and 2^{nd} order schemes
- fluvial regime over a bump
- general scheme





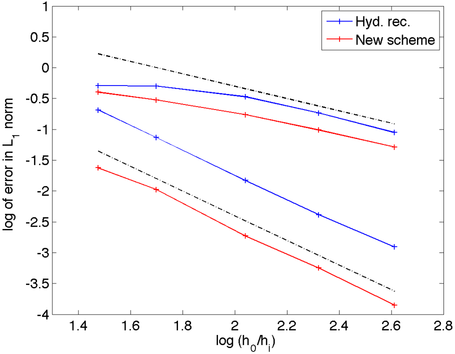
Transcritical regime with shock

- HLL & kinetic fluxes
- general scheme



Parabolic bowl

- Kinetic fluxes
- 1^{st} and 2^{nd} order (in space & time) schemes
- general scheme (anim)



Conclusion & outlook

- Scheme without reconstruction
  - simple & efficient
  - can be used with various solvers
  - significant improvement w.r.t. hyd. rec.
  - remaining degrees of freedom (all equilibria ?)
  - valid in 2d (and 3d)
  - entropy satisfying for H enough large
- Extension to Navier-Stokes
  - kinetic interpretation [Audusse,Bristeau,Perthame,JSM 11]
  - more complex source terms