# Exact solutions to ideal hydrodynamics of inelastic gases: global existence and singularities

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The motion of the dilute gas where the characteristic hydrodynamic length scale of the flow is sufficiently large and the viscous and heat conduction terms can be neglected is governed by the systems of equations of ideal granular hydrodynamics (e.g. Brilliantov, Pöschel, *Kinetic theory of granular gases*, 2004.):

$$\partial_t \rho + \mathrm{div}_x(\rho u) = 0,$$
$$\partial_t(\rho u) + \mathrm{Div}_x(\rho u \otimes u) = -\nabla_x p,$$
$$\partial_t T + (u, \nabla_x T) + (\gamma - 1) T \mathrm{div}_x u = -\Lambda \rho T^{3/2},$$

$\rho$ is the gas density,

$u = (u_1, ..., u_n)$ is the velocity,

$T$ is the temperature, $p = R\rho T$ is the pressure (the constant $R$ is a adiabatic invariant, for the sake of simplicity we set $R = 1$ ),

$\gamma$ is the adiabatic index,

$\Lambda = const > 0$.

The system is given in $\mathbb{R} \times \mathbb{R}^n$, $n \geq 1$.

**No nontrivial constant states!**

There exists a solution (the homogeneous cooling state) with constant $\rho, u$ and $p \neq 0$:

$$T = T(t) = (\frac{\Lambda \rho_0 t}{2} + T(0)^{-1/2})^{-2},$$

where $T(0)$ is the initial value of temperature (the Haff's law).

## Blow-up result (R.,2012)

Total mass:

$$M = \int_{\mathbb{R}^n} \rho \, dx,$$

momentum

$$P = \int_{\mathbb{R}^n} \rho u \, dx,$$

moment of inertia

$$G(t) = \frac{1}{2} \int_{\mathbb{R}^n} \rho(t,x)|x|^2 \, dx$$

**Theorem 1** *Let $P \neq 0$ and $m$ is sufficiently small, $n \geq 1$. Then the classical solutions to the Cauchy problem with finite moment of inertia either develop a singularity within a finite time or $\|\rho\|_{L^\infty} \to \infty$ as $t \to \infty$.*

# Family of exact solutions in 1D (Fouxon, Meerson, Assaf, Livne, 2007)

Lagrangian mass coordinate:

$$m(x,t) = \int_0^x \rho(\xi,t)\, d\xi$$

The system in the Lagrangian coordinates:

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right), \qquad \frac{\partial v}{\partial m} = \frac{\partial p}{\partial m}, \quad \frac{\partial p}{\partial m} = -\mu p, \quad \mu = \frac{\Lambda}{\gamma\sqrt{2}}.$$

Solution:

$$p = 2A\cos(\mu m), \quad A = \text{const}.$$

$$\rho(m,t) = \frac{\rho(m,0)}{(1 - \mu t\sqrt{A\rho(m,0)}\cos \mu m)^2}.$$

$$\rho(0,t) \sim \text{const}\,(t_* - t)^{-2}, \quad t \to t_*.$$

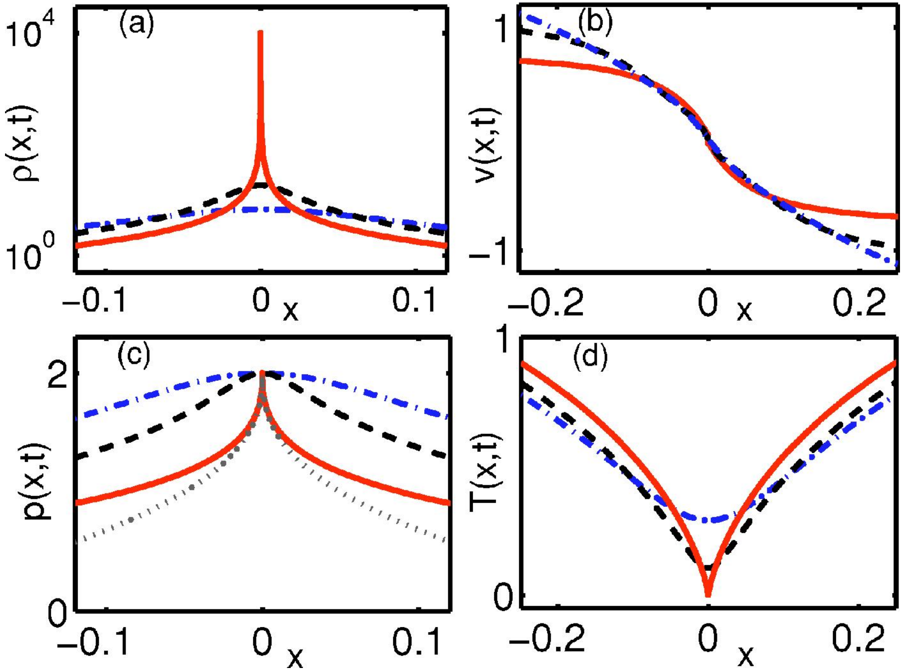


Fig. 1: Finite mass solution

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# Family of self-similar solutions in 1D (R.,2012)

Let us find a solution that depends on $\xi = x - at$, $a = const$.

$$u(\xi) = \frac{c_1}{\rho(\xi)} + a, \; c_1 = const$$

$$c_1 u(\xi) + p(\xi) = c_2, \; c_2 = const > 0$$

ODE for the density:

$$\rho'(\xi) = \frac{\Lambda}{c_1} \frac{\rho^(\xi)\,((c_2 - ac_1)\rho - c_1^2)^{3/2}}{c_1^2(\gamma+1) - \gamma(c_2 - ac_1)\rho},$$

or

$$z'(\xi) = -\frac{\Lambda}{c_1(c_2 - ac_1)} \frac{z^{3/2}(\xi)(z(\xi) + c_1^2)}{c_1^2 - \gamma z(\xi)},$$

where $z = (c_2 - ac_1)\rho - c_1^2$.

$$x = f(z) := \frac{2c_1(c_2 - ac_1)}{\Lambda} \frac{1}{\sqrt{z}} + \frac{2(\gamma+1)(c_2 - ac_1)}{c_1} \arctan \frac{\sqrt{z}}{c_1} + c_3, \; c_3 = const.$$

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## Steady state



Let us choose $a = 0$, $c_1 = \mathrm{sign} x\, k$, $k > 0$ and construct an even solution

$$\bar{z}(x) = \begin{cases} z_-(x), & x \in (x_0, +\infty), \\ z_+(x), & x \in (-\infty, -x_0), \end{cases}$$

where $z_-(x)$ and $z_+(x)$ correspond to $c_1 = -k$ and $c_1 = k$, respectively. Here $x_0 = x_*$ if the constant $c_3$ for $z_+(x)$ is chosen such that $x_* > 0$ and $x_0 = 0$, otherwise.

The density, pressure and velocity can be found as

$$\bar{\rho} = \frac{\bar{z} + k^2}{c_2}, \quad \bar{p} = c_2 - \frac{k^2}{\bar{\rho}}, \quad \bar{u} = \frac{\mathrm{sign} x\, k}{\bar{\rho}}.$$

