On the Management of Vehicular Traffic

HYP2012

Massimiliano D. Rosini
mrosini@icm.edu.pl
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1 Introduction to Vehicular Traffic

2 Mathematics

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The fundamental traffic variables

Along a road it can be measured:

- the traffic density $\rho$: number of vehicles per unit space
- the velocity $v$: distance covered by vehicles per unit time
- the traffic flow $f$: number of vehicles per unit time
The fundamental traffic variables

Along a road it can be measured:

- the **traffic density** \( \rho \): number of vehicles per unit space
- the **velocity** \( v \): distance covered by vehicles per unit time
- the **traffic flow** \( f \): number of vehicles per unit time

**What are the relations between** \( \rho \), \( v \) **and** \( f \)?
Relations between $\rho$, $v$, $f$

Vehicles with the same length $L$ and velocity $v$ move equally spaced

The distance between vehicles and the density do not change. The number of vehicles passing the observer in $\tau$ hours is the number of vehicles in $[x - \tau v, x]$ at time $t - \tau$ and therefore

$$f = \frac{\rho [x - (x - \tau v)]}{\tau} = \rho v$$
Relations between $\rho$, $v$, $f$

If no entries or exits are present in $[a, b]$, then

$$\int_a^b \rho(T, y) \, dy = \int_a^b \rho(t_0, y) \, dy + \int_{t_0}^T f(t, a) \, dt - \int_{t_0}^T f(t, b) \, dt$$

or equivalently

$$\int_{t_0}^T \int_a^b \left[ \partial_t \rho(t, x) - \partial_x f(t, x) \right] \, dx \, dt = 0$$

Since $a$, $b$, $T$ and $t_0$ are arbitrary we deduce

\[ \partial_t \rho + \partial_x f = 0 \] 

scalar conservation law:
Relations between $\rho$, $v$, $f$

\[ f = \rho \, v \]

and

\[ \partial_t \rho + \partial_x f = 0 \]

2 equations

3 unknown variables

\[ \implies \text{necessary a further independent equation} \]
Relations between $\rho$, $v$, $f$

\[ f = \rho \, v \quad \text{and} \quad \partial_t \rho + \partial_x f = 0 \]

2 equations \{ 3 unknown variables \} $\implies$ necessary a further independent equation

LWR:

\[ v = v(\rho) \]

with $v : [0, \rho_m] \rightarrow [0, v_m]$ decreasing, $v(0) = v_m$ and $v(\rho_m) = 0$

Greenshields:

\[ f \quad \text{with} \quad f_{\text{free max}} \]

\[ \rho \quad \text{and} \quad \rho_{\text{free max}} \]
The resulting system is then

\[
\begin{align*}
\text{conservation} & \quad \partial_t \rho + \partial_x f(\rho) = 0 \quad & (t, x) \in \mathbb{R} \times ]0, +\infty[ \\
\text{initial datum} & \quad \rho(0, x) = \rho_0(x) \quad & x \in ]0, +\infty[ 
\end{align*}
\]
If there is an **entry**, say sited at $x = 0$, we have to add the equation

$$f(\rho(t, 0)) = q_b(t)$$

If there is a **restriction** (traffic lights, toll gates, construction sites, etc.), say sited at $x = x_c$, we have to add the equation

$$f(\rho(t, x_c)) \leq q_c(t)$$

The resulting system is then

- **conservation**
  $$\partial_t \rho + \partial_x f(\rho) = 0 \quad (t, x) \in ]0, +\infty[$$

- **initial datum**
  $$\rho(0, x) = \rho_0(x) \quad x \in ]0, +\infty[$$

- **entry**
  $$f(\rho(t, 0)) = q_b(t) \quad t \in ]0, +\infty[$$
Resulting system

If there is an **entry**, say sited at $x = 0$, we have to add the equation

$$f(\rho(t, 0)) = q_b(t)$$

If there is a **restriction** (traffic lights, toll gates, construction sites, etc.), say sited at $x = x_c$, we have to add the equation

$$f(\rho(t, x_c)) \leq q_c(t)$$

The resulting system is then

<table>
<thead>
<tr>
<th>conservation</th>
<th>$\partial_t \rho + \partial_x f(\rho) = 0$</th>
<th>$(t, x) \in ]0, +\infty[$</th>
</tr>
</thead>
<tbody>
<tr>
<td>initial datum</td>
<td>$\rho(0, x) = \rho_o(x)$</td>
<td>$x \in ]0, +\infty[$</td>
</tr>
<tr>
<td>entry</td>
<td>$f(\rho(t, 0)) = q_b(t)$</td>
<td>$t \in ]0, +\infty[$</td>
</tr>
<tr>
<td>constraint</td>
<td>$f(\rho(t, x_c)) \leq q_c(t)$</td>
<td>$t \in ]0, +\infty[$</td>
</tr>
</tbody>
</table>
Conservation law + unilateral constraint

\[
\begin{aligned}
(CCP) \quad & \left\{ \begin{array}{l}
\partial_t \rho + \partial_x f(\rho) = 0 \\
\rho(0, x) = \rho_0(x) \\
f(\rho(t, 0)) \leq F(t)
\end{array} \right. \\
& x \in \mathbb{R}, \quad t \in \mathbb{R}_+
\end{aligned}
\]

