On the Management of Vehicular Traffic

HYP2012

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The fundamental traffic variables

Along a road it can be measured:

- the **traffic density** ρ: number of vehicles per unit space
- the **velocity** v: distance covered by vehicles per unit time
- the **traffic flow** f: number of vehicles per unit time

The fundamental traffic variables

Along a road it can be measured:

- the **traffic density** ρ: number of vehicles per unit space
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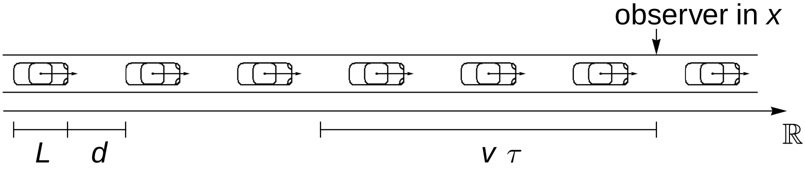
What are the relations between ρ, v and f?

Vehicles with the same length L and velocity v move equally spaced



The distance between vehicles and the density do not change.
The number of vehicles passing the observer in τ hours is the number of vehicles in $[x - \tau\, v, x]$ at time $t - \tau$ and therefore

$$f = \frac{\rho\,[x - (x - \tau\, v)]}{\tau} = \rho\, v$$

Relations between ρ, v, f

If no entries or exits are present in $[a,b]$, then

$$\underbrace{\int_a^b \rho(T,y)\,dy}_{\substack{\text{cars in } [a,b] \\ \text{at time } t=T}} = \underbrace{\int_a^b \rho(t_o,y)\,dy}_{\substack{\text{cars in } [a,b] \\ \text{at time } t=t_o}} + \underbrace{\int_{t_o}^T f(t,a)dt}_{\substack{\text{cars entering} \\ \text{in } [a,b]}} - \underbrace{\int_{t_o}^T f(t,b)dt}_{\substack{\text{cars exiting} \\ \text{from } [a,b]}}$$

or equivalently

$$\int_{t_o}^T \int_a^b \left[\partial_t \rho(t,x) - \partial_x f(t,x)\right]\, dx\, dt = 0\,.$$

Since a, b, T and t_o are arbitrary we deduce

scalar conservation law: $\boxed{\partial_t \rho + \partial_x f = 0}$

Relations between ρ, v, f

$$\boxed{f = \rho\, v} \qquad \text{and} \qquad \boxed{\partial_t \rho + \partial_x f = 0}$$

$$\left.\begin{array}{r}\text{2 equations}\\ \text{3 unknown variables}\end{array}\right\} \Longrightarrow \text{necessary a further independent equation}$$

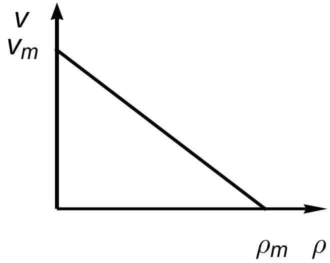
$$f = \rho\, v \qquad \text{and} \qquad \partial_t \rho + \partial_x f = 0$$

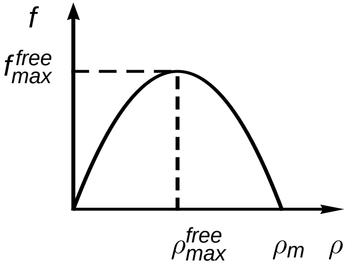
2 equations
3 unknown variables } $\Longrightarrow$ necessary a further independent equation

LWR:
$$v = v(\rho)$$

with $v : [0, \rho_m] \to [0, v_m]$ decreasing, $v(0) = v_m$ and $v(\rho_m) = 0$

Greenshields:





Resulting system

The resulting system is then

conservation	$\partial_t \rho + \partial_x f(\rho) = 0$	$(t,x) \in \mathbb{R} \times]0,+\infty[$
initial datum	$\rho(0,x) = \rho_o(x)$	$x \in]0,+\infty[$

Resulting system

If there is an **entry**, say sited at $x = 0$, we have to add the equation

$$\boxed{f\left(\rho(t,0)\right) = q_b(t)}$$

The resulting system is then

$$\begin{array}{rrr} \text{conservation} & \partial_t \rho + \partial_x f(\rho) = 0 & (t,x) \in \,]0,+\infty[^2 \\ \text{initial datum} & \rho(0,x) = \rho_o(x) & x \in \,]0,+\infty[\\ \text{entry} & f\left(\rho(t,0)\right) = q_b(t) & t \in \,]0,+\infty[\end{array}$$

Resulting system

If there is an **entry**, say sited at $x = 0$, we have to add the equation

$$\boxed{f(\rho(t,0)) = q_b(t)}$$

If there is a **restriction** (traffic lights, toll gates, construction sites, etc.), say sited at $x = x_c$, we have to add the equation

$$\boxed{f(\rho(t,x_c)) \leq q_c(t)}$$

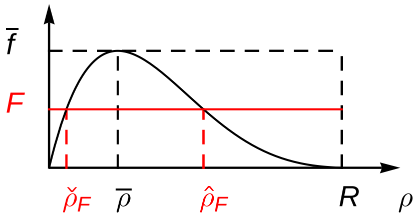
The resulting system is then

conservation	$\partial_t \rho + \partial_x f(\rho) = 0$	$(t,x) \in \left]0,+\infty\right[^2$
initial datum	$\rho(0,x) = \rho_o(x)$	$x \in \left]0,+\infty\right[$
entry	$f(\rho(t,0)) = q_b(t)$	$t \in \left]0,+\infty\right[$
constraint	$f(\rho(t,x_c)) \leq q_c(t)$	$t \in \left]0,+\infty\right[$

Conservation law+unilateral constraint

$$(CCP) \qquad \begin{cases} \partial_t \rho + \partial_x f(\rho) = 0 & x \in \mathbb{R},\ t \in \mathbb{R}_+ \\ \rho(0,x) = \rho_o(x) & x \in \mathbb{R} \\ f(\rho(t,0)) \leq F(t) & t \in \mathbb{R}_+ \end{cases}$$

- $f \in \mathcal{L}ip([0,R];[0,+\infty[)$, $f(0) = f(R) = 0$, $\exists \overline{\rho}$ s.t. $f'(\rho)\,(\overline{\rho} - \rho) > 0$
- $\rho_o \in \mathbf{L}^\infty(\mathbb{R};[0,R])$
- $F \in \mathbf{L}^\infty(\mathbb{R}_+;[0,f(\overline{\rho})])$

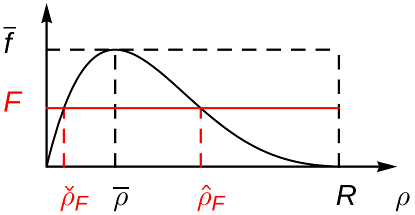


The Riemann solver $\mathcal{R}^F$

$$(CRP) \qquad \begin{cases} \partial_t \rho + \partial_x f(\rho) = 0 \\ \rho(0,x) = \rho_o(x) \\ f(\rho(t,0)) \le F \end{cases} \qquad \rho_o(x) = \begin{cases} \rho^l & \text{if } x < 0 \\ \rho^r & \text{if } x > 0 \end{cases}$$

Definition (Colombo–Goatin '07)

If $f(\mathcal{R}(\rho^l, \rho^r))(0)) \le F$, then
$$\mathcal{R}^F(\rho^l, \rho^r) = \mathcal{R}(\rho^l, \rho^r).$$
Otherwise
$$\mathcal{R}^F(\rho^l, \rho^r) = \begin{cases} \mathcal{R}(\rho^l, \hat{\rho}_F) & \text{if } x < 0 \\ \mathcal{R}(\check{\rho}_F, \rho^r) & \text{if } x > 0. \end{cases}$$



$\Longrightarrow$ non classical shock at $x = 0$

Entropy conditions

Definition (Colombo–Goatin '07)

$\rho \in \mathbf{L}^\infty$ is a weak entropy solution to (CCP) if

- $\forall \varphi \in \mathbf{C_c^1}$, $\varphi \geq 0$, and $\forall k \in [0, R]$

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} \left(|\rho - k| \partial_t + \Phi(\rho, k) \partial_x\right) \varphi \, dx \, dt + \int_{\mathbb{R}} |\rho_o - k| \varphi(0, x) \, dx$$
$$+2 \int_{\mathbb{R}_+} \left(1 - \frac{F(t)}{f(\bar{\rho})}\right) f(k) \, \varphi(t, 0) \, dt \geq 0$$

- $f(\rho(t, 0-)) = f(\rho(t, 0+)) \leq F(t)$ for a.e. $t > 0$

where $\Phi(a, b) = \operatorname{sgn}(a - b) \; (f(a) - f(b))$
and $\rho(t, 0\pm)$ the measure theoretic traces implicitly defined by