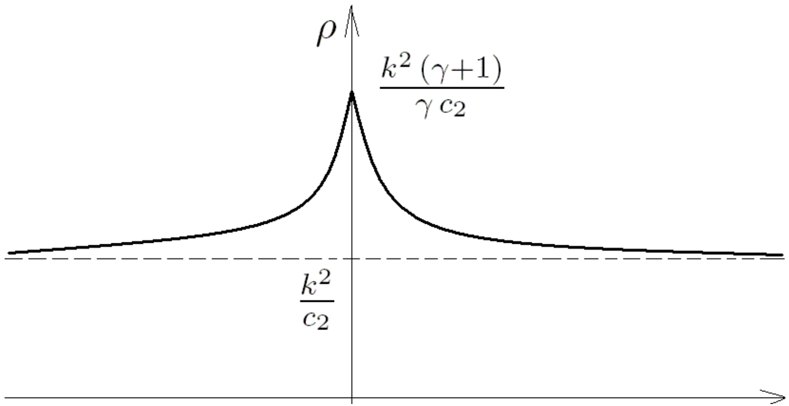


Fig. 2: The density

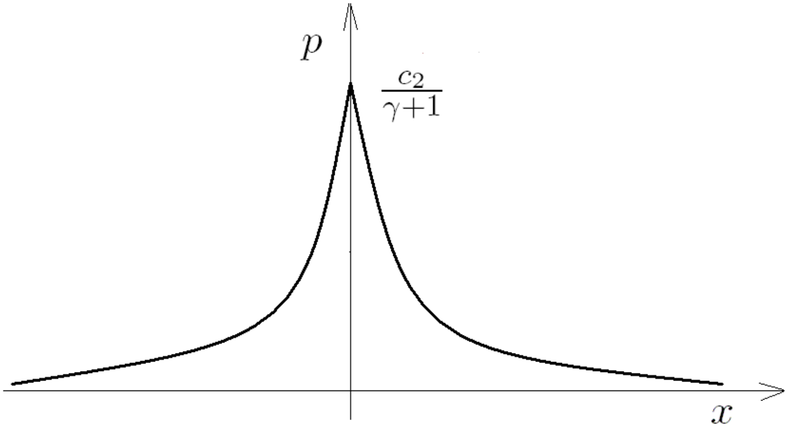


Fig. 3: The pressure

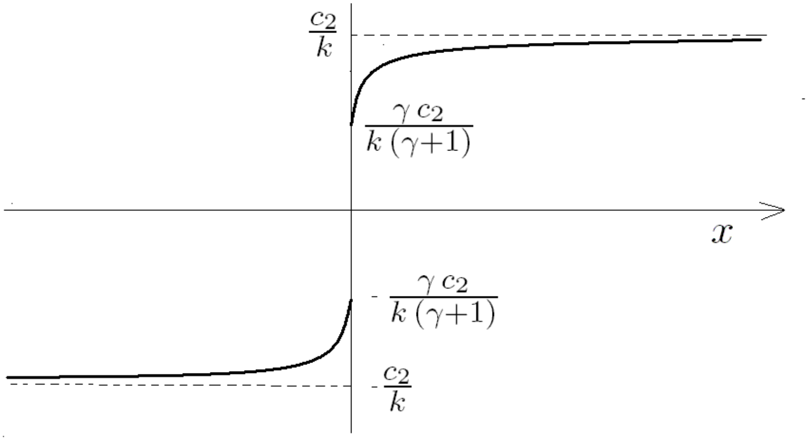


Fig. 4: The velocity

## Multi D (R.,2012)



Here for any spatial dimensions we construct a simple family of solutions having a singularity in the density whereas other components are continuous.

$$u(t,x) = \alpha(t)\,x, \quad \rho(t,x) = \beta(t)\,|x|^q, \quad p(t,x) = s(t)\,|x|^l,$$

where $x$ is a radius-vector of point, we obtain

$$q = -1, \qquad l = 1, \qquad \beta(t) = \beta_0 = const \geq 0,$$

and $\alpha(t)$, $s(t) \geq 0$ satisfy the following system of nonlinear ODE:

$$\alpha'(t) + \alpha^2(t) + \frac{s(t)}{\beta_0} = 0,$$

$$s'(t) + (\gamma + 1)\,n\;s(t)\,\alpha(t) = -\Lambda\,\beta^{1/2}(t)\,s^{3/2}(t).$$