- \( f \in \mathcal{L}ip([0, R]; [0, +\infty[), \) \( f(0) = f(R) = 0, \exists \bar{\rho} \) s.t. \( f'(\rho) (\bar{\rho} - \rho) > 0 \)
- \( \rho_0 \in \mathbf{L}^\infty(\mathbb{R}; [0, R]) \)
- \( F \in \mathbf{L}^\infty(\mathbb{R}_+; [0, f(\bar{\rho})]) \)
The Riemann solver $\mathcal{R}^F$

\[(CRP)\]
\[
\begin{aligned}
& \partial_t \rho + \partial_x f(\rho) = 0 \\
& \rho(0, x) = \rho_o(x) \\
& f(\rho(t, 0)) \leq F
\end{aligned}
\]

Definition (Colombo–Goatin ’07)

If \( f(\mathcal{R}(\rho^l, \rho^r))(0) \leq F \), then
\[
\mathcal{R}^F(\rho^l, \rho^r) = \mathcal{R}(\rho^l, \rho^r).
\]

Otherwise
\[
\mathcal{R}^F(\rho^l, \rho^r) = \begin{cases}
\mathcal{R}(\rho^l, \hat{\rho}_F) & \text{if } x < 0 \\
\mathcal{R}(\hat{\rho}_F, \rho^r) & \text{if } x > 0
\end{cases}
\]

\[\Rightarrow\ \text{non classical shock at } x = 0\]
Entropy conditions

**Definition (Colombo–Goatin ’07)**

\( \rho \in L^\infty \) is a weak entropy solution to (CCP) if

- \( \forall \varphi \in C^1_c, \varphi \geq 0 \), and \( \forall k \in [0, R] \)

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}} (|\rho - k| \partial_t + \Phi(\rho, k) \partial_x) \varphi \, dx \, dt + \int_{\mathbb{R}} |\rho_0 - k| \varphi(0, x) \, dx \\
+ 2 \int_{\mathbb{R}^+} \left( 1 - \frac{F(t)}{f(\bar{\rho})} \right) f(k) \varphi(t, 0) \, dt \geq 0
\]

- \( f(\rho(t, 0-)) = f(\rho(t, 0+)) \leq F(t) \) for a.e. \( t > 0 \)

where \( \Phi(a, b) = \text{sgn}(a - b) \,(f(a) - f(b)) \)

and \( \rho(t, 0\pm) \) the measure theoretic traces implicitly defined by

\[
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^{+\infty} \int_{X_c+\varepsilon} |\rho(t, x) - \rho(t, 0+)\| \varphi(t, x) \, dx \, dt = 0 \quad \forall \varphi \in C^1_c(\mathbb{R}^2; \mathbb{R})
\]
Entropy conditions

**Definition (Colombo–Goatin ’07)**

\( \rho \in L^\infty \) is a weak entropy solution to (CCP) if

\[
\forall \varphi \in C^1_c, \varphi \geq 0, \text{ and } \forall k \in [0, R]
\]

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}} (|\rho - k| \partial_t + \Phi(\rho, k) \partial_x) \varphi \, dx \, dt + \int_{\mathbb{R}} |\rho_0 - k| \varphi(0, x) \, dx
\]

\[
+ 2 \int_{\mathbb{R}^+} \left( 1 - \frac{F(t)}{f(\rho)} \right) f(k) \varphi(t, 0) \, dt \geq 0
\]

\[
f(\rho(t, 0^-)) = f(\rho(t, 0^+)) \leq F(t) \text{ for a.e. } t > 0
\]

where \( \Phi(a, b) = \text{sgn}(a - b) \ (f(a) - f(b)) \)

and \( \rho(t, 0^\pm) \) the measure theoretic traces implicitly defined by

\[
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^\infty \int_{X_c - \varepsilon}^{X_c} |\rho(t, x) - \rho(t, 0^-)| \varphi(t, x) \, dx \, dt = 0 \quad \forall \varphi \in C^1_c(\mathbb{R}^2; \mathbb{R})
\]
Definition (Colombo–Goatin ’07)

\( \rho \in L^\infty \) is a weak entropy solution to (CCP) if

- \( \forall \varphi \in C^1_c, \varphi \geq 0, \) and \( \forall k \in [0, R] \)

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}} (|\rho - k| \partial_t + \Phi(\rho, k) \partial_x) \varphi \, dx \, dt + \int_{\mathbb{R}} |\rho_0 - k| \varphi(0, x) \, dx \\
+ 2 \int_{\mathbb{R}^+} \left( 1 - \frac{F(t)}{f(\overline{\rho})} \right) f(k) \varphi(t, 0) \, dt \geq 0
\]

- \( f(\rho(t, 0^-)) = f(\rho(t, 0^+)) \leq F(t) \) for a.e. \( t > 0 \)


Padua 28.06.2012
M.D. Rosini (ICM, Warsaw University)
Well-posedness in BV

constraint $\implies$ TV($\rho$) explosion

Example

$\rho_0(x) \equiv \bar{\rho}$ $\implies$ $\rho(t, x) = \begin{cases} 
\bar{\rho} & x < (f(\hat{\rho}_F) - f(\bar{\rho})) / (\hat{\rho}_F - \bar{\rho}) \\
\hat{\rho}_F & (f(\hat{\rho}_F) - f(\bar{\rho})) / (\hat{\rho}_F - \bar{\rho}) < x < 0 \\
\bar{\rho} & 0 < x < (f(\check{\rho}_F) - f(\bar{\rho})) / (\check{\rho}_F - \bar{\rho}) \\
\hat{\rho}_F & x > (f(\check{\rho}_F) - f(\bar{\rho})) / (\check{\rho}_F - \bar{\rho}) 
\end{cases}$
Well–posedness in BV

\[ \text{constraint} \implies TV(\rho) \text{ explosion} \]

We consider the set

\[ \left\{ \rho \in L^1 : \Psi(\rho) \in BV \right\} \]

\[ \Psi(\rho) = \text{sgn}(\rho - \overline{\rho}) \left( f(\overline{\rho}) - f(\rho) \right) \]

(cfr. Temple '82, Coclite–Risebro '05...)
Theorem (Colombo–Goatin ’07)

\( F \in BV. \) There exists a semigroup \( S^F : \mathbb{R}_+ \times D \rightarrow D \) for (CCP) s.t.