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_0^{+\infty} \int_{x_c}^{x_c + \varepsilon} |\rho(t, x) - \rho(t, 0+)| \; \varphi(t, x) \; \mathrm{d}x \; \mathrm{d}t = 0 \quad \forall \varphi \in \mathbf{C_c^1}(\mathbb{R}^2; \mathbb{R})$$

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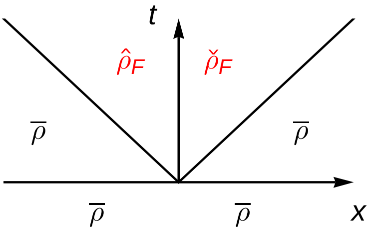
- $f(\rho(t, 0-)) = f(\rho(t, 0+)) \leq F(t)$ for a.e. $t > 0$

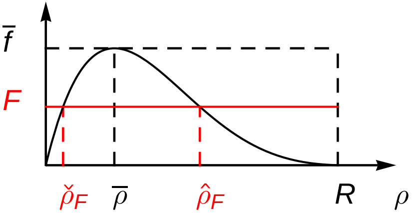
(Cfr. conservation laws with discontinuous flux function:
Baiti–Jenssen '97, Karlsen–Risebro–Towers '03, Karlsen–Towers '04, Coclite–Risebro '05, Andreianov–Goatin–Seguin '10...)

constraint $\Longrightarrow \mathrm{TV}(\rho)$ explosion

Example

$$\rho_o(x) \equiv \overline{\rho} \quad \Longrightarrow \quad \rho(t,x) = \begin{cases} \overline{\rho} & x < (f(\hat{\rho}_F) - f(\overline{\rho})) \,/\, (\hat{\rho}_F - \overline{\rho}) \\ \hat{\rho}_F & (f(\hat{\rho}_F) - f(\overline{\rho})) \,/\, (\hat{\rho}_F - \overline{\rho}) < x < 0 \\ \check{\rho}_F & 0 < x < (f(\check{\rho}_F) - f(\overline{\rho})) \,/\, (\check{\rho}_F - \overline{\rho}) \\ \overline{\rho} & x > (f(\check{\rho}_F) - f(\overline{\rho})) \,/\, (\check{\rho}_F - \overline{\rho}) \end{cases}$$



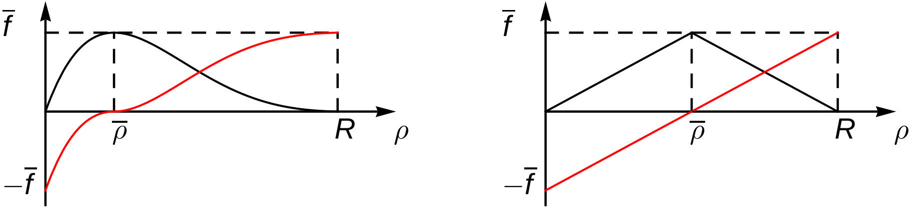


Well–posedness in BV

$$\text{constraint} \Longrightarrow \mathrm{TV}(\rho) \text{ explosion}$$

We consider the set

$$\left\{\rho \in \mathbf{L^1} \colon \Psi(\rho) \in \mathbf{BV}\right\} \qquad \Psi(\rho) = \mathrm{sgn}(\rho - \overline{\rho})\ \left(f(\overline{\rho}) - f(\rho)\right)$$



(cfr. Temple '82, Coclite–Risebro '05...)

Well–posedness in BV

Theorem (Colombo–Goatin '07)

$F \in \mathbf{BV}$. *There exists a semigroup* $S^F : \mathbb{R}_+ \times \mathcal{D} \to \mathcal{D}$ *for (CCP) s.t.*

- $\mathcal{D} \supseteq \left\{\rho \in \mathbf{L^1} \colon \Psi(\rho) \in \mathbf{BV}\right\}$
- $\left\| S^F_t \rho - S^F_t \rho' \right\|_{\mathbf{L^1}} \leq \|\rho - \rho'\|_{\mathbf{L^1}} \; \forall \rho, \rho' \in \mathcal{D}$

Proof.

- Wave–front tracking
- Glimm functional *ad hoc*

$$\Upsilon(\rho^n, F^n) = \sum_{\alpha} \left|\Psi(\rho^n_{\alpha+1}) - \Psi(\rho^n_\alpha)\right| + 5 \sum_{t_\beta \geq 0} \left|F^n_{\beta+1} - F^n_\beta\right| + \gamma$$

- Doubling of variables method with constraint

□

Well–posedness in $\mathbf{L}^\infty$

Theorem (Andreianov–Goatin–Seguin '10)

$\forall \rho_o \in \mathbf{L}^\infty$ *and* $\forall F \in \mathbf{L}^\infty$ $\exists!$ *weak entropy solution.*
If $F, F' \in \mathbf{L}^\infty$, $\rho_o, \rho_o' \in \mathbf{L}^\infty$ *and* $\rho_o - \rho_o' \in \mathbf{L}^1$*:*

$$\left\|\rho(t) - \rho'(t)\right\|_{\mathbf{L}^1} \leq \left\|\rho_o - \rho_o'\right\|_{\mathbf{L}^1} + 2\left\|F - F'\right\|_{\mathbf{L}^1}$$

Proof.

Truncation + regularization + finite propagation speed □

The initial–boundary value problem

Colombo–Goatin–Rosini '10:
Generalization of previous results to

$$(CIBVP) \quad \begin{cases} \partial_t \rho + \partial_x f(\rho) = 0 & x \in \mathbb{R}_+, t \in \mathbb{R}_+ \\ \rho(0,x) = \rho_o(x) & x \in \mathbb{R}_+ \\ f(\rho(t,0+)) = q(t) & t \in \mathbb{R}_+ \\ f(\rho(t,\bar{x})) \leq F(t) & t \in \mathbb{R}_+ \end{cases}$$

(CIBVP) can be used as a basic brick to describe

- merging road
- sequence of traffic lights
- work sites

and optimization of related cost functionals.

Well–posedness for IBVP

Definition (Colombo–Goatin–Rosini '10)

$\rho \in \mathbf{L}^\infty$ is a weak entropy solution to (CIBVP) if

- $\forall \varphi \in \mathbf{C}^1_\mathbf{c}$, $\varphi \geq 0$, and $\forall k \in [0, R]$

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} \left(|\rho - k| \partial_t + \Phi(\rho, k) \partial_x \right) \varphi \, dx \, dt + \int_{\mathbb{R}} |\rho_o - k| \varphi(0, x) \, dx$$
$$+ \int_{\mathbb{R}_+} \operatorname{sgn}({f_*}^{-1}(q(t)) - k) (f(\rho(t, 0+)) - f(k)) \, \varphi(t, 0) \, dt$$
$$+2 \int_{\mathbb{R}_+} \left(1 - \frac{F(t)}{f(\overline{\rho})} \right) f(k) \, \varphi(t, \overline{x}) \, dt \geq 0$$

- $f(\rho(t, \overline{x}-)) = f(\rho(t, \overline{x}+)) \leq F(t)$ for a.e. $t > 0$

$$f_* = f|_{[0, \overline{\rho}]}$$

Well–posedness for IBVP

Theorem (Colombo–Goatin–Rosini '10)

$\forall F, q \in \mathbf{BV}$, $\rho_o \in \mathcal{D}$ $\exists!$ *entropy weak solution to (CIBVP). Moreover,*

$$\left\|\rho(t) - \rho'(t)\right\|_{\mathbf{L}^1} \leq \left\|\rho_o - \rho'_o\right\|_{\mathbf{L}^1} + \left\|q - q'\right\|_{\mathbf{L}^1} + 2\left\|F - F'\right\|_{\mathbf{L}^1}$$

Proof.