# Solution with a constraint



Let us introduce a new dependent variable $z(t,x)$ as follows: $z = \rho - \phi(t)T^{-\frac{1}{2}}$, $\phi(t)$ is an arbitrary differentiable function function. Thus, in the variables $z, T, u$ the system takes the form

$$\partial_t z + \mathrm{div}(zu) - \frac{\Lambda}{2}\,\phi(t)\,z + (\gamma+1)\,\phi(t)\,T^{-\frac{1}{2}}\,\mathrm{div}\,\mathrm{u} + (2\phi'(t) + \Lambda\phi^2(t))\,T^{-\frac{1}{2}} = 0,$$

$$\partial_t u + (u, \nabla)\,u = -\frac{1}{z + \phi(t)T^{-\frac{1}{2}}}\,\nabla_x(zT + \phi(t)T^{-\frac{1}{2}}),$$

$$\partial_t T + (u, \nabla T) + (\gamma - 1)\,T\,\mathrm{div}\,u = -\Lambda z\,T^{3/2} - \Lambda\,\phi(t)\,T.$$

We consider particular class of solutions characterized by property $z = 0$. There are two possibilities:

- $\gamma = -1$ and $\phi(t)$ is a solution to ordinary differential equation

$$\phi'(t) = -\frac{\Lambda}{2}\phi^2(t).$$

- $u(t, x) = \alpha(t)x + \beta(t)$ and $\phi(t)$ is a solution to ordinary differential equation

$$\phi'(t) = -\frac{\gamma + 1}{2}\phi(t)\,\mathrm{tr}\,\alpha(t) - \frac{\Lambda}{2}\phi^2(t).$$

The first possibility is **the case of Chaplygin gas**, where the state equation has the form

$$p = -A^2\rho^{-1}, \quad A = const > 0.$$

The system is reduced to a couple of equations

$$\partial_t u + (u, \nabla)\, u = T^{\frac{1}{2}}\, \nabla(T^{\frac{1}{2}}),$$

$$\partial_t T^{\frac{1}{2}} + (u,\, \nabla T^{\frac{1}{2}}) - T^{\frac{1}{2}} \mathrm{div}\, u = -\Lambda\, \phi(t)\, T^{\frac{1}{2}},$$

where $\phi(t)$ can be found explicitly. The negative pressure ensures the hyperbolicity of system. In 1D case this system as any system of two equations can be written in the Riemann invariants, this allows to apply the technique usual for ordinary gas dynamics.

In the second case, for an arbitrary $\gamma$ (physically it makes sense to require $(1 < \gamma \le 1 + \frac{2}{n})$, nevertheless formally it is possible to consider any constant $\gamma$), the equation **can be satisfied only for** $u(t,x) = \alpha(t)x + \beta(t)$. $T$ has to solve

$$(\partial_t \alpha(t) + \alpha^2(t))x + (\partial_t \beta(t) + \alpha(t)\beta(t)) = -\frac{1}{2}\nabla T,$$

$$\partial_t T + ((\alpha x + \beta), \nabla T) + (\gamma - 1)T \operatorname{tr} \alpha(t) = -\Lambda\, \phi(t)\, T,$$

and the structure of the field of velocity requires a special structure of the field of temperature, namely,

$$T(t,x) = x^T A(t)\, x + (B(t), x) + C(t).$$

Thus, we get a system of $\frac{3n^2+5n+4}{2}$ nonlinear differential equations for components of a square matrix $\alpha(t)$, a square symmetric matrix $A(t)$, vectors $\beta(t)$ and $B(t)$, the scalar functions $C(t)$ and $\phi(t)$. This system can be explicitly (in the simplest symmetric cases) or numerically integrated, one can study its qualitative behavior. Thus, the component of density can be found as

$$\rho(t,x) = \frac{\phi(t)}{(x^T A(t)\, x + (B(t), x) + C(t))^{1/2}}.$$

Thus, if the singularity arises with time in a point, it has a form of $|x - x_0|^{-1}$ and therefore it is integrable for $n > 1$. Nevertheless, the total mass is infinite for this solution, since integral $M(t) = \int\limits_{\mathbb{R}^n} \rho\, dx$ diverges as $|x| \to \infty$.