- \( D \supseteq \{ \rho \in L^1 : \Psi(\rho) \in BV \} \)
- \( \| S_t^F \rho - S_t^F \rho' \|_{L^1} \leq \| \rho - \rho' \|_{L^1} \) \( \forall \rho, \rho' \in D \)

Proof.

- Wave–front tracking
- Glimm functional \( ad \ hoc \)

\[
\Upsilon(\rho^n, F^n) = \sum_{\alpha} |\Psi(\rho^n_{\alpha+1}) - \Psi(\rho^n_{\alpha})| + 5 \sum_{t_\beta \geq 0} |F^n_{\beta+1} - F^n_\beta| + \gamma
\]

- Doubling of variables method with constraint
Theorem (Andreianov–Goatin–Seguin ’10)

∀ \rho_o \in L^\infty \text{ and } \forall F \in L^\infty \exists ! \text{ weak entropy solution.}

If \( F, F' \in L^\infty \), \( \rho_o, \rho'_o \in L^\infty \) and \( \rho_o - \rho'_o \in L^1 \): 

\[
\| \rho(t) - \rho'(t) \|_{L^1} \leq \| \rho_o - \rho'_o \|_{L^1} + 2 \| F - F' \|_{L^1}
\]

Proof.

Truncation + regularization + finite propagation speed
The initial–boundary value problem

Colombo–Goatin–Rosini ’10:
Generalization of previous results to

\[
\begin{aligned}
\partial_t \rho + \partial_x f(\rho) &= 0 \\
\rho(0, x) &= \rho_o(x) \\
f(\rho(t, 0^+)) &= q(t) \\
f(\rho(t, x)) &\leq F(t)
\end{aligned}
\]

\((CIBVP)\)

(CIBVP) can be used as a basic brick to describe

- merging road
- sequence of traffic lights
- work sites

and optimization of related cost functionals.


**Definition (Colombo–Goatin–Rosini ’10)**

\( \rho \in L^\infty \) is a weak entropy solution to (CIBVP) if

- \( \forall \varphi \in C^1_c, \varphi \geq 0 \), and \( \forall k \in [0, R] \)

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}} (|\rho - k| \partial_t + \Phi(\rho, k) \partial_x) \varphi \, dx \, dt + \int_{\mathbb{R}} |\rho_0 - k| \varphi(0, x) \, dx
\]

\[
+ \int_{\mathbb{R}^+} \text{sgn}(f_*^{-1}(q(t)) - k)(f(\rho(t, 0^+)) - f(k)) \varphi(t, 0) \, dt
\]

\[
+ 2 \int_{\mathbb{R}^+} \left(1 - \frac{F(t)}{f'(\bar{\rho})}\right) f(k) \varphi(t, \bar{x}) \, dt \geq 0
\]

- \( f(\rho(t, \bar{x}^-)) = f(\rho(t, \bar{x}^+)) \leq F(t) \) for a.e. \( t > 0 \)

\[ f_* = f|_{[0, \bar{\rho}]} \]
Theorem (Colombo–Goatin–Rosini ’10)

\[ \forall F, q \in BV, \rho_o \in D \exists! \text{ entropy weak solution to (CIBVP). Moreover,} \]

\[ \| \rho(t) - \rho'(t) \|_{L^1} \leq \| \rho_o - \rho'_o \|_{L^1} + \| q - q' \|_{L^1} + 2 \| F - F' \|_{L^1} \]

Proof.

- Wave–front tracking
- Glimm functional \textit{ad hoc}

\[ \gamma = \sum_{\alpha} |\psi(\rho^n_{\alpha+1}) - \psi(\rho^n_{\alpha})| + 2 \sum |q^n_{\beta+1} - q^n_{\beta}| \]

\[ + 5 \sum |F^n_{\beta+1} - F^n_{\beta}| + \gamma_o + \gamma_c \]

- Doubling of variables method with constraint and boundary
Well–posedness for IBVP

Theorem (Colombo–Goatin–Rosini ’10)

\[ \forall F, q \in BV, \rho_o \in D \exists! \text{ entropy weak solution to (CIBVP)}. \]

Moreover,

\[ \| \rho(t) - \rho'(t) \|_{L^1} \leq \| \rho_o - \rho'_o \|_{L^1} + \| q - q' \|_{L^1} + 2 \| F - F' \|_{L^1} \]

\[ \Rightarrow \text{ optimization of cost functionals} \]
Introduction to Vehicular Traffic

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Numerical Examples

Crowd Accidents

The Model

Corridor with One Exit

Corridor with Two Exits
LWR (Lightill–Witham ’55, Richards ’56)

assumptions:

- the number of cars is conserved
- \( v = v(\rho) = \left(1 - \frac{\rho}{R}\right)V \)

\[ \partial_t \rho + \partial_x (\rho \, v) = 0 \]

\( t \in \mathbb{R}_+ \): time \quad \rho = \rho(t, x) : \text{mean density} \\
\( x \in \mathbb{R} \): space \quad v = v(t, x) : \text{mean velocity} \\

\( V \): maximal speed \quad \( R \): maximal density (traffic jam)
Cost functional: queue length

Queue length for $\text{BV}$ data and $F(t) \equiv F \equiv \text{const}$:

$$A_c(\rho) = \{ x \in [0, \overline{x}] : \psi(\rho(\xi^+)) = \overline{f} - F \text{ for a.e. } \xi \in [x, \overline{x}] \}$$

and

$$\mathcal{L}(\rho(t)) = \begin{cases} \overline{x} - \inf A_c(\rho(t)) & \text{if } A_c(\rho(t)) \neq \emptyset \\ 0 & \text{if } A_c(\rho(t)) = \emptyset \end{cases}$$
Cost functional: queue length

Queue length for \( \text{BV} \) data and \( F(t) \equiv F \equiv \text{const} \):

\[
A_c(\rho) = \{ x \in [0, \bar{x}] : \psi(\rho(\xi+)) = \bar{f} - F \text{ for a.e. } \xi \in [x, \bar{x}] \}
\]

and

\[
L(\rho(t)) = \begin{cases} 
\bar{x} - \inf A_c(\rho(t)) & \text{if } A_c(\rho(t)) \neq \emptyset \\
0 & \text{if } A_c(\rho(t)) = \emptyset
\end{cases}
\]

Upper semicontinuity (Colombo–Goatin–Rosini ’10)

The map \( L \) is upper semicontinuous with respect to the \( L^1 \)–norm.

\[ \implies \text{existence of maximizers for queue length!} \]
The total variation of traffic speed weighted by $\rho(x) \in [0, 1]$

$$\mathcal{J}(\rho) = \int_0^T \int_{\mathbb{R}^+} p(x) \, d|\partial_x v(\rho)| \, dt$$

Lower semicontinuity (Colombo–Goatin–Rosini ’10)

The map $\mathcal{J}$ is lower semicontinuous with respect to the $L^1$–norm.

$\implies$ existence of minimizers for stop & go waves
Cost functional: travel times

If $\rho_0 = 0$ and $\text{supp}(q) \subseteq [0, \tau_0]$, then $Q_{in} = \int_0^{\tau_0} q(t) \, dt$ and

\[
T_a(x) = \frac{1}{Q_{in}} \int_{\mathbb{R}^+} t f(\rho(t, x)) \, dt \\
T_t(x) = \frac{1}{Q_{in}} \int_{\mathbb{R}^+} t (f(\rho(t, x)) - f(\rho(t, 0))) \, dt
\]

Lipschitz continuity (Colombo–Goatin–Rosini ’10)

The maps $T_a(x)$ and $T_t(x)$ are Lipschitz continuous with respect to the $L^1$–norm.

$\Rightarrow$ existence of maximizers and minimizers for travel times
Cost functional: $\rho$ dependent functional

Fix $T > 0$ and $b > a > 0$

$$\mathcal{F}(\rho) = \int_0^T \int_a^b \varphi(\rho(t, x)) \, w(t, x) \, dx \, dt$$

where $\varphi$ can be chosen

- $\varphi(\rho) = (v(\rho) - \bar{v})^2$, to have vehicles travelling at a speed as near as possible to a desired optimal speed $\bar{v}$ along a given road segment $[a, b]$
- $\varphi(\rho) = f(\rho)$, to maximize the traffic flow along $[a, b]$

Lipschitz continuity (Colombo–Goatin–Rosini ’10)

$\exists$ initial/boundary data and/or of the constraint that optimize $\mathcal{F}$.
Example: toll gate

Colombo–Goatin–Rosini ’09:

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho \ (1 - \rho)) &= 0 \quad \text{(LWR)} \\
\rho(0, x) &= 0.3 \ \chi_{[0.2, 1]}(x) \\
f(\rho(t, 1)) &\leq 0.1
\end{align*}
\]
Example: toll gate
Example: toll gate

$T$: the time necessary for all vehicles to pass the toll gate
Left: 3D diagram
Right: the level curves
Wave–front tracking scheme

For simple initial data is a good alternative to precisely compute shock position and exit times

WFT solution with $\overline{x} = 0$, $\rho_0 = \chi_{[-0.9,-0.3]}$, $F = 0.2$
Wave–front tracking VS Lax–Friedrichs

Wave–Front Tracking

<table>
<thead>
<tr>
<th>$\Delta \rho$</th>
<th>Exit Time</th>
<th>CPU Time (s)</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.00e-03</td>
<td>4.79564272</td>
<td>0.32</td>
<td>-1.90e-02 %</td>
</tr>
<tr>
<td>2.00e-03</td>
<td>4.79615273</td>
<td>0.59</td>
<td>-8.40e-03 %</td>
</tr>
<tr>
<td>1.00e-03</td>
<td>4.79640870</td>
<td>1.18</td>
<td>-3.07e-03 %</td>
</tr>
<tr>
<td>5.00e-04</td>
<td>4.79653693</td>
<td>2.36</td>
<td>-3.94e-04 %</td>
</tr>
<tr>
<td>2.50e-04</td>
<td>4.79660132</td>
<td>4.95</td>
<td>9.49e-04 %</td>
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<tr>
<td>1.25e-04</td>
<td>4.79656903</td>
<td>10.60</td>
<td>2.76e-04 %</td>
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<tr>
<td>6.25e-05</td>
<td>4.79655291</td>
<td>24.48</td>
<td>-6.06e-05 %</td>
</tr>
</tbody>
</table>