- Wave–front tracking
- Glimm functional *ad hoc*

$$\Upsilon = \sum_\alpha \left|\Psi(\rho^n_{\alpha+1}) - \Psi(\rho^n_\alpha)\right| + 2\sum \left|q^n_{\beta+1} - q^n_\beta\right| + 5\sum \left|F^n_{\beta+1} - F^n_\beta\right| + \gamma_o + \gamma_c$$

- Doubling of variables method with constraint and boundary

□

Well–posedness for IBVP

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$\forall F, q \in \mathbf{BV}$, $\rho_o \in \mathcal{D}$ $\exists!$ *entropy weak solution to (CIBVP). Moreover,*

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$\Longrightarrow$ optimization of cost functionals

LWR (Lightill–Witham ’55, Richards ’56)

assumptions:
- the number of cars is conserved
- $v = v(\rho) = \left(1 - \frac{\rho}{R}\right) V$

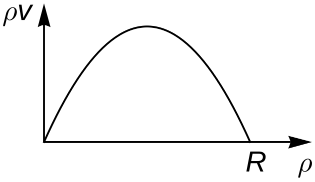
$$\partial_t \rho + \partial_x (\rho\, v) = 0$$

$t \in \mathbb{R}_+$: time

$x \in \mathbb{R}$: space

$\rho = \rho(t,x)$: mean density

$v = v(t,x)$: mean velocity



V maximal speed

R maximal density (traffic jam)

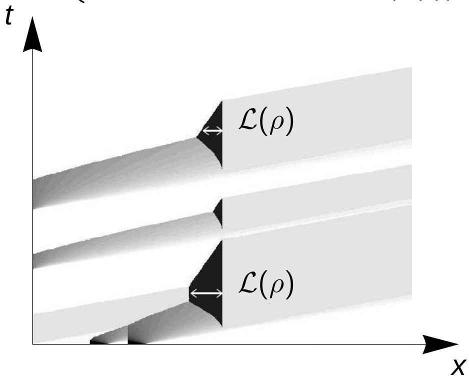
Cost functional: queue length

Queue length for **BV** data and $F(t) \equiv F \equiv const$:

$$A_c(\rho) = \{x \in [0, \bar{x}[\ : \ \Psi(\rho(\xi+)) = \bar{f} - F \text{ for a.e. } \xi \in [x, \bar{x}[\}$$

and

$$\mathcal{L}(\rho(t)) = \begin{cases} \bar{x} - \inf A_c(\rho(t)) & \text{if } A_c(\rho(t)) \neq \emptyset \\ 0 & \text{if } A_c(\rho(t)) = \emptyset \end{cases}$$



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Upper semicontinuity (Colombo–Goatin–Rosini '10)

The map $\mathcal{L}$ is upper semicontinuous with respect to the $\mathbf{L^1}$–norm.

$\Longrightarrow$ existence of maximizers for queue length!

Cost functional: stop & go waves

The total variation of traffic speed weighted by $p(x) \in [0,1]$

$$\mathcal{J}(\rho) = \int_0^T \int_{\mathbb{R}_+} p(x)\, d|\partial_x v(\rho)|\, dt$$

Lower semicontinuity (Colombo–Goatin–Rosini '10)

The map $\mathcal{J}$ is lower semicontinuous with respect to the $\mathbf{L}^1$–norm.

$\Longrightarrow$ existence of minimizers for stop & go waves

Cost functional: travel times

If $\rho_o = 0$ and $\mathrm{supp}(q) \subseteq [0, \tau_o]$, then $Q_{in} = \int_0^{\tau_o} q(t)\, dt$ and

$$\text{mean arrival time} \qquad T_a(x) = \frac{1}{Q_{in}} \int_{\mathbb{R}_+} t\, f(\rho(t,x))\, dt$$

$$\text{mean travel time} \qquad T_t(x) = \frac{1}{Q_{in}} \int_{\mathbb{R}_+} t\ \left(f(\rho(t,x)) - f(\rho(t,0))\right)\ dt$$

Lipschitz continuity (Colombo–Goatin–Rosini ’10)

The maps $T_a(x)$ and $T_t(x)$ are Lipschitz continuous with respect to the $\mathbf{L^1}$–norm.

$\Longrightarrow$ existence of maximizers and minimizers for travel times

Cost functional: ρ dependent functional

Fix $T > 0$ and $b > a > 0$

$$\mathcal{F}(\rho) = \int_0^T \int_a^b \varphi(\rho(t,x))\, w(t,x)\, dx\, dt$$

where φ can be chosen

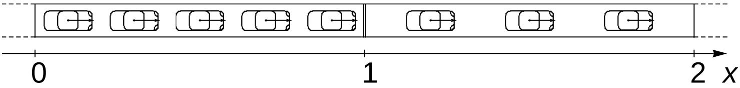
- $\varphi(\rho) = (v(\rho) - \bar{v})^2$, to have vehicles travelling at a speed as near as possible to a desired optimal speed $\bar{v}$ along a given road segment $[a,b]$
- $\varphi(\rho) = f(\rho)$, to maximize the traffic flow along $[a,b]$

Lipschitz continuity (Colombo–Goatin–Rosini ’10)

$\exists$ initial/boundary data and/or of the constraint that optimize $\mathcal{F}$.

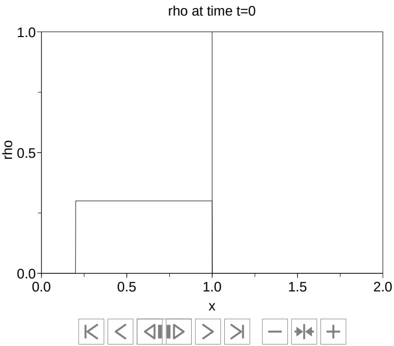
Example: toll gate

Colombo–Goatin–Rosini '09:



$$\begin{aligned} &\partial_t \rho + \partial_x (\rho\,(1-\rho)) = 0 \qquad \text{(LWR)} \\ &\rho(0,x) = 0.3\,\chi_{[0.2,1]}(x) \\ &f(\rho(t,1)) \leq 0.1 \end{aligned}$$

Example: toll gate



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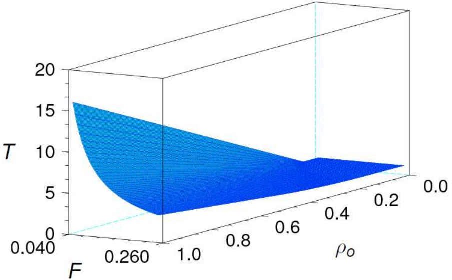
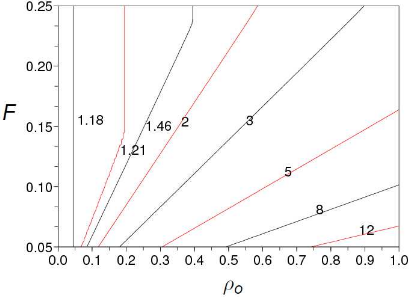
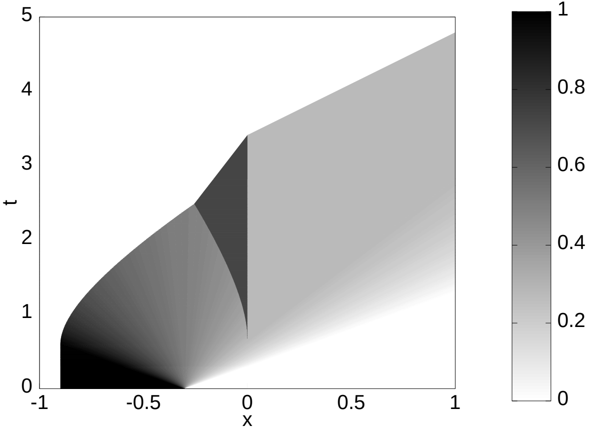




T: the time necessary for all vehicles to pass the toll gate
Left: 3D diagram
Right: the level curves

For simple initial data is a good alternative to precisely compute shock position and exit times



WFT solution with $\bar{x} = 0$, $\rho_o = \chi_{[-0.9,-0.3]}$, $F = 0.2$

Wave–front tracking VS Lax–Friedrichs

Wave–Front Tracking

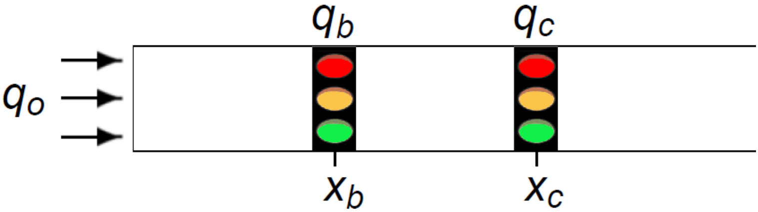
$\Delta\rho$	Exit Time	CPU Time (s)	Relative Error
4.00e-03	4.79564272	0.32	-1.90e-02 %
2.00e-03	4.79615273	0.59	-8.40e-03 %
1.00e-03	4.79640870	1.18	-3.07e-03 %
5.00e-04	4.79653693	2.36	-3.94e-04 %
2.50e-04	4.79660132	4.95	9.49e-04 %
1.25e-04	4.79656903	10.60	2.76e-04 %
6.25e-05	4.79655291	24.48	-6.06e-05 %

Lax–Friedrichs

Δx	Exit Time	CPU Time (s)	Relative Error
4.00e-03	4.94600000	1.69	3.12e-00 %
2.00e-03	4.87000000	5.18	1.53e-00 %
1.00e-03	4.83300000	18.90	7.60e-01 %
5.00e-04	4.81475000	73.40	3.79e-01 %
2.50e-04	4.80562500	295.99	1.89e-01 %
1.25e-04	4.80100000	1213.41	9.27e-02 %
6.25e-05	4.79878125	5264.29	4.64e-02 %

(Colombo–Goatin–Rosini ’10)