## Chaplygin gas, $\gamma = -1$

This model of gas dynamics is known as the Chaplygin gas. The Chaplygin system is known to be hyperbolic, linearly degenerate, weakly stable. The sound speed is $c = \frac{A}{\rho}$. The acoustic fields are linearly degenerate, in the terminology of hyperbolic systems.

The system can be reduced to

$$\partial_t \rho + \nabla(\rho u) = 0,$$

$$\partial_t \rho + \nabla(\rho u \otimes u) - \phi(t) \nabla \left( \frac{1}{\rho} \right) = 0,$$

It is similar to the Chaplygin gas system, the only difference is in the known multiplier $\phi$, for the Chaplygin gas $\phi = 1$. For $n = 1$ it can be written in the Riemann invariants terms of functions $u, T^{\frac{1}{2}}$, as

$$\partial_t s + r \partial_x s = \frac{\Lambda \phi(t)}{4}(s - r),$$

$$\partial_t r + s \partial_x r = \frac{\Lambda \phi(t)}{4}(r - s),$$

where $s = u - T^{\frac{1}{2}}$, $r = u + T^{\frac{1}{2}}$.

Let us consider **the Riemann problem for the Caplygin gas system** (Brenier, 2005):

$$(u,T)=\begin{cases}(u_L,T_L), & x<0,\\(u_R,T_R), & x>0.\end{cases}$$

Since the system is linear degenerate, the jumps are contact discontinuities and move along characteristics. Thus, if

$$u_L<u_R+T_L^{\frac{1}{2}}+T_R^{\frac{1}{2}},$$

then the solution is

$$(u,T)=\begin{cases}(u_L,\frac{T_L}{(\frac{\Lambda}{2}t+1)^2}), & x<(u_L-T_L^{\frac{1}{2}})t,\\(u_M,T_M), & (u_L-T_L^{\frac{1}{2}})t<x<(u_R+T_R^{\frac{1}{2}})t,\\(u_R,\frac{T_R}{(\frac{\Lambda}{2}t+1)^2}), & x>(u_R+T_R^{\frac{1}{2}})t,\end{cases}$$

with

$$u_M=\frac{u_L+u_R+(T_R^{\frac{1}{2}}-T_L^{\frac{1}{2}})(\frac{\Lambda}{2}t+1)}{2},\quad T_M^{\frac{1}{2}}=\frac{u_R-u_L+(T_R^{\frac{1}{2}}+T_L^{\frac{1}{2}})(\frac{\Lambda}{2}t+1)}{2}.$$

# Solutions with uniform deformation, arbitrary $\gamma$



$$\alpha'(t) + \alpha^2(t) + A(t) = 0$$

$$\beta'(t) + 2\alpha(t)\beta(t) + \frac{1}{2}B(t) = 0$$

$$A'(t) + 2A(t)\alpha(t) + (\gamma - 1)\operatorname{tr}\alpha(t)A(t) + \Lambda\phi(t)A(t) = 0$$

$$B'(t) + 2A(t)\beta(t) + B(t)\alpha(t) + (\gamma - 1)\operatorname{tr}\alpha(t)B(t) + \Lambda\phi(t)B(t) = 0$$

$$C'(t) + (B(t), \beta(t)) + (\gamma - 1)\operatorname{tr}\alpha(t)C(t) + \Lambda\phi(t)C(t) = 0$$

$$\phi'(t) = -\frac{\gamma + 1}{2}\,\phi(t)\,\operatorname{tr}\alpha(t) - \frac{\Lambda}{2}\phi^2(t).$$

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Let us consider a simplest case with $A(t) = a(t)\mathbb{I}$, $\alpha(t) = \alpha_1(t)\mathbb{I}+\mathbb{W}$, $B(t) = 0$, $\beta(t) = 0$, where $\mathbb{I}$ is the unit matrix, $\mathbb{W}$ is a skew-symmetric matrix.