Lax–Friedrichs

<table>
<thead>
<tr>
<th>$\Delta x$</th>
<th>Exit Time</th>
<th>CPU Time (s)</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.00e-03</td>
<td>4.94600000</td>
<td>1.69</td>
<td>3.12e-00 %</td>
</tr>
<tr>
<td>2.00e-03</td>
<td>4.87000000</td>
<td>5.18</td>
<td>1.53e-00 %</td>
</tr>
<tr>
<td>1.00e-03</td>
<td>4.83300000</td>
<td>18.90</td>
<td>7.60e-01 %</td>
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<tr>
<td>5.00e-04</td>
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<td>3.79e-01 %</td>
</tr>
<tr>
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<td>1.89e-01 %</td>
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<tr>
<td>1.25e-04</td>
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<td>1213.41</td>
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<td>6.25e-05</td>
<td>4.79878125</td>
<td>5264.29</td>
<td>4.64e-02 %</td>
</tr>
</tbody>
</table>

(Colombo–Goatin–Rosini ’10)
Synchronizing traffic lights

Colombo–Goatin–Rosini ’10:

\[
\begin{align*}
\partial_t \rho + \partial_x f(\rho) &= 0 \\
\rho(0, x) &= 0 \\
f(\rho(t, 0)) &= q_o(t) \\
f(\rho(t, x_b)) &\leq q_b(t) \\
f(\rho(t, x_c)) &\leq q_c(t)
\end{align*}
\]

\[
\begin{align*}
f(\rho) &= \rho(1 - \rho) \\
x_b &= 1 \\
x_c &= 2 \\
q_o &= f(\rho_o) \chi[0,4] \\
\rho_o &= 0.01, 0.1, 0.2, 0.3, 0.4, 0.5 \\
q_b &= 0.25 \chi[0,1] \cup [2,3] \cup [4,5] \cup [6,7] \\
q_c^\tau(t) &= q_b(t - \tau)
\end{align*}
\]

Padua 28.06.2012

M.D. Rosini (ICM, Warsaw University)
Synchronizing traffic lights

Two solutions:

\[
\rho_0 = 0.1, \tau = 1.23
\]

\[
\rho_0 = 0.6, \tau = 0.34
\]
Synchronizing traffic lights

The lower graphs corresponding to the lower inflows.
Perspectives

- Rigorous study of general fluxes and non-classical problems
- Improve numerical techniques for non-classical problems
- Control problems
- Extension to 2-phase models or the Aw–Rascle model
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<table>
<thead>
<tr>
<th>YEAR</th>
<th>DEAD</th>
<th>CITY</th>
<th>NATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1711</td>
<td>245</td>
<td>Lyon</td>
<td>France</td>
</tr>
<tr>
<td>1872</td>
<td>19</td>
<td>Ostrów</td>
<td>Poland</td>
</tr>
<tr>
<td>1876</td>
<td>278</td>
<td>Brooklyn</td>
<td>USA</td>
</tr>
<tr>
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</table>
1 Introduction to Vehicular Traffic
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3 Applications to LWR
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7 Corridor with One Exit
8 Corridor with Two Exits
Our Results

The 1–D macroscopic model for pedestrian flows presented in

R.M. Colombo, M.D. Rosini
Pedestrian Flows and Nonclassical Shocks

describes the fall in a door through-flow due to the rise of panic, as well as the Braess’ paradox.

From the physical point of view, the main assumption of this model was experimentally confirmed two years later by studying the unique video of the crowd accident on the Jamarat bridge of 2006.

D. Helbing, A. Johansson, H. Z. Al-Abideen
Dynamics of crowd disasters: An empirical study.
Physical Review E, 2007

From the analytical point of view, this model is one of the few examples of a conservation law in which nonclassical solutions have a physical motivation and a global existence result for the Cauchy problem with large data is available.
Our Results

Stability, existence and optimal management problems.

M.D. Rosini
Nonclassical Interactions Portrait in a Macroscopic Pedestrian Flow Model

R.M. Colombo, M.D. Rosini
Existence of Nonclassical Cauchy Problem Modeling Pedestrian Flows

R.M. Colombo, P. Goatin, G. Maternini, M.D. Rosini
Using conservation Laws in Pedestrian Modeling
acts of the congress 2009 SIDT International Conference

R.M. Colombo, G. Facchi, G. Maternini, M.D. Rosini
On the Continuum Modeling of Crowds

R.M. Colombo, P. Goatin, M.D. Rosini
A macroscopic model for pedestrian flows in panic situations

R.M. Colombo, P. Goatin, G. Maternini, M.D. Rosini
Macrosopic Models for Pedestrian Flows

R.M. Colombo, P. Goatin, M.D. Rosini
On the Modeling and Management of Traffic
ESAIM: Mathematical Modelling and Numerical Analysis 45 (2011) 853–872
Phenomenon: Evacuation of a corridor through an exit door.

Basic assumptions: The total number of pedestrians is conserved. \( v = v(\rho) \).