Colombo–Goatin–Rosini '10:



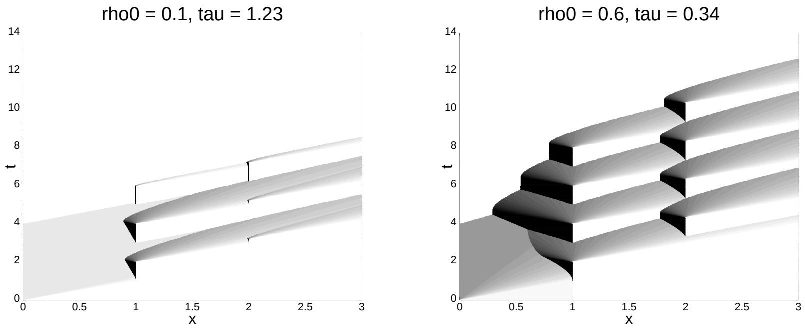
$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0 \\ \rho(0,x) = 0 \\ f(\rho(t,0)) = q_o(t) \\ f(\rho(t,x_b)) \le q_b(t) \\ f(\rho(t,x_c)) \le q_c(t) \end{cases}$$

with

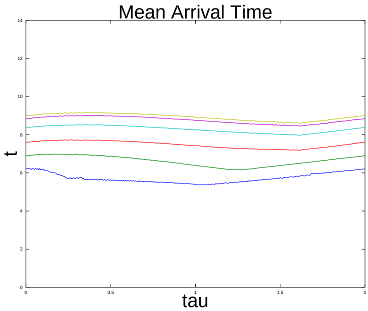
$$\begin{aligned} f(\rho) &= \rho(1-\rho) \\ x_b &= 1 \\ x_c &= 2 \\ q_o &= f(\rho_o)\,\chi_{[0,4]} \\ \rho_o &= 0.01,\ 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5 \\ q_b &= 0.25\,\chi_{[0,1]\cup[2,3]\cup[4,5]\cup[6,7]} \\ q_c^\tau(t) &= q_b(t-\tau) \end{aligned}$$

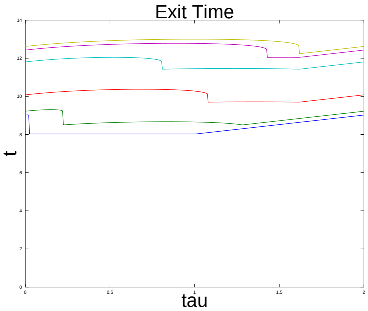
Synchronizing traffic lights

Two solutions:



Synchronizing traffic lights





The lower graphs corresponding to the lower inflows.

Perspectives

- Rigorous study of general fluxes and non–classical problems
- Improve numerical techniques for non–classical problems
- Control problems
- Extension to 2–phase models or the Aw–Rascle model

Crowd Accidents

YEAR	DEAD	CITY	NATION
1711	245	Lyon	France
1872	19	Ostrów	Poland
1876	278	Brooklyn	USA
1883	12	Brooklyn	USA
1883	180	Sunderland	England
1896	1,389	Moscow	Russia
1903	602	Chicago	USA
1908	16	Barnsley	England
1913	73	Michigan	USA
1941	4,000	Chongqing	China
1942	354	Genoa	Italy
1943	173	London	England
1946	33	Bolton	England
1956	124	Yahiko	Japan
1971	66	Glasgow	England
1979	11	Cincinnati	USA
1982	66	Moscow	Russia
1985	39	Brussels	Belgium
1988	93	Tripureswhor	Nepal
1989	96	Sheffield	England
1990	1,426	Al-Mu'aysam	Saudi Arabia
1991	40	Orkney	South Africa
1991	42	Chalma	Mexico
1993	21	Hong Kong	Cina
1993	73	Madison	USA
1994	270	Mecca	Saudi Arabia
1994	113	Nagpur	India
1996	82	Guatemala City	Guatemala

YEAR	DEAD	CITY	NATION
1998	70	Kathmandu	Nepal
1998	118	Mecca	Saudi Arabia
1999	53	Minsk	Belarus
2001	43	Henderson	USA
2001	126	Accra	Ghana
2003	21	Chicago	USA
2003	100	West Warwick	USA
2004	194	Buenos Aires	Argentina
2004	251	Mecca	Saudi Arabia
2005	300	Wai	India
2005	265	Maharashtra	India
2005	1,000	Baghdad	Iraq
2006	345	Mecca	Saudi Arabia
2006	74	Pasig City	Philippines
2006	51	Ibb	Yemen
2007	12	Chililabombwe	Zambia
2008	12	Mexico City	Mexico
2008	23	Omdurman	Sudan
2008	147	Jodhpur	India
2008	162	Himachal Pradesh	India
2008	147	Jodhpur	India
2009	19	Abidjan	Côte d'Ivoire
2010	71	Kunda	India
2010	63	Amsterdam	Netherlands
2010	21	Duisburg	Germany
2010	347	Phnom Penh	Cambodia
2011	102	Kerala	India
2011	16	Haridwar	India

font: //en.wikipedia.org

Our Results

The 1–D macroscopic model for pedestrian flows presented in

- R.M.Colombo, M.D.Rosini
  Pedestrian Flows and Nonclassical Shocks
  Mathematical Methods in the Applied Sciences, 28 (2005), no. 13, 1553–1567

describes the fall in a door through-flow due to the rise of panic, as well as the Braess' paradox.

From the physical point of view, the main assumption of this model was experimentally confirmed two years later by studying the unique video of the crowd accident on the Jamarat bridge of 2006.

- D. Helbing, A. Johansson, H. Z. Al-Abideen
  Dynamics of crowd disasters: An empirical study.
  Physical Review E, 2007

From the analytical point of view, this model is one of the few examples of a conservation law in which nonclassical solutions have a physical motivation and a global existence result for the Cauchy problem with large data is available.

Our Results

Stability, existence and optimal management problems.

M.D.Rosini
Nonclassical Interactions Portrait in a Macroscopic Pedestrian Flow Model
Journal of Differential Equations 246 (2009) 408–427

R.M.Colombo, M.D.Rosini
Existence of Nonclassical Cauchy Problem Modeling Pedestrian Flows
Journal of Nonlinear Analysis-B: Real World Applications 10 (2009) 2716–2728

R.M.Colombo, P.Goatin, G.Maternini, M.D.Rosini
Using conservation Laws in Pedestrian Modeling
acts of the congress *2009 SIDT International Conference*

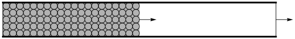
R.M.Colombo, G.Facchi, G.Maternini, M.D.Rosini
On the Continuum Modeling of Crowds
Proceedings of Symposia in Applied Mathematics 67-2 (2009) 517–526

R.M.Colombo, P.Goatin, M.D.Rosini
A macroscopic model for pedestrian flows in panic situations
Gakuto International Series. Mathematical Sciences and Applications 32 (2010) 255–272

R.M.Colombo, P.Goatin, G.Maternini, M.D.Rosini
Macroscopic Models for Pedestrian Flows
in Proceedings of the International Conference Big Events and Transport, Venice (2010) 11–22

R.M.Colombo, P.Goatin, M.D.Rosini
On the Modeling and Management of Traffic
ESAIM: Mathematical Modelling and Numerical Analysis 45 (2011) 853–872

Phenomenon: Evacuation of a corridor through an exit door.



Basic assumptions: The total number of pedestrians is conserved.
$v = v(\rho)$.

Simplify: 1D.

Write a model: Conservation law + Nonclassical Shocks.

Qualitative properties: When-where-how-why does panic arise?
Does the model describe reduced outflows?
Does the model describe the Braess' paradox?

First Attempt

The total number of pedestrians is conserved

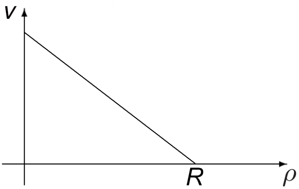
$$\frac{d}{dt}\int_a^b \rho\, dx = \frac{d}{dx}\int_a^b \rho v(\rho)\, dx$$

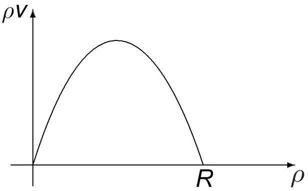
$$\Updownarrow$$

Conservation law

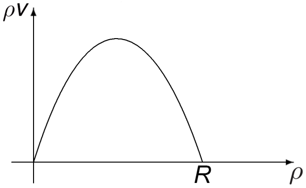
$$\partial_t \rho + \partial_x \left(\rho v(\rho)\right) = 0$$

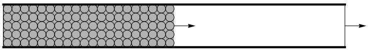
$v = v(\rho)$
(LWR)





Classical solution





$$\partial_t \rho + \partial_x (\rho v(\rho)) = 0$$

Problems

- No panic states.
- No transition to panic. (Maximum Principle)
- Drop in the door outflow.

