Loss of strict hyperbolicity in Riemann solutions for vertical three-phase flow in porous media

Panters Rodriguez Bermudez
Dan Marchesin

Instituto Nacional de Matemática Pura e Aplicada
Rio de Janeiro, Brazil.

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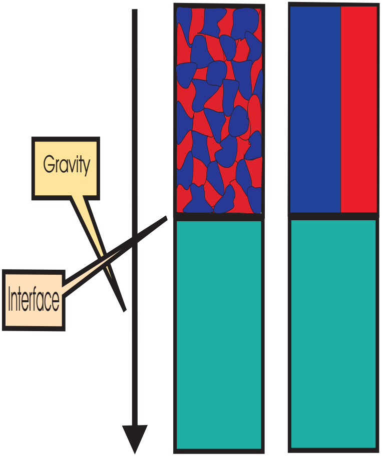
Historical Review

- The horizontal two-phase flow injection problem was solved by Buckley and Leverett in 1942.
- The B-L equation for two-phase flow with gravity is solved through Oleinik's construction, (*ex.*, Proskurowski (1981)).
- Isaacson, Marchesin, Plohr, Temple, Paes Leme, Seabra, De Souza, Furtado, etc, contributed to solve the R-P for immiscible horizontal three-phase flow.
- We present a class of Riemann solutions for immiscible three-phase flow with gravity.

Physical Problem and Simplifications

- The porosity ϕ and the absolute permeability of the rock k are constant.
- The fluids are immiscible and there is no mass interchange between phases.
- The flow occurs uniformly in the vertical direction filling the entire porous medium.
- The fluids are incompressible.
- There are no sources or sinks.



Mass Conservation and Darcy's Law.

Conservation of mass

$$\frac{\partial}{\partial t}\phi s_i + \frac{\partial}{\partial x}\mathbf{u}_i = 0 \quad i = 1, 2, 3, \tag{1}$$

where s_i denotes saturation and $\mathbf{u}_i$ is the velocity of each phase.

Darcy's law

$$\mathbf{u}_i = -k\frac{k_{r,i}}{\mu_i}\left(\cancel{\frac{\partial p}{\partial x}} - \rho_i g\right) \quad i = 1, 2, 3, \tag{2}$$

p is the pressure, $k_{r,i}$ is the relative permeability, μ_i is the viscosity and ρ_i is the density for each phase i.

Deriving the System of Conservation Laws.

$$\mathbf{u}_i = \mathbf{u} f_i + k \Lambda_i \sum_{j \neq i} f_j \rho_{ij} g, \quad i = 1, 2, 3, \tag{3}$$

where $\rho_{ij} = \rho_i - \rho_j$.

Mobilities and fractional flow functions

$$\Lambda_i = k_{r,i}/\mu_i, \qquad f_i = \frac{\Lambda_i}{\sum_{j=1}^{3} \Lambda_j}, \quad i = 1, 2, 3.$$

We use the quadratic Corey permeability model

$$k_{r,i}(s_i) = s_i^2. \tag{4}$$

System of Conservation Laws

Dimensionless equations for vertical three-phase flow

$$\frac{\partial s_i}{\partial t} + \frac{\partial}{\partial x}\big(\alpha f_i(s_1, s_2) + G_i(s_1, s_2)\big) = 0, \quad i = 1, 2, 3, \tag{5}$$

where

$$\alpha = \frac{\mathbf{u}}{\mathrm{u}_{\mathrm{ref}}} = \frac{\mathbf{u}\ \mu_{\mathrm{ref}}}{K_{\mathrm{ref}}\ \rho_{\mathrm{ref}}\ g}$$

is the convection/gravity ratio (later we will set $\alpha = 0$).
Since $\sum s_i = 1$ there is a redundant equation.

Gravitational Fluxes

$$\begin{aligned} G_1 &= k\Lambda_1\big((1 - f_1)\rho_{13} + f_2\rho_{32}\big), \\ G_2 &= k\Lambda_2\big((1 - f_2)\rho_{21} + f_3\rho_{13}\big), \\ G_3 &= k\Lambda_3\big((1 - f_3)\rho_{32} + f_1\rho_{21}\big). \end{aligned}$$

Umbilic and quasi-umbilic points

Umbilic point

A coinc. point S^* is an umbilic point of $S_t + [F(S)]_x = 0$ if

(H1) $dF(S^*)$ is diagonalizable.

