Steady and self similar inviscid flow

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Two-dimensional conservation laws

Consider $U : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^m$, such that

$$U_t + f^x(U)_x + f^y(U)_y = 0.$$

Isentropic compressible Euler equations with density ρ, horizontal velocity u, vertical velocity v, and pressure p are

$$\begin{bmatrix} \rho \\ \rho u \\ \rho v \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \end{bmatrix}_x + \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \end{bmatrix}_y = 0.$$

Pressure law $p := p(\rho)$ satisfies $p_\rho := c^2 > 0$, $c_\rho > -1$.

Steady and self similar reduction

Many experiments, e.g. regular reflection (four shocks meeting at a point), and Mach reflection (two shocks and a contact), correspond to steady flow such that, to first order, is constant along rays emanating from a distinguished point. Therefore,

$$U(t,x,y) = U(\phi), \qquad \phi = \angle(x,y) \in [0, 2\pi).$$

For literature on regular reflection, see [Chen-Feldman 2010, Elling-Liu 2008, Čanić-Keyfitz-Lieberman 2000, Zheng 2006, Henderson-Menikoff 1998, Elling 2009, Elling 2009, Elling 2010]. For literature on Mach reflection, see [Ben-Dor 1992, Ben-Dor 2006, Hornung 1986, Hunter-Tesdall 2002, Vasilev-Kraiko 1999, Skews 1997].

Steady and self similar reduction

Not all configurations of waves are possible - e.g. triple points (three shocks meeting at a point) are not possible [Neumann 1943, Courant-Friedrichs 1948, Henderson-Menikoff 1998, Serre 2007]. What configurations are possible?

Previous results on multi-dimensional Riemann problems

A related question is which function space to consider. We consider solutions that are small L^∞ perturbations of a constant supersonic state, and are able to prove that such solutions are necessarily BV. This is crucial because it is known [Rauch 1986] that BV is not well suited to multi-dimensional conservation laws, in contrast to the satisfactory theory of well posedness for the Cauchy problem for functions of small BV norm for 1-dimensional strictly hyperbolic conservation laws [Glimm 1965, Glimm-Lax 1970, Bianchini-Bressan 2001].

Previous uniqueness results

Our results show uniqueness in L^∞ of self similar (i.e., functions of x/t only) solutions to 1-dimensional strictly hyperbolic conservation laws, generalizing a result of [Heibig 1990] which required genuine nonlinearity. Though uniqueness does not hold backward in time, we are still able to prove small L^∞ solutions are BV. (For related uniqueness results, see [Dafermos 2008, Bressan-Goatin 1999, Bressan-LeFloch 1997, Bressan-Crasta-Piccoli 2000, Liu-Yang 1999, Oleinik 1959, Kruzkov 1970, Smoller 1969]).

Summary of main result

We have the following description of steady and self-similar Euler flows U that are sufficiently L^∞-close to a constant background state $\overline{U} = (\rho, Mc, 0)$ with Mach number $M > 1$ (supersonic), defining *Mach angle* $\mu = \arcsin \frac{1}{M}$:

1. they are necessarily BV,
2. they are constant outside six narrow sectors whose center lines are $(1:0)$, $(\cos\mu : \sin\mu)$, $(\cos\mu : -\sin\mu)$,
3. in the $(1:0)$ forward and backward sectors U is constant on each side of a single contact discontinuity (which may vanish),

Summary of main result

4. in the forward $(\cos\mu : \pm\sin\mu)$ sectors U is constant on each side of a single shock or single rarefaction wave (which may vanish),
5. in the backward $(\cos\mu : \pm\sin\mu)$ sectors U can have an infinite or any finite number of shocks and compression waves, but
5a. two consecutive compression waves with a gap are not possible, and
5b. the shock set (on the unit circle) is discrete, with each shock having constant neighborhoods on each side whose size is lower-bounded proportionally to the shock strength.

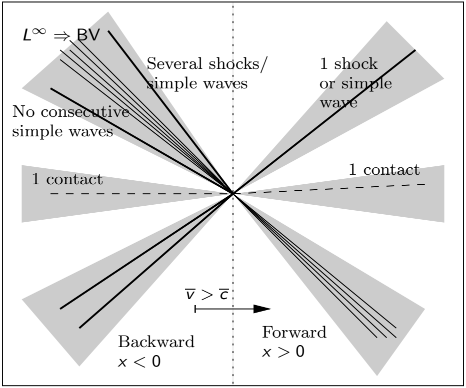


Figure: U must be constant outside narrow sectors specified by eigenvalues evaluated at $\overline{U}$. Linearly degenerate sectors: at most one contact discontinuity. Genuinely nonlinear forward sectors: at most one shock or simple wave. Genuinely nonlinear backward sectors: infinitely many waves possible, but no consecutive simple waves. Here we have taken the background state to have horizontal velocity $(\overline{v}, 0)$ and sound speed $\overline{c}$.