Classical Vs Nonclassical

A classical shock has to satisfy the Lax conditions

$$q'(\rho_l) \geq \frac{q(\rho_r) - q(\rho_l)}{\rho_r - \rho_l} \geq q'(\rho_r)$$

while a nonclassical one does not satisfy them.

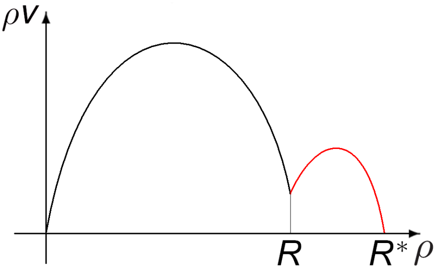
P. G. LeFloch.
Hyperbolic systems of conservation laws.
Lectures in Mathematics ETH Zürich.
Birkhäuser Verlag, Basel, 2002.

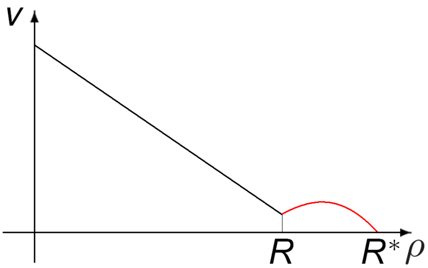
Nonclassical Solution

Solution

- Introduce panic states $]R, R^*]$.
- Extend the fundamental diagram and the speed law.

  Introduce Nonclassical Shocks $\Rightarrow$ No Maximum Principle $\Rightarrow$ Transition to panic.





Nonclassical Solution

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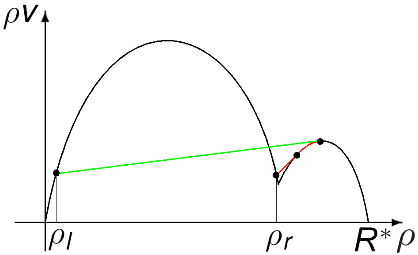
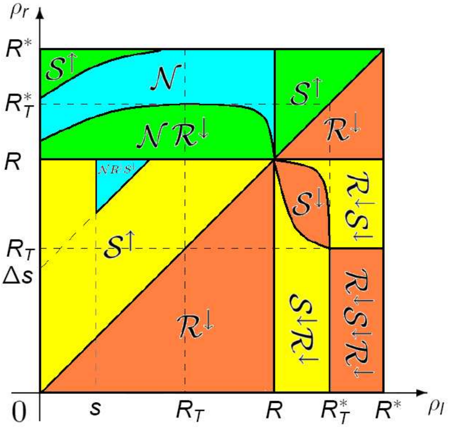
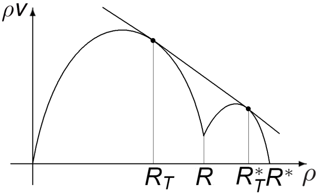


Nonclassical Riemann Solver

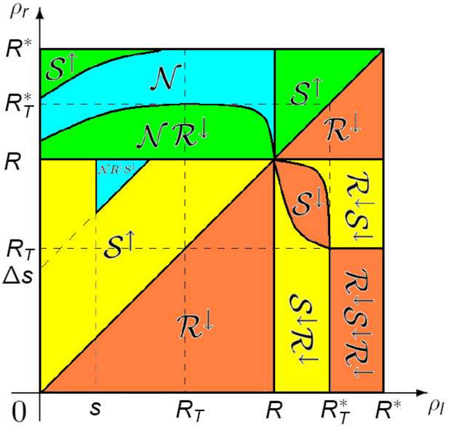




$$\partial_t \rho + \partial_x (\rho v(\rho)) = 0$$

$$\rho(0,x) = \begin{cases} \rho_l & \text{if } x < 0 \\ \rho_r & \text{if } x > 0 \end{cases}$$

Nonclassical Riemann Solver



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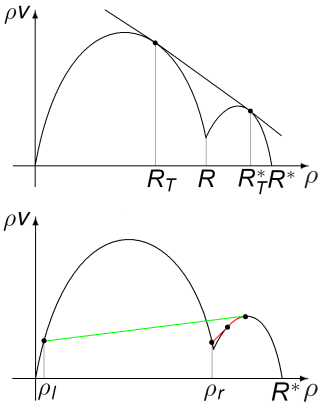
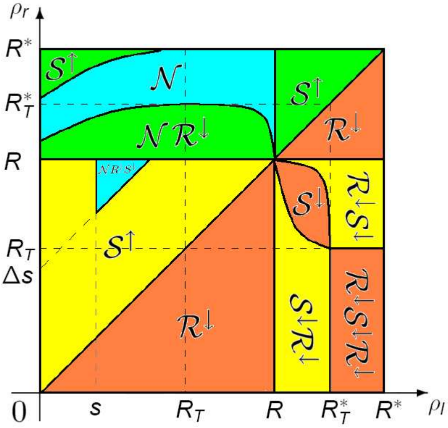
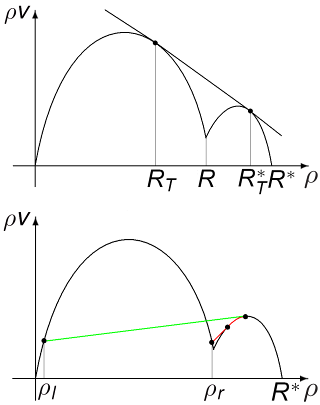


The Maximum Principle is violated!





$$\partial_t \rho + \partial_x (\rho v(\rho)) = 0$$

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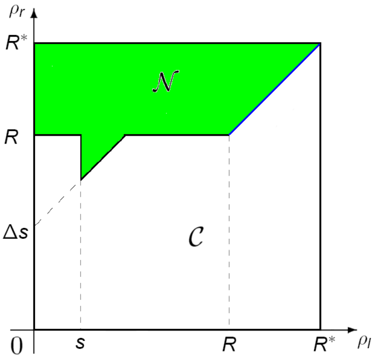
The Maximum Principle is violated!

Properties of the Riemann Solver

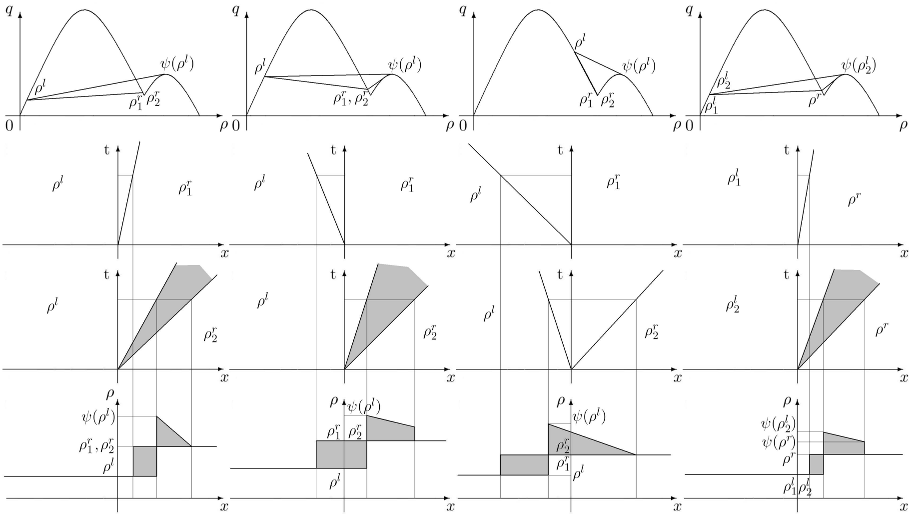
Theorem (Colombo & Rosini, M2AS, 2005)

The Riemann Solver so defined is

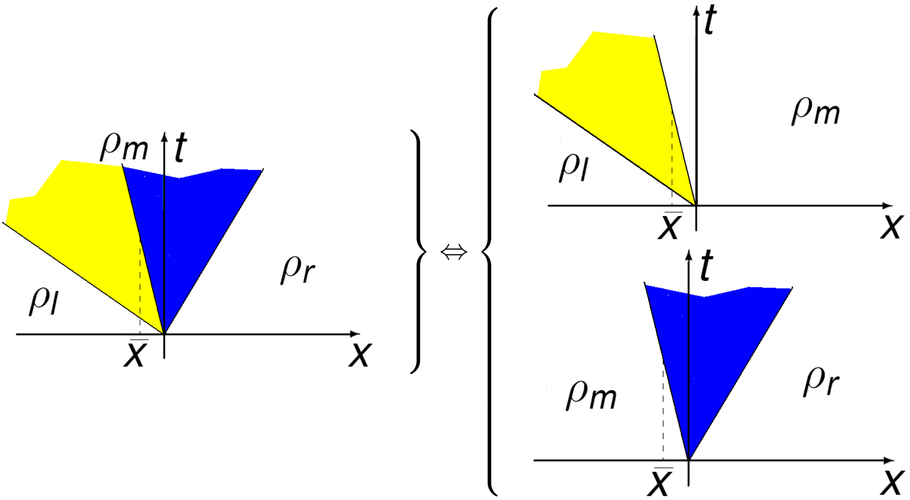
- $\mathbf{L^1_{loc}}$-continuous in $\mathcal{C}$, in $\mathcal{N}$ and along the blue segment,
- consistent in $\mathcal{C}$ and separately in $\mathcal{N}$.