Simplify: 1D.

Write a model: Conservation law + Nonclassical Shocks.

Qualitative properties: When-where-how-why does panic arise? Does the model describe reduced outflows? Does the model describe the Braess’ paradox?
First Attempt

The total number of pedestrians is conserved

\[ \frac{d}{dt} \int_{a}^{b} \rho \, dx = \frac{d}{dx} \int_{a}^{b} \rho v(\rho) \, dx \]

\[ \iff \]

\[ \partial_t \rho + \partial_x (\rho v(\rho)) = 0 \]

\[ v = v(\rho) \]

(LWR)

Classical solution
Classical Solution - Application

\[ \partial_t \rho + \partial_x (\rho v(\rho)) = 0 \]

Problems

- No panic states.
- No transition to panic. (Maximum Principle)
- Drop in the door outflow.
A classical shock has to satisfy the Lax conditions

\[ q'(\rho_l) \geq \frac{q(\rho_r) - q(\rho_l)}{\rho_r - \rho_l} \geq q'(\rho_r) \]

while a nonclassical one does not satisfy them.

---

P. G. LeFloch.  
*Hyperbolic systems of conservation laws.*  
Lectures in Mathematics ETH Zürich.  
Nonclassical Solution

Solution

- Introduce panic states \([R, R^*]\).
- Extend the fundamental diagram and the speed law.

Introduce Nonclassical Shocks \(\Rightarrow\) No Maximum Principle \(\Rightarrow\) Transition to panic.
Introduce panic states \([R, R^*]\).

Extend the fundamental diagram and the speed law.

Introduce **Nonclassical Shocks** \(\Rightarrow\) No Maximum Principle \(\Rightarrow\) Transition to panic.
Nonclassical Riemann Solver

\[ \partial_t \rho + \partial_x (\rho v(\rho)) = 0 \]

\[ \rho(0, x) = \begin{cases} 
\rho_l & \text{if } x < 0 \\
\rho_r & \text{if } x > 0 
\end{cases} \]
Nonclassical Riemann Solver

\[ \partial_t \rho + \partial_x (\rho v(\rho)) = 0 \]

\[ \rho(0, x) = \begin{cases} 
\rho_l & \text{if } x < 0 \\
\rho_r & \text{if } x > 0 
\end{cases} \]

The Maximum Principle is violated!
Nonclassical Riemann Solver

\[ \partial_t \rho + \partial_x \left( \rho v(\rho) \right) = 0 \]

\[ \rho(0, x) = \begin{cases} 
\rho_l & \text{if } x < 0 \\
\rho_r & \text{if } x > 0 
\end{cases} \]

The Maximum Principle is violated!
Theorem (Colombo & Rosini, M2AS, 2005)

The Riemann Solver so defined is

- \( L^1_{\text{loc}} \)-continuous in \( C \), in \( \mathcal{N} \) and along the blue segment,
- consistent in \( C \) and separately in \( \mathcal{N} \).
The R.S. is not $L^1_{loc}$-continuous in $[0, R_*]^2$
Consistency

\[ \begin{array}{c}
\rho_m \\
\rho_l \\
\rho_r
\end{array} \quad \Leftrightarrow \quad \begin{array}{c}
\rho_m \\
\rho_l \\
\rho_r
\end{array} \]
The R.S. is not consistent in $[0, R_*)^2$
Properties of the Riemann Solver

Theorem (Rosini, JDE, 2009)
There exists $W > 1$ such that the weighted total variation
$TV_w : BV(\mathbb{R}; \mathbb{R}) \rightarrow [0, +\infty[$ represented in the picture does not increase after an interaction.
Theorem (Colombo & Rosini, NARWA, 2009)

For any $\bar{\rho} \in (L^1 \cap BV)(\mathbb{R}; [0, R_*])$, the Cauchy problem

$$\begin{cases}
\partial_t \rho + \partial_x (\rho v(\rho)) = 0 \\
\rho(0, x) = \bar{\rho}(x)
\end{cases}$$

admits a nonclassical weak solution $\rho = \rho(t, x)$ defined for all $t \in \mathbb{R}_+$. Moreover:

$$TV(\rho(t)) \leq W \cdot TV(\bar{\rho})$$

$$\rho(t, x) \leq \max \{\|\bar{\rho}\|_{L^\infty}, R_*\}$$

$$\bar{\rho}(\mathbb{R}) \subseteq [0, R] \quad TV(\bar{\rho}) < \Delta s \quad \Rightarrow \quad \rho(t, x) \in [0, R] \text{ and is a classical solution.}$$

weak solution:

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} (\rho \partial_t \varphi + q(\rho) \partial_x \varphi) \, dx \, dt + \int_{\mathbb{R}} \bar{\rho} \varphi(0, x) \, dx = 0$$

$$q(\rho) = \rho v(\rho) \quad \forall \varphi \in C_c^1(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$$
Theorem (Colombo & Rosini, NARWA, 2009)

For any \( \bar{\rho} \in (L^1 \cap BV)(\mathbb{R}; [0, R_\ast]) \), the Cauchy problem

\[
\begin{aligned}
\partial_t \rho + \partial_x (\rho v(\rho)) &= 0 \\
\rho(0, x) &= \bar{\rho}(x)
\end{aligned}
\]

admits a nonclassical weak solution \( \rho = \rho(t, x) \) defined for all \( t \in \mathbb{R}_+ \).