We find an asymptotic solution at the point $t = t_*$ of the singularity appearance.

First we dwell at the case $\mathbb{W} = 0$. We get a system of 4 equations:

$$\alpha_1'(t) + \alpha_1^2(t) + a(t) = 0$$
$$a'(t) + ((2 + n(\gamma - 1))\alpha_1(t)a(t) + \Lambda\phi(t))a(t) = 0$$
$$C'(t) + (n(\gamma - 1)\alpha_1(t)C(t) + \Lambda\phi(t))C(t) = 0$$
$$\phi'(t) = -\frac{n}{2}(\gamma + 1)\,\phi(t)\,\alpha_1(t) - \frac{\Lambda}{2}\phi^2(t).$$

In 2D we can consider a vortex solution with $\alpha(t) = \alpha_1(t)\mathbb{I} + \mathbb{W}$,

$$\mathbb{W} = \alpha_2(t)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

## Asymptotics at the point of singularity

In order to analyze the occurrence of blow-up for solutions, we build local series of the form:

$$\mathbf{x} = \Psi(\lambda, s, t) = \lambda\tau^{s}(1 + h(\tau, \ln\tau)),$$

where $\tau = t - t_*$ and $h(\tau; \ln\tau)$ is a power series in its argument which vanishes as $\tau \to 0$.

The notation $\lambda\tau^s$ refers to the vector whose i-th component is $\lambda_i\tau^{s_i}$.

In order to obtain **the leading behavior $\lambda\tau^s$** of the solution around $t_*$ we look for all the truncations $\hat{\mathbf{f}}$ of the vector field $\mathbf{f} = \hat{\mathbf{f}} + \breve{\mathbf{f}}$ such that the dominant behavior $\mathbf{x} = \lambda\tau^s, \quad \lambda \in \mathbb{C}^n$ is an exact solution of the truncated system $\dot{\mathbf{x}} = \hat{\mathbf{f}}(\mathbf{x})$ and

$$\breve{\mathbf{f}}(\lambda\tau^s) \sim \breve{\lambda}\tau^{s+\breve{s}-1}, \quad \breve{s} \in \mathbb{Q}^n, \quad \breve{s}_i > 0,$$

as $\tau \to 0$.

Each truncation defines a **balance** $(\lambda, s)$ and every balance corresponds to the first term $\lambda\tau^s$ in an expansion around movable singularities. For such an expansion to describe a general solution, the $\Psi$-series must contain $n-1$ arbitrary constants in addition to the arbitrary parameter $t_*$. The position in the power series where these arbitrary constants appear is given by **the resonances**.

Each balance defines a new set of resonances. They are given by the eigenvalues of the matrix $R$:

$$R = -D\hat{\mathbf{f}}(\lambda) - diag(s),$$

where $D\hat{\mathbf{f}}(\lambda)$ is the Jacobian matrix evaluated on $\lambda$. The resonances are labeled $r_i$, $i = 1, ..., n$ with $r_1 = -1$.

**A general solution is a formal solution $\mathbf{x} = \Psi(\lambda, s, t)$ with balance $(\lambda, s)$ such that $r_j > 0$ for all $j > 1$.**

That is, the $\Psi$-series built on that balance contains $(n-1)$ arbitrary coefficients (the final arbitrary constants being the singularity position $t_*$).