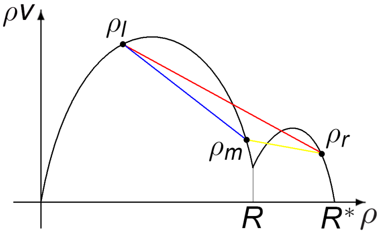
The R.S. is not L^1_{loc}-continuous in $[0, R_*]^2$

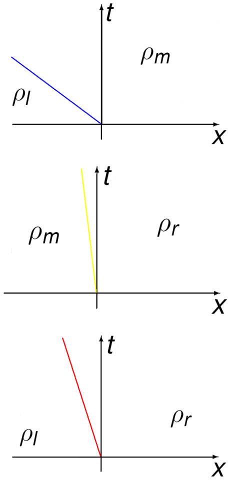


Consistency



The R.S. is not consistent in $[0, R_*]^2$

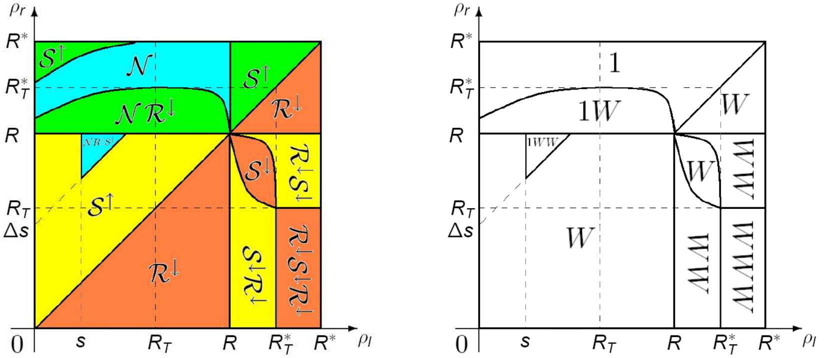




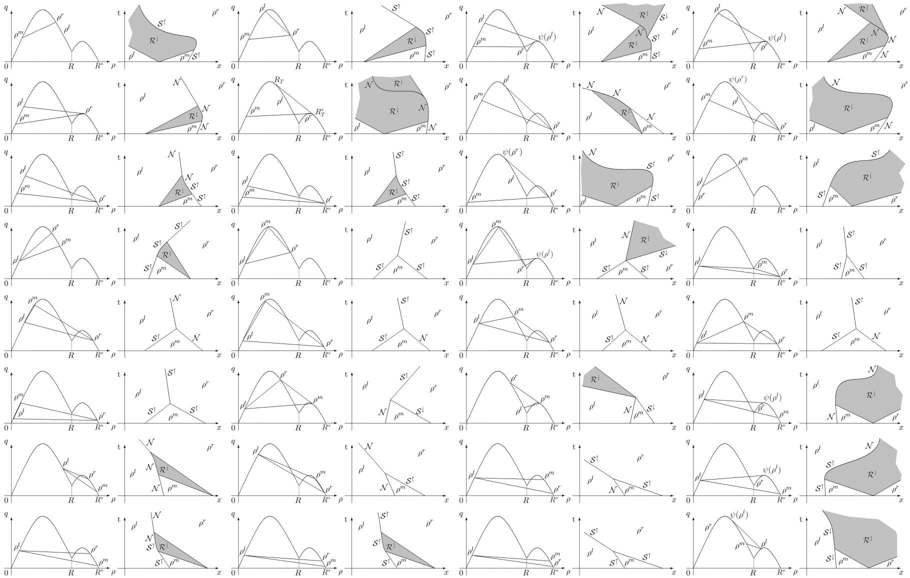
Properties of the Riemann Solver

Theorem (Rosini, JDE, 2009)

There exists $W > 1$ such that the weighted total variation $\mathrm{TV_w}: \mathbf{BV}(\mathbb{R};\mathbb{R}) \to [0,+\infty[$ represented in the picture does not increase after an interaction.



Proof



Theorem (Colombo & Rosini, NARWA, 2009)

For any $\bar\rho \in (L^1 \cap \mathbf{BV})(\mathbb{R}; [0, R_*])$, the Cauchy problem

$$\begin{cases} \partial_t \rho + \partial_x(\rho v(\rho)) = 0 \\ \rho(0,x) = \bar\rho(x) \end{cases}$$

admits a nonclassical weak solution $\rho = \rho(t,x)$ defined for all $t \in \overline{\mathbb{R}}_+$. Moreover:

$$\begin{array}{rcl} \mathrm{TV}(\rho(t)) & \leq & W \cdot \mathrm{TV}(\bar\rho) \\ \rho(t,x) & \leq & \max\{\|\bar\rho\|_{\mathbf{L}^\infty}, R_T^*\} \\ \left.\begin{array}{l} \bar\rho(\mathbb{R}) \subseteq [0,R] \\ \mathrm{TV}(\bar\rho) < \Delta s \end{array}\right\} & \Rightarrow & \rho(t,x) \in [0,R] \text{ and is a classical solution.} \end{array}$$

weak solution:

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} (\rho\, \partial_t \varphi + q(\rho)\, \partial_x \varphi)\, dx\, dt + \int_{\mathbb{R}} \bar\rho\, \varphi(0,x)\, dx = 0$$

$q(\rho) = \rho v(\rho)$ $\qquad \forall \varphi \in \mathbf{C}^1_{\mathbf{c}}(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$

Theorem (Colombo & Rosini, NARWA, 2009)

For any $\bar{\rho} \in (L^1 \cap \mathbf{BV})(\mathbb{R}; [0, R_*])$, the Cauchy problem

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Proof.

Wave–front tracking + TV_{w} + Helly's Theorem □

Theorem (Colombo & Rosini, 2008)

Not L^1-continuity in $[0, R_*]^2$.

Proof.

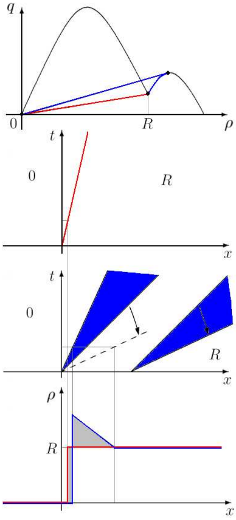
$$\rho(0,x) = \begin{cases} 0 & x \in \,]-\infty, 0[\\ R & x \in [0, +\infty[\end{cases}$$

$$\rho_n(0,x) = \begin{cases} 0 & x \in \,]-\infty, 0[\\ R + 1/n & x \in [0, 1] \\ R & x \in \,]1, +\infty[\end{cases}$$

$$\lim_{n\to\infty} \|\rho(0) - \rho_n(0)\|_{L^1(\mathbb{R};[0,R_*])} = 0$$

$$\lim_{n\to\infty} \|\rho(t) - \rho_n(t)\|_{L^1(\mathbb{R};[0,R_*])} \neq 0$$

□



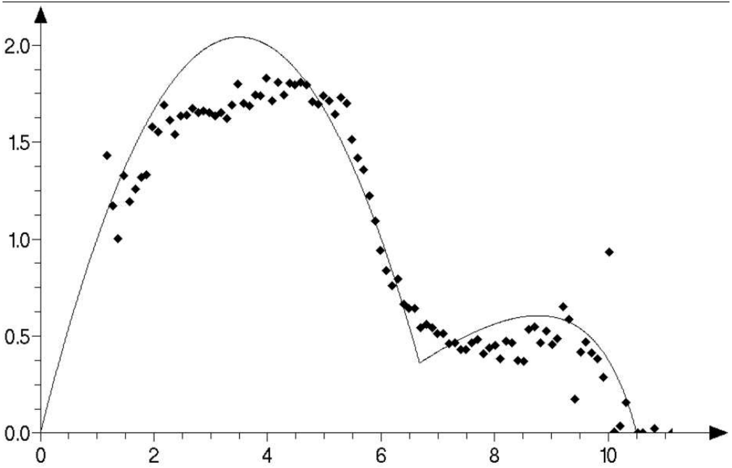
Criticisms

- 1D
- lack of continuous dependence

Criticisms

- 1D $\left(\begin{array}{l}\text{Wave Front Tracking and the theory}\\ \text{for nonclassical shocks are only 1D}\end{array}\right)$
- lack of continuous dependence (because of s and Δs)

Positive Aspects



D. Helbing, A. Johansson, H. Z. Al-Abideen
Dynamics of crowd disasters: An empirical study.
Physical Review E, 2007

Numerical approximation

C. Chalons.
Numerical approximation of a macroscopic model of pedestrian flows.
SIAM Journal on Scientific Computing, 2005.