(H2) There is a neighborhood $\mathcal{V}$ of S^* such that $dF(S)$ has distinct eigenvalues for all $S \in \mathcal{V} - S^*$.

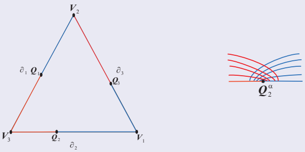
Quasi-umbilic point

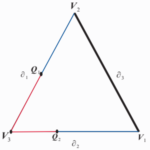
A coincidence point S^* is a quasi-umbilic point $\Leftrightarrow$ (H2) holds but (H1) fails.

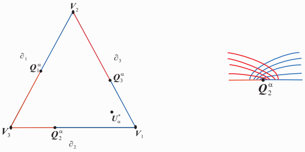
Coincidence diagonalization curve

A coincidence diagonalization curve is a curve of coincidence points along which condition (H2) fails but condition (H1) holds.

Coincidence Points







$\alpha = 0,\ \rho_1 > \rho_2 > \rho_3$

V_i are umbilic points, Q_i are quasi-umbilic points.

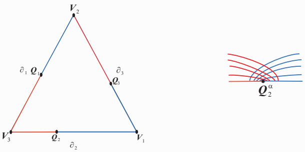
$\alpha = 0,\ \rho_1 = \rho_2 \neq \rho_3$

V_3 is an umbilic point, Q_1, Q_2 are quasi-umbilic points, $\partial 3$ is a coincidence diagonalization line.

$\alpha \neq 0$ small, $\rho_1 > \rho_2 > \rho_3$

V_i and U_α^* are umbilic points, Q_i^α are quasi-umbilic points.

Coincidence Points



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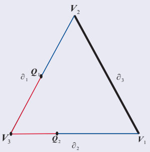

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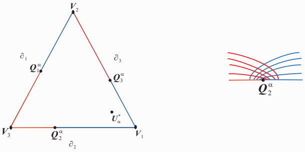

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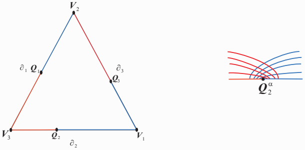

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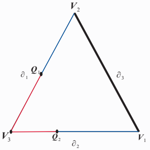

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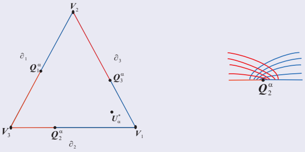

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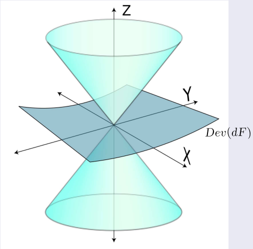
Schaeffer-Shearer Cone and Deviator operator

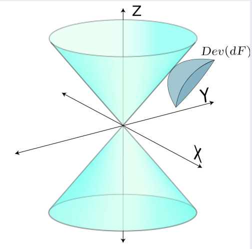
$$Dev(dF(s_1,s_2)) = dF - \frac{1}{2}tr(dF)I = \begin{bmatrix} X & Y+Z \\ Y-Z & X \end{bmatrix} \qquad (6)$$

$$(s_1,s_2) \longrightarrow (X,Y,Z)$$

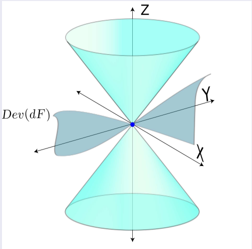
Umbilic point



Quasi-umbilic point



Coinc. diag. line



Wave groups and Riemann solutions

- The Riemann solutions consist of two wave groups separated by a constant state.
- Each wave group consists on a sequence of rarefaction waves and adjacent shock waves.
- Shock waves must satisfy the Generalized Lax conditions.
- No 1-wave is preceded by a 2-wave.
- Parameterized by wave curves (Liu).

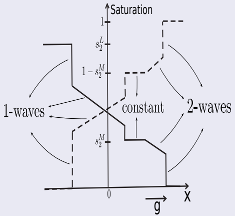


Figure: Wave groups in Riemann solutions

Generalized Lax Shock Waves and TSR

Generalized Lax 1-shock and 2-shock waves

$$\text{1-shock:} \quad \lambda_1(S^+) \leq \sigma \leq \lambda_1(S^-), \quad \text{and} \quad \sigma < \lambda_2(S^+). \tag{7}$$
$$\text{2-shock:} \quad \lambda_2(S^+) \leq \sigma \leq \lambda_2(S^-), \quad \text{and} \quad \lambda_1(S^-) < \sigma. \tag{8}$$

At most one equality in (7), (8).