Moreover:

\[
\begin{align*}
TV(\rho(t)) &\leq W \cdot TV(\bar{\rho}) \\
\rho(t, x) &\leq \max\{\|\bar{\rho}\|_{L^\infty}, R^*_T\} \\
\bar{\rho}(\mathbb{R}) &\subseteq [0, R] \\
TV(\bar{\rho}) &< \Delta s \quad \Rightarrow \quad \rho(t, x) \in [0, R] \text{ and is a classical solution.}
\end{align*}
\]

Proof.

Wave–front tracking + \( TV_w \) + Helly’s Theorem
Theorem (Colombo & Rosini, 2008)
Not $L^1$-continuity in $[0, R_*]^2$.

Proof.

\[
\rho(0, x) = \begin{cases} 
0 & x \in ]-\infty, 0[ \\
R & x \in [0, +\infty[ 
\end{cases}
\]

\[
\rho_n(0, x) = \begin{cases} 
0 & x \in ]-\infty, 0[ \\
R + 1/n & x \in [0, 1] \\
R & x \in ]1, +\infty[ 
\end{cases}
\]

\[
\lim_{n \to \infty} \|\rho(0) - \rho_n(0)\|_{L^1(\mathbb{R}; [0, R_*])} = 0
\]

\[
\lim_{n \to \infty} \|\rho(t) - \rho_n(t)\|_{L^1(\mathbb{R}; [0, R_*])} \neq 0
\]
Criticisms

- 1D
- lack of continuous dependence
1D

(Wave Front Tracking and the theory for nonclassical shocks are only 1D)

lack of continuous dependence (because of $s$ and $\Delta s$)
Positive Aspects

D. Helbing, A. Johansson, H. Z. Al-Abideen
Dynamics of crowd disasters: An empirical study.
Physical Review E, 2007
C. Chalons.
Numerical approximation of a macroscopic model of pedestrian flows.
*SIAM Journal on Scientific Computing, 2005.*

It devised an efficient numerical scheme to approximate the solutions of our model, making it practically usable.
Citation of our model

D. Amadori and M. Di Francesco.
The one-dimensional Hughes model for pedestrian flow: Riemann-type solutions.

C. Appert-Rolland, P. Degond, and S. Motsch.
Two-way multi-lane traffic model for pedestrians in corridors.
*Networks and Heterogeneous Media*, 2011.

MAM Azahar, M.S. Sunar, A. Bade, and D. Daman.
Crowd simulation for ancient Malacca virtual walkthrough.

N. Bellomo and C. Dogbe.
On the modeling of traffic and crowds: a survey of models, speculations, and perspectives.

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Étude mathématique et numérique d’équations hyperboliques non-linéaires: couplage de modèles et chocs non classiques.

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C. Chalons.
Numerical approximation of a macroscopic model of pedestrian flows.

C. Chalons.
Transport-equilibrium schemes for pedestrian flows with nonclassical shocks.

Study on evacuation behaviors at t-shaped intersection by force-driving cellular automata model.
Citation of our model

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*ESAIM COCV*, 2011.

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B. Piccoli and A. Tosin.
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Cellular automaton simulation of counter flow with paired pedestrians.

Study on mechanics of crowd jam based on the cusp-catastrophe model.

Modeling crowd evacuation of a building based on seven methodological approaches.
Introduction to Vehicular Traffic

Mathematics

Applications to LWR

Numerical Examples

Crowd Accidents

The Model

Corridor with One Exit

Corridor with Two Exits
Corridor with one exit

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho v(\rho)) &= 0 \\
\rho(0, x) &= \rho_o(x) \\
q(\rho(t, L)) &\leq p(\rho(t, L))
\end{align*}
\]
Corridor with one exit

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho v(\rho)) &= 0 \\
\rho(0, x) &= \rho_o(x) \\
q(\rho(t, L)) &\leq p(\rho(t, L))
\end{align*}
\]

Riemann Problems and Wave Front Tracking
The panic arises when:

- the amount of people is large;
- the door is small;
- the initial distance of the people from the door is small.
\[
\begin{align*}
\frac{\partial_t \rho}{\partial_t \rho} + \frac{\partial_x q(\rho)}{\partial_x q(\rho)} &= 0 \\
\rho(0, x) &= \bar{\rho} \cdot \chi_{[a,b]}(x) \\
q(\rho(t, O)) &\leq p_O \\
q(\rho(t, L)) &\leq \rho(\rho(t, L))
\end{align*}
\]

\[ (t, x) \in ]0, +\infty[ \times ]-\infty, L[ \]
\[ x \in ]-\infty, L[ \]
\[ t \in ]0, +\infty[ \]
Braess’ paradox

\[ t_H - t_F \approx \frac{p_M}{p_m} > 1 \]
Corridor with one exit

One exit

One exit and one door
Braess’ paradox
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8 Corridor with Two Exits
Corridor with two exits