It devised an efficient numerical scheme to approximate the solutions of our model, making it practically usable.

Citation of our model

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A. Johansson, D. Helbing, H.Z. Al-Abideen, and S. Al-Bosta.
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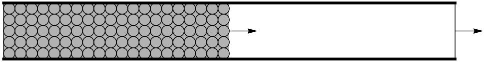
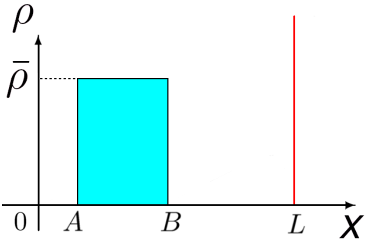
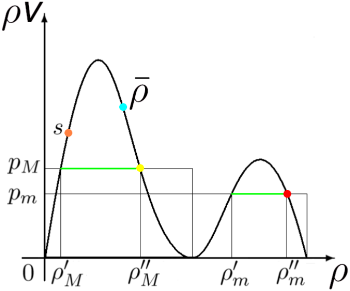
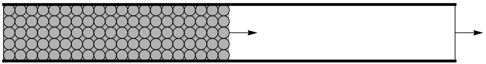


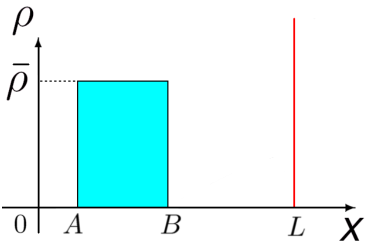
$$\begin{cases} \partial_t \rho + \partial_x(\rho v(\rho)) = 0 \\ \rho(0,x) = \rho_o(x) \\ q\,(\rho(t,L)) \le p\,(\rho(t,L)) \end{cases}$$



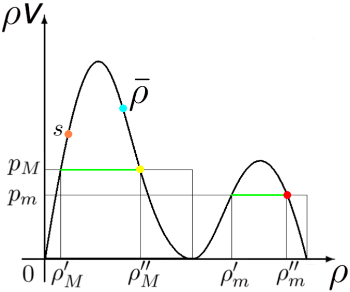


Corridor with one exit

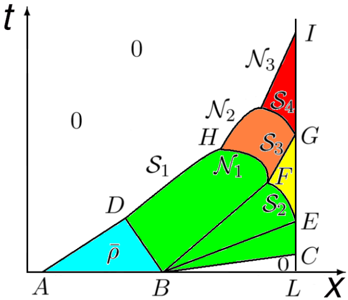


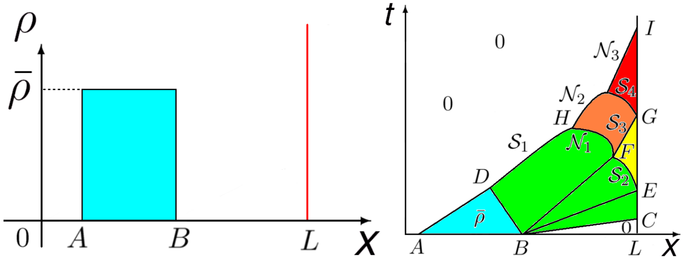


$$\begin{cases} \partial_t \rho + \partial_x (\rho v(\rho)) = 0 \\ \rho(0,x) = \rho_o(x) \\ q\,(\rho(t,L)) \le p\,(\rho(t,L)) \end{cases}$$



Riemann Problems and Wave Front Tracking

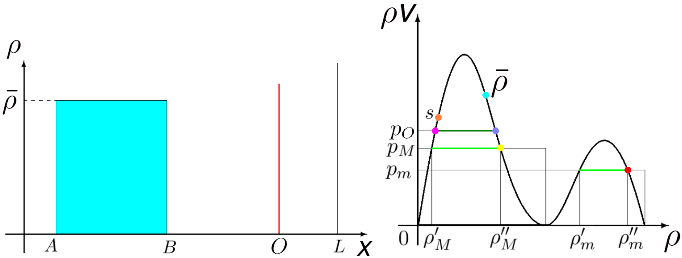




The panic arise when:

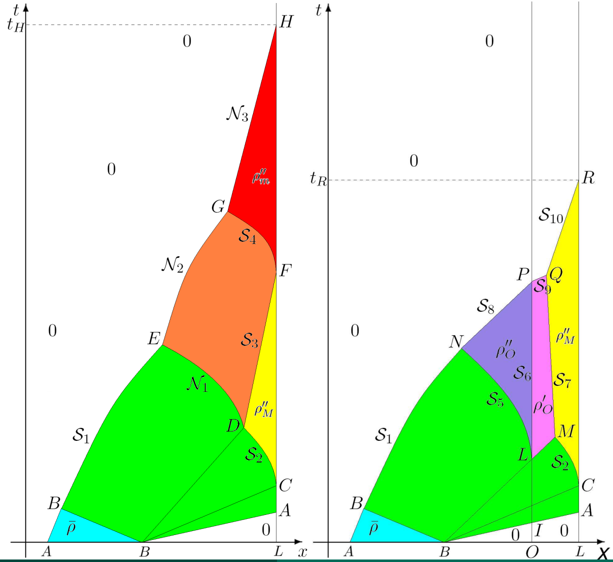
- the amount of people is large;
- the door is small;
- the initial distance of the people from the door is small.

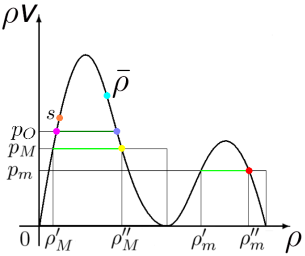
Corridor with one exit and one door



$$
\begin{cases}
\partial_t \rho + \partial_x q(\rho) = 0 & (t,x) \in \,]0,+\infty[\times]-\infty, L[\\
\rho(0,x) = \bar\rho \cdot \chi_{[a,b]}(x) & x \in \,]-\infty, L[\\
q\left(\rho(t,O)\right) \le p_O & t \in \,]0,+\infty[\\
q\left(\rho(t,L)\right) \le p\left(\rho(t,L)\right) & t \in \,]0,+\infty[
\end{cases}
$$

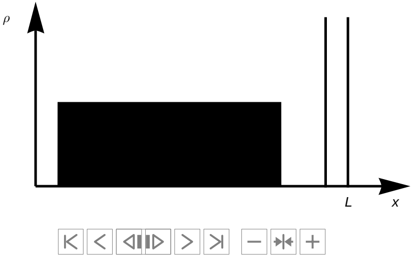
Braess' paradox





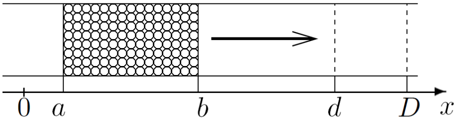
$$\frac{t_H - t_F}{t_R - t_F} \approx \frac{p_M}{p_m} > 1$$

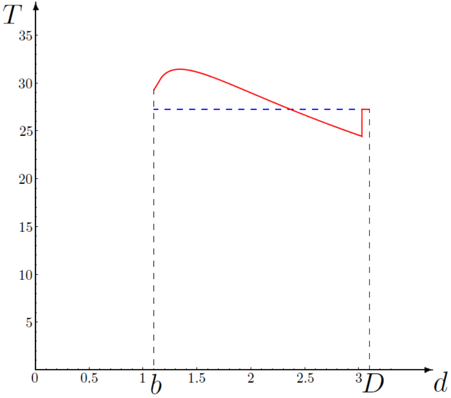
Corridor with one exit



One exit

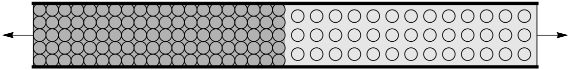
One exit and one door


0
a
b
d
D
x


T
35
30
25
20
15
10
5
0
0.5
1
b
1.5
2
2.5
3
D
d

Corridor with two exits

People evacuating a corridor $[-1, 1]$ with two exits in $x = \pm 1$



conservation law $$\partial_t \rho - \partial_x \left(\rho v(\rho) \frac{\varphi_x}{|\varphi_x|} \right) = 0$$

eikonal equation $$|\partial_x \varphi| = c(\rho)$$

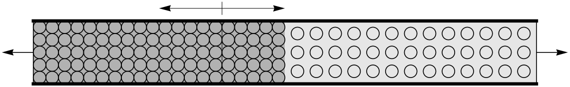
Dirichlet boundary conditions $$\rho(t, \pm 1) = \varphi(t, \pm 1) = 0$$

initial condition $$\rho(0, x) = \begin{cases} \rho_L & x \in]-1, 0[\\ \rho_R & x \in]0, 1[\end{cases}$$