**Theorem 2** ***(Goriely, 2001)*** *Consider the polynomial system*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n,$$

*a nonlinear polynomial and assume that the general solution can be locally expanded in a convergent* $\Psi$*-series. Then the two following statements are equivalent:*

*a) There exists an open set of initial conditions* $X_0 \subseteq \mathbb{R}^n$ *such that for all* $x_0 \in X_0$*, there exists* $t_* \in \mathbb{R}$ *for which* $|\mathbf{x}(t; x_0)| \to \infty$ *as* $t \to t_*$*;*

*b) There exists a general solution* $\mathbf{x} = \Psi(\lambda, s, t, x_0, t_*)$ *with* $\lambda \in \mathbb{R}^n$

To find main terms of asymptotic at the point of singularity we consider a quasihomogeneous truncation

$$\alpha_1'(t) + \alpha_1^2(t) = 0$$

$$a'(t) + ((2 + n(\gamma - 1))\alpha_1(t)a(t) + \Lambda\phi(t))a(t) = 0$$

$$C'(t) + (n(\gamma - 1)\alpha_1(t)C(t) + \Lambda\phi(t))C(t) = 0$$

$$\phi'(t) = -\frac{n}{2}(\gamma + 1)\,\phi(t)\,\alpha_1(t) - \frac{\Lambda}{2}\phi^2(t).$$

The main term of asymptotics as $t \longrightarrow t_* - 0$:

$$\alpha_1(t) = (t - t_*)^{-1},$$

$$A(t) = A_0(t - t_*)^{2(n-2)}, \, A_0 = \mathrm{const}$$

$$C(t) = C_0(t - t_*)^{2(n-1)},$$

$$\phi(t) = \frac{2 - n(\gamma + 1)}{\Lambda}(t - t_*)^{-1}, \quad C_0 = \mathrm{const}$$