Triple Shock Rule (TSR)

Assume that the states S_1, S_2 and S_3 satisfy $S_2 \in \mathcal{H}(S_1)$, $S_3 \in \mathcal{H}(S_2)$ and $\sigma(S_1, S_2) = \sigma(S_2, S_3)$ then $S_3 \in \mathcal{H}(S_1)$ and $\sigma(S_1, S_3) = \sigma(S_1, S_2) = \sigma(S_2, S_3)$.

Some bifurcation manifolds

i-Inflection manifold

States S such that $\nabla\lambda_i(S)\cdot r_i(S) = 0$

i-Boundary extension manifold

States S for which exist S' such that

$$S \in \mathcal{H}(S') \quad \text{with } S' \text{ on the boundary and} \quad \lambda_i(S) = \sigma(S, S').$$

(i, j)-Double contact manifold

States S for which exist states S' such that

$$S' \in \mathcal{H}(S) \quad \text{with} \quad \lambda_i(S) = \sigma(S, S') = \lambda_j(S'),$$

Simplified cases

Buoyancy driven flow $\alpha = 0$

Three distinct densities

$\rho_3 < \rho_2 < \rho_1$

EHD

$\rho_3 < \rho_2 = \rho_1$

ELD

$\rho_3 = \rho_2 < \rho_1$

- Solutions for general buoyancy-driven problem (three fluids with distinct densities) are “superpositions” of the solutions of simplified cases EHD and ELD.

Rarefaction curves for EHD: $\alpha = 0$, $\rho_1 = \rho_2 > \rho_3$

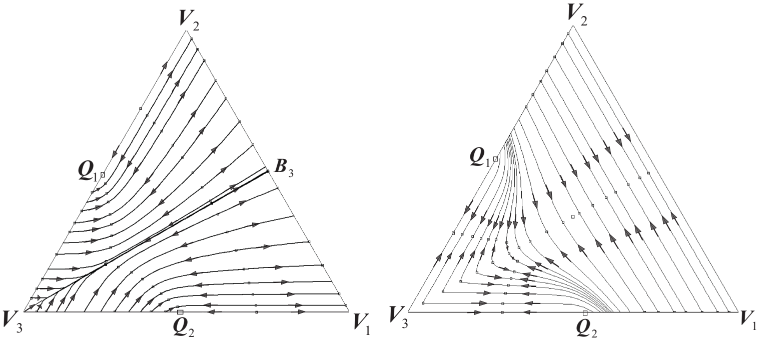
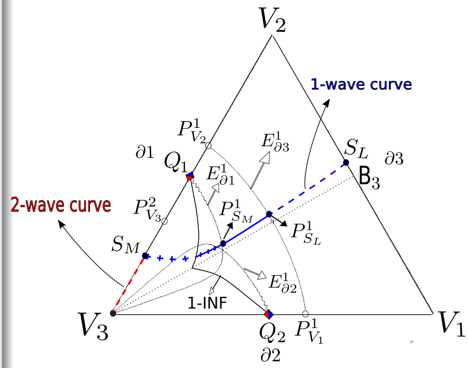


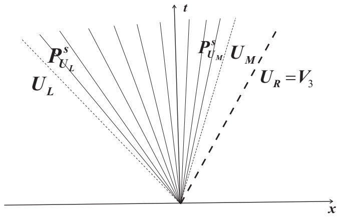
Figure: Integral curves for both families. The arrows indicate increasing characteristic speed. The dots represent the Inflection curves.

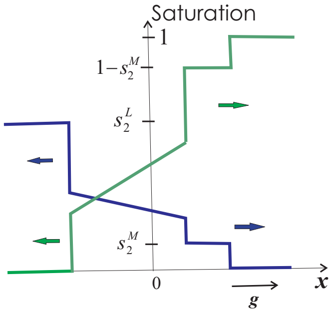
Wave curve for EHD. Left state in $\partial 3$, right state V_3.