People evacuating a corridor $[-1, 1]$ with two exits in $x = \pm 1$

![Diagram of a corridor with two exits]

conservation law

$$\partial_t \rho - \partial_x \left( \rho v(\rho) \frac{\varphi_x}{|\varphi_x|} \right) = 0$$

eikonal equation

$$|\partial_x \varphi| = c(\rho)$$

Dirichlet boundary conditions

$$\rho(t, \pm 1) = \varphi(t, \pm 1) = 0$$

initial condition

$$\rho(0, x) = \begin{cases} 
\rho_L & x \in [-1, 0[
\rho_R & x \in ]0, 1[
\end{cases}$$

c: $\mathbb{R} \rightarrow \mathbb{R}$ is the cost function
People evacuating a corridor \([-1, 1]\) with two exits in \(x = \pm 1\)

\[
\begin{align*}
\text{corridor with two exits} \\
\text{conservation law} & \quad \partial_t \rho + \partial_x F(t, x, \rho) = 0 \\
\text{eikonal equation} & \quad \int_{-1}^{\xi(t)} c(\rho(t, y)) \, dy = \int_{\xi(t)}^{1} c(\rho(t, y)) \, dy \\
\text{Dirichlet boundary conditions} & \quad \rho(t, \pm 1) = \varphi(t, \pm 1) = 0 \\
\text{initial condition} & \quad \rho(0, x) = \begin{cases} 
\rho_L & x \in ]-1, 0[ \\
\rho_R & x \in ]0, 1[ 
\end{cases} \\
\text{c: } \mathbb{R} \to \mathbb{R} \text{ is the cost function} & \quad F(t, x, \rho) = \text{sgn}(x - \xi(t)) f(\rho)
\end{align*}
\]
Corridor with two exits

**Definition (Goatin–Nader–Rosini ’12)**

\( \rho \in C^0(\mathbb{R}^+; BV \cap L^1(\Omega)) \) is an entropy weak solution if

\[
\forall k \in [0, 1] \quad 0 \leq \psi \in C^\infty_c
\]

\[
\int_0^{+\infty} \int_{-1}^1 (|\rho - k| \partial_t + \Phi(t, x, \rho, k) \partial_x) \psi \, dx \, dt + \int_{-1}^1 |\rho_0(x) - k| \psi(0, x) \, dx \\
+ 2 \int_0^{+\infty} f(k) \psi(t, \xi(t)) \, dt + \text{sgn}(k) \int_0^{+\infty} (f(\rho(t, 1-)) - f(k)) \psi(t, 1) \, dt \\
+ \text{sgn}(k) \int_0^{+\infty} (f(\rho(t, -1+)) - f(k)) \psi(t, -1) \, dt \geq 0
\]

\[
\Phi(t, x, \rho, k) = \text{sgn}(\rho - k)(F(t, x, \rho) - F(t, x, k))
\]
Cost function

\[ c(\rho) = \frac{1}{v(\rho)} \]

D. Amadori and M. Di Francesco.
The one-dimensional Hughes model for pedestrian flow: Riemann-type solutions.

On the Hughes’ model for pedestrian flow: the one-dimensional case.

R. L. Hughes.
A continuum theory for the flow of pedestrians.

R. L. Hughes.
The flow of human crowds.
Cost function

\[ c(\rho) = \frac{1}{v(\rho)} \]

We assume

\[ c \in C^0([0, 1]; [1, +\infty[), \; c(0) = 1 \; \text{and} \; c' \geq 0 \]

and optimize w.r.t. \( c \) that we interpret as the strategy used by or imposed to the pedestrians to reach the exits.

D. Amadori and M. Di Francesco.
The one-dimensional Hughes model for pedestrian flow: Riemann-type solutions.

On the Hughes’ model for pedestrian flow: the one-dimensional case.

R. L. Hughes.
A continuum theory for the flow of pedestrians.

R. L. Hughes.
The flow of human crowds.
Cost function

Proposition (Goatin–Nader–Rosini ’12)

\[ c_0(\rho) = \begin{cases} 
1 & \rho < \frac{1}{2} \\
2\rho & \rho \geq \frac{1}{2}
\end{cases} \]

optimize the evacuation for \( \rho_L, \rho_R > \frac{1}{2} \) and

\[ T_{\text{exit}}^{c_0} = \begin{cases} 
\frac{1}{1 - \rho_L} & \rho_R \leq \rho_L \leq \frac{1}{2} \\
\frac{1}{1 - \rho_R} & \rho_L \leq \rho_R \leq \frac{1}{2} \\
1 + 2\rho_L & \rho_R \leq \frac{1}{2} \leq \rho_L \\
1 + 2\rho_R & \rho_L \leq \frac{1}{2} \leq \rho_R \\
2(\rho_L + \rho_R) & \frac{1}{2} \leq \rho_L, \rho_R
\end{cases} \]

Does not exist any cost that optimize the evacuation for any initial data.
Numerical example (two exits)

\[
\begin{align*}
\rho_L &= \frac{18}{20} \\
\rho_R &= \frac{11}{20}
\end{align*}
\]
Numerical example (two exits)

\[
\rho_L = \frac{18}{20} \quad \rho_R = \frac{11}{20}
\]
Numerical example (two exits)

Figure: Case $c(\rho) = 1/\nu(\rho)$: We can observe positive densities appearing on the right of $x = \xi(t)$, representing people changing advise and inverting their route. The numerically computed exit time is $T_{exit} = 2.542$. 
Figure: Case \( c(\rho) = c_0(\rho) \): Oscillations have disappeared and \( \rho(t, \xi(t)\pm) \equiv 0 \). The numerically computed exit time has improved to \( T_{exit} = 2.474 \).
Numerical example (two exits)

**Figure:** Case $c(\rho) = 1$: This choice corresponds to a panicking crowd: people are moving towards the closer exit regardless of the densities. The numerically computed exit time has increased to $T_{\text{exit}} = 2.572$. 
THANK YOU FOR YOUR ATTENTION