$$s = diag(-1, 2(2 - n), 2(1 - n), -1)$$

$$\lambda = (1, \ A_0, \ C_0, \ \frac{2 - n(\gamma + 1)}{\Lambda})$$

In 2D in case of vortex solution we get

$$\alpha_2(t) = \alpha_2^0(t - t_*)^{-2}, \quad \alpha_2^0 = \mathrm{const}$$

Thus,

$$\rho(t, 0) \sim \mathrm{const}(t - t_*)^{-n}.$$

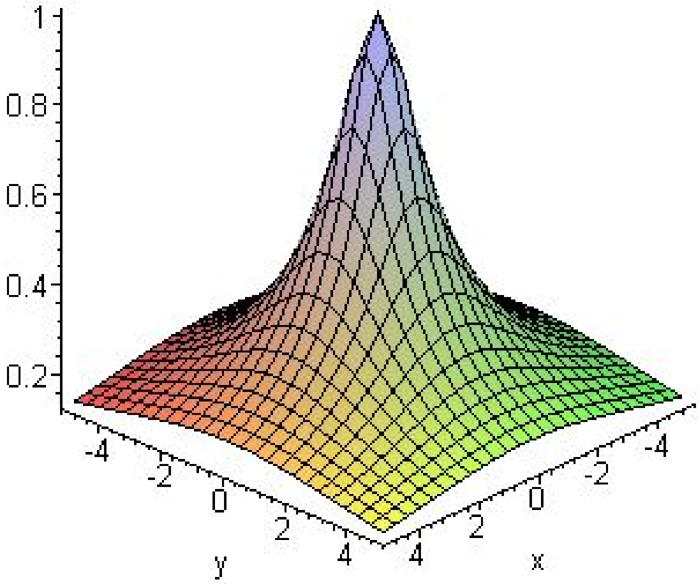


Fig. 5: The initial density

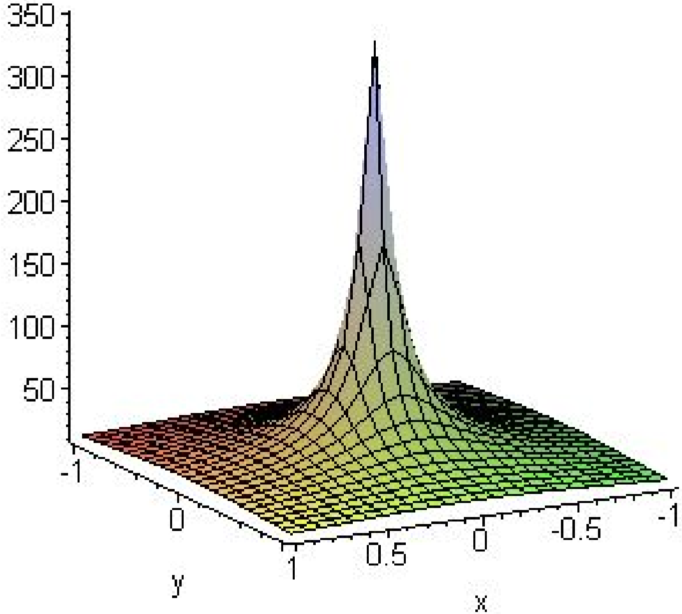


Fig. 6: The density near the blow-up time, symmetric case

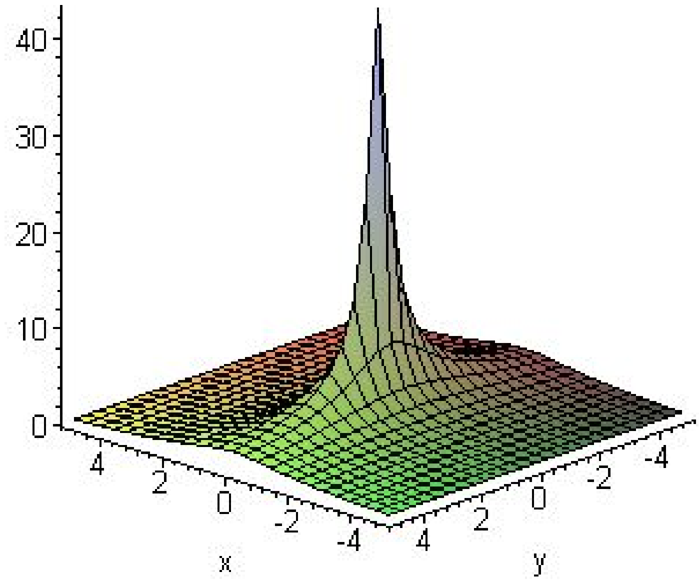


Fig. 7: The density near the blow-up time, asymmetric case

# References

[1] Brenier Y. (2005) Solutions with Concentration to the Riemann Problem for the One-Dimensional Chaplygin Gas Equations, *J. math. fluid mech.* 7, 326–331.

[2] Brilliantov N V, Pöschel T *Kinetic theory of granular gases* (Oxford: Oxford University Press), 2004.

[3] Bento M. C., Bertolami, O., Sen, A. A. (2002) Generalized Chaplygin gas, accelerated expansion and dark energy-matter unification, *Phys. Rev.* 66 (4): 043507.

[4] Gorini V, Kamenshchik A, Moschella U, Pasquier V (2004) The Chaplygin gas as a model for dark energy, http://arxiv.org/pdf/gr-qc/0403062.

[5] Fouxon I, Meerson B, Assaf M, Livne E (2007) Formation and evolution of density singularities in ideal hydrodynamics of freely

cooling inelastic gases: A family of exact solutions *Physics of Fluids* **19**, 093303

[6] Fouxon I (2000) Finite-time collapse and localized states in the dynamics of dissipative gases *Phys. Rev. E***80** 010301(R)

[7] Kamenshchik A. Yu., Moschella U., Pasquier V. (2001) An alternative to quintessence, *Phys.Lett.* B511 265-268.

[8] Luding S (2009) Towards dense, realistic granular media in 2D *Nonlinearity* 22 R101

[9] Rozanova O (2012) Formation of singularities in solutions to ideal hydrodynamics of freely cooling inelastic gases, *Nonlinearity,* 25, 1547-1558.

[10] Sedov L I (1982) *Similarity and dimensional methods in mechanics* (Moscow: Mir).

[11] Serre D (2009) Multidimensional Shock Interaction for a Chaplygin Gas *Arch. Rational Mech. Anal.* 191 539577.