- $P^1_{S_L}$ satisfies $\sigma(S_L, P^1_{S_L}) = \lambda_1(P^1_{S_L}) < 0$.
- $P^1_{S_M}$ satisfies $\sigma(P^1_{S_M}, S_M) = \lambda_1(P^1_{S_M}) > 0$.
- $\sigma(P^1_{S_M}, S_M) < \sigma(S_M, V_3)$ (speed compatibility!!!)
- Dominant phase in S_L remains "dominant" along the solution: The triangles V_3–B_3–V_2 and V_3–B_3–V_1 are invariant in the solution.

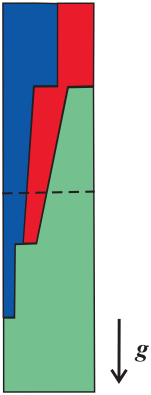


Example of Riemann solution in EHD





$\rho = \rho > \rho$

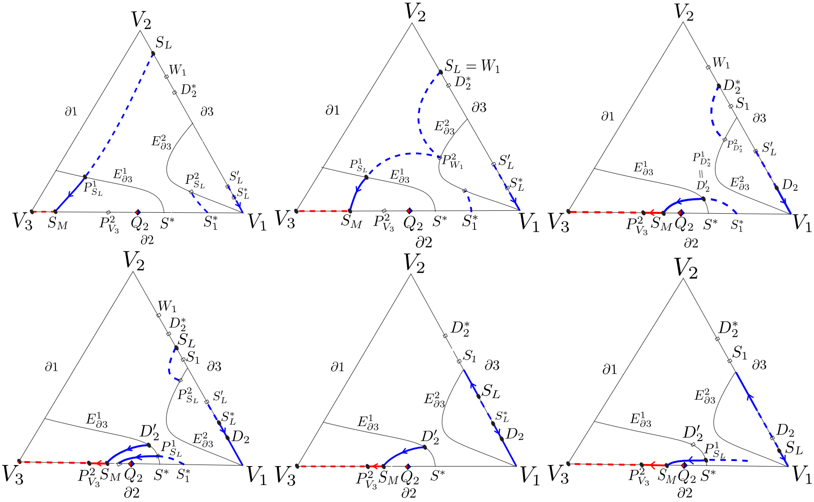


blue fluid dominant

$\Updownarrow$

$s/\mu > s/\mu$

Wave curves in ELD: $\alpha = 0$, $\rho_3 = \rho_2 < \rho_1$



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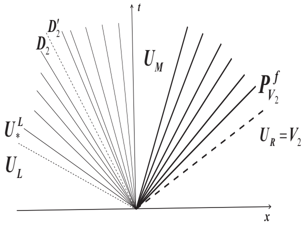
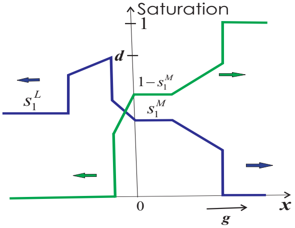
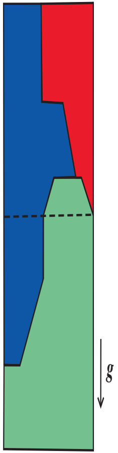
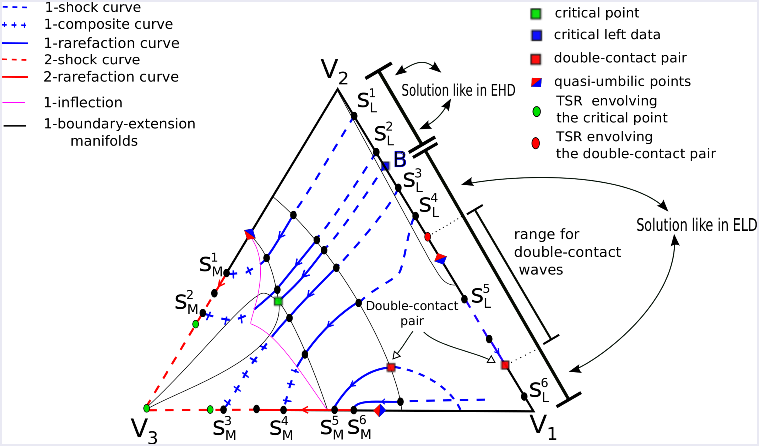




$\rho > \rho = \rho$



RP for three-distinct densities



THANKS