$c : \mathbb{R} \to \mathbb{R}$ is the cost function

Corridor with two exits

People evacuating a corridor $[-1,1]$ with two exits in $x = \pm 1$



conservation law	$\partial_t \rho + \partial_x F(t,x,\rho) = 0$
eikonal equation	$\int_{-1}^{\xi(t)} c\left(\rho(t,y)\right) dy = \int_{\xi(t)}^{1} c\left(\rho(t,y)\right) dy$
Dirichlet boundary conditions	$\rho(t,\pm 1) = \varphi(t,\pm 1) = 0$
initial condition	$\rho(0,x) = \begin{cases} \rho_L & x \in]-1,0[\\ \rho_R & x \in]0,1[\end{cases}$

$c : \mathbb{R} \to \mathbb{R}$ is the cost function $\qquad F(t,x,\rho) = \operatorname{sgn}(x - \xi(t)) f(\rho)$

Corridor with two exits

Definition (Goatin–Nader–Rosini '12)

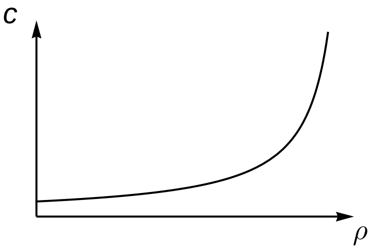
$\rho \in \mathbf{C^0}\left(\mathbb{R}^+; \mathbf{BV} \cap \mathbf{L^1}(\Omega)\right)$ is an entropy weak solution if $\forall k \in [0,1]$
$0 \leq \psi \in \mathbf{C_c^\infty}$

$$\int_0^{+\infty}\int_{-1}^{1} \left(|\rho - k|\partial_t + \Phi(t,x,\rho,k)\partial_x\right)\psi \, dx \, dt + \int_{-1}^{1} |\rho_0(x) - k|\psi(0,x) \, dx$$
$$+ 2\int_0^{+\infty} f(k)\psi\left(t,\xi(t)\right) \, dt + \operatorname{sgn}(k)\int_0^{+\infty} \left(f\left(\rho(t,1-)\right) - f(k)\right)\psi(t,1) \, dt$$
$$+ \operatorname{sgn}(k)\int_0^{+\infty} \left(f\left(\rho(t,-1+)\right) - f(k)\right)\psi(t,-1) \, dt \geq 0$$

$$\Phi(t,x,\rho,k) = \operatorname{sgn}(\rho - k)\left(F(t,x,\rho) - F(t,x,k)\right)$$

Cost function

$$c(\rho) = \frac{1}{v(\rho)}$$



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Cost function

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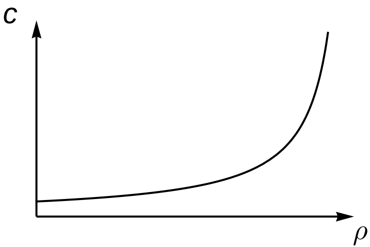

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We assume

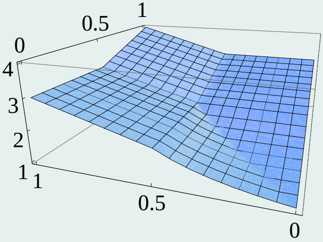
$$c \in \mathbf{C^0}([0,1];[1,+\infty[),\ c(0) = 1 \text{ and } c' \geq 0$$

and optimize w.r.t. c that we interpret as the strategy used by or imposed to the pedestrians to reach the exits

Proposition (Goatin–Nader–Rosini '12)

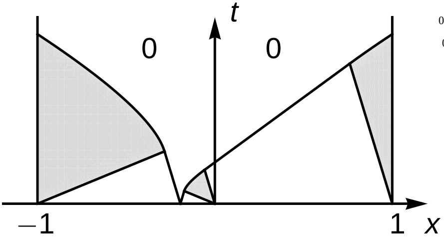
$$c_o(\rho) = \begin{cases} 1 & \rho < 1/2 \\ 2\rho & \rho \geq 1/2 \end{cases}$$ *optimize the evacuation for* $\rho_L, \rho_R > 1/2$ *and*

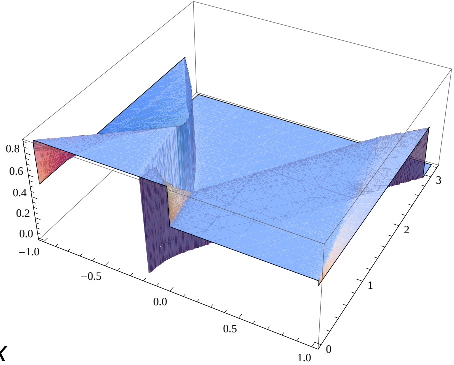
$$T_{exit}^{c_o} = \begin{cases} \dfrac{1}{1-\rho_L} & \rho_R \leq \rho_L \leq \dfrac{1}{2} \\ \dfrac{1}{1-\rho_R} & \rho_L \leq \rho_R \leq \dfrac{1}{2} \\ 1 + 2\rho_L & \rho_R \leq \dfrac{1}{2} \leq \rho_L \\ 1 + 2\rho_R & \rho_L \leq \dfrac{1}{2} \leq \rho_R \\ 2(\rho_L + \rho_R) & \dfrac{1}{2} \leq \rho_L, \rho_R \end{cases}$$



Does not exist any cost that optimize the evacuation for any initial data.

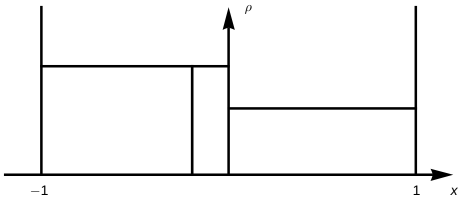
Numerical example (two exits)





$$\rho_L = \frac{18}{20} \qquad \rho_R = \frac{11}{20}$$

Numerical example (two exits)



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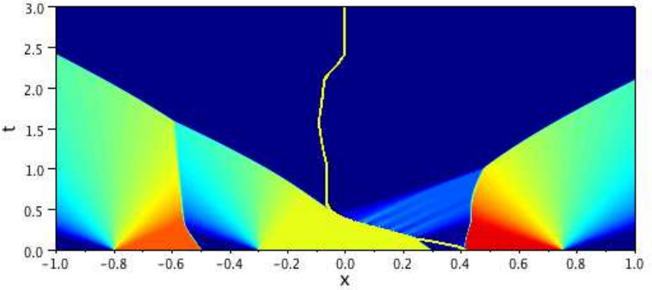

Figure: Case $c(\rho) = 1/v(\rho)$: We can observe positive densities appearing on the right of $x = \xi(t)$, representing people changing advise and inverting their route. The numerically computed exit time is $T_{exit} = 2.542$.

Numerical example (two exits)

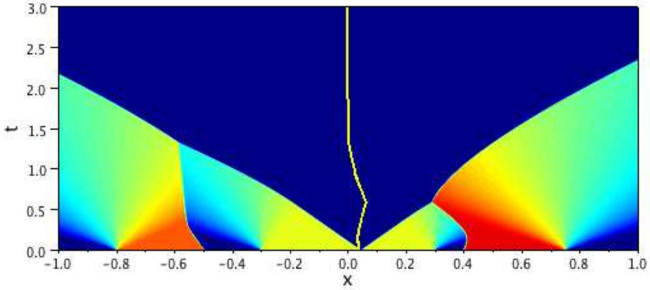

Figure: Case $c(\rho) = c_o(\rho)$: Oscillations have disappeared and $\rho(t, \xi(t)\pm) \equiv 0$. The numerically computed exit time has improved to $T_{exit} = 2.474$.

Numerical example (two exits)

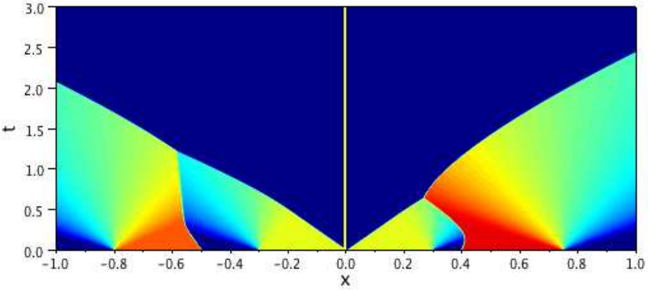

Figure: Case $c(\rho) = 1$: This choice corresponds to a panicking crowd: people are moving towards the closer exit regardless of the densities. The numerically computed exit time has increased to $T_{exit} = 2.572$.

THANK YOU
FOR YOUR
ATTENTION