Change to V

We assume that $f^x_U(\overline{U})$ is non-singular, which we can do in the case of the Euler equations without loss of generality by picking our background state $\overline{U}$ to have velocity $\overline{v}$ horizontal and supersonic. In this case f^x is a local diffeomorphism which maps the small neighborhood of $\overline{U}$ under consideration to $\mathcal{P}_\epsilon := \left\{ V \in \mathbb{R}^m \middle| ||V - \overline{V}||_{L^\infty} \leq \epsilon \right\}$ with $V := f^x(U), f := f^y \circ (f^x)^{-1}$. One easily verifies we obtain a new entropy-entropy flux pair (e, q) with e uniformly convex. The weak form then is, with $\xi := y/x$,

$$\begin{cases} \big(f(V) - \xi V\big)_\xi + V = 0 & \\ \big(q(V) - \xi e(V)\big)_\xi + e(V) \leq 0 & : \mathrm{Supp}(\Phi) \subset \{x > 0\} \\ \big(q(V) - \xi e(V)\big)_\xi + e(V) \geq 0 & : \mathrm{Supp}(\Phi) \subset \{x < 0\} \end{cases} .$$

Note that all our results will apply to self similar (that is, functions of x/t) solutions to one-dimensional conservation laws, as this is the appropriate weak form for that problem.

Strict hyperbolicity

We assume that the system is strictly hyperbolic, that is the matrix $f_V(V)$ has m distinct, real eigenvalues $\{\lambda^\alpha(V)\}_{\alpha=1}^m$ for all $V \in \mathcal{P}_\epsilon$. These eigenvalues are smooth functions of V, and we have smooth right and left eigenvectors of $f_V(V)$ satisfying the normalization

$$l^\alpha(V) r^\beta(V) = \delta_{\alpha\beta}.$$

Moreover, we assume that each field is either genuinely nonlinear, i.e.,

$$\lambda^\alpha_V(V) r^\alpha(V) > 0 \qquad \forall V \in \mathcal{P}_\epsilon;$$

or linearly degenerate, i.e.,

$$\lambda^\alpha_V(V) r^\alpha(V) \equiv 0 \qquad \forall V \in \mathcal{P}_\epsilon.$$

Averaged matrix

By defining the averaged matrix

$$\hat{A}(V^{\pm}) := \int_0^1 f_V(sV^+ + (1-s)V^-)ds,$$

and with the proper choice of version of V, the weak form is equivalent to

$$\left(\hat{A}\big(V(\xi_1), V(\xi_2)\big) - \xi_1 I\right)\big(V(\xi_2) - V(\xi_1)\big) = \int_{\xi_1}^{\xi_2} V(\xi_2) - V(\eta)d\eta$$

for all ξ_1, ξ_2. By smoothness of $f_V(V)$, $\hat{A}$ has m distinct real eigenvalues $\left\{\hat{\lambda}^{\alpha}(V^{\pm})\right\}$ and eigenvectors $\hat{l}^{\alpha}(V^{\pm})$ and $\hat{r}(V^{\pm})$ satisfying the same normalization.

Left and right sequences

Since we do not assume $V \in BV$, V may not have well defined left or right limits at any point ξ. Consider a pair of sequences $\left\{\tilde{\xi}_k^-\right\}, \left\{\tilde{\xi}_k^+\right\}$ both converging to ξ, with $\tilde{\xi}_k^- < \tilde{\xi}_k^+$. Since V has values in the compact set $\mathcal{P}_\epsilon$, we may choose subsequences $\{\xi_k^-\}$ and $\{\xi_k^+\}$ such that $\{V(\xi_k^\pm)\} \to V^\pm$. Assuming no ambiguity in which sequences are meant, in this context we define for any function g

$$[g(V)] := g(V^+) - g(V^-).$$

If we define

$$J(g(V);\xi) := \sup|[g(V)]|,$$

where the sup is over all such sequences $\{\xi_k^\pm\}$, we have that $J(g(V);\xi) = 0$ if and only if $g \circ V$ is continuous at ξ.

Rankine-Hugoniot condition

It follows that

$$\Big(\hat{A}\big(V(\xi_k^+), V(\xi_k^-)\big) - \xi_k^- I\Big)\big(V(\xi_k^+) - V(\xi_k^-)\big) = \mathcal{O}(|\xi_k^+ - \xi_k^-|)$$

and in the limit $k \to \infty$ we have

$$(\hat{A}(V^{\pm}) - \xi I)[V] = 0.$$

Therefore, $[V] \parallel \hat{r}^{\alpha}(V^{\pm})$ and $\xi = \hat{\lambda}^{\alpha}(V^{\pm})$, which is the usual Rankine-Hugoniot condition for shocks. Therefore, we can still apply it even when the solution is not smooth on either side of ξ; more specifically when it does not even have left and right limits.

$(f_V - \xi I)$ nonsingular $\implies$ V constant

If V is differentiable at ξ, then we would have

$$(f_V(V(\xi)) - \xi I)V_\xi(\xi) = 0.$$

Therefore if $(f_V(V(\xi)) - \xi I)$ is nonsingular; that is, if ξ is not an eigenvalue, this would imply $V_\xi = 0$. The bulk of this study is to make this argument work without assuming any differentiability properties of V, while classifying where various features can occur depending on the spectrum of $f_V(\overline{V})$.

Theorem 1

Theorem

Suppose V is continuous on an interval $I = (\xi_1, \xi_2)$ and that ξ is not an eigenvalue of $f_V(V(\xi))$ for any $\xi \in I$. Then V is constant on I.

Proof.

Fix some $\xi \in I$. We claim that V must be Lipschitz at ξ. Suppose not. Then we can choose a sequence $\{h_n\} \to 0$ (with $h_n \neq 0$) such that

$$0 < \left| \frac{V(\xi + h_n) - V(\xi)}{h_n} \right| \nearrow \infty.$$

Divide both sides by $|V(\xi + h_n) - V(\xi)|$ to obtain

Proof of Theorem 1

Proof ctd.

$$\left(\hat{A}\big(V(\xi), V(\xi + h_n)\big) - \xi I\right) \frac{V(\xi + h_n) - V(\xi)}{|V(\xi + h_n) - V(\xi)|}$$
$$= \frac{1}{|V(\xi + h_n) - V(\xi)|} \mathcal{O}(|h_n|) = o(1).$$

$f_V(V(\xi)) - \xi I$ regular $\implies$ for h_n sufficiently small
$\hat{A}\big(V(\xi), V(\xi + h_n)\big) - \xi I$ uniformly regular. $\Rightarrow\Leftarrow$
Therefore, V must be Lipschitz on $I \implies V$ differentiable a.e.
with zero derivative $\implies V$ constant on I. □

Theorem 2

Theorem

Consider an interval $I = (\xi_1, \xi_2)$*. There is a* $\delta_s = \delta_s(\epsilon) > 0$*, with*

$$\delta_s \downarrow 0 \quad as \quad \epsilon \downarrow 0,$$

so that

$$\forall \alpha \in \{1, ..., m\} \forall x \in I : |\lambda^\alpha(V(\xi)) - \xi| > \delta_s \tag{1}$$

implies V is constant on I. [Here we do not require continuity, but a stronger bound on the spectrum.]

Sectors

Theorem 2 allows us to construct $2m$ thin sectors I^α (one forward and one backward for each $\alpha = 1..m$) around each eigenvalue of $f_V(\overline{V})$ outside which V must be constant. In a linearly degenerate sector, we expect contact discontinuities, and in genuinely nonlinear sectors we expect simple waves and/or shocks.

Degenerate sectors

Theorem

There can be at most one contact discontinuity in a degenerate (forward or backward) sector I^α.

Sketch of proof.

Claim: $\xi \mapsto \lambda^\alpha(V(\xi))$ is continuous in the α-sector - only possible jumps in V at ξ are between two states on a contact curve, on which $\lambda^\alpha(V^\pm) = \xi$. $\implies \{\xi \in I^\alpha | \lambda^\alpha(V(\xi)) = \xi\}$ is closed $\implies$ its complement is countable union of open intervals. Easy to see its complement could be at most two open intervals, so this resonant set is at most a single closed interval.

Sketch of proof ctd.

If we could take a derivative, we'd have a contradiction, since the strong form differentiated is

$$(f_V - \xi I)V_\xi = 0,$$

and so the only possibility in I^α sector is $V_\xi \parallel r^\alpha(V(\xi))$, which contradicts the differentiated form of $\lambda^\alpha(V(\xi)) = \xi$, i.e.

$$\lambda^\alpha_V(V(\xi))V_\xi = 1,$$

since λ^α is linearly degenerate. Clearly we cannot just take a derivative. However, we don't really need a derivative for the contradiction, only a single sequence converging to ξ_0 on which the difference quotients converge. Is this too much to ask?

Degenerate Sectors

Sketch of proof ctd.

Theorem (Saks 1937) For any finite (real valued) function F, the set of points x at which

$$\lim_{h \to 0+} |F(x+h) - F(x)|/h = +\infty$$

is of measure zero.

Apply this to $\xi \mapsto I^{\alpha}(\overline{V})(V(\xi) - V(\xi_0))$.
$\xi \mapsto \hat{I}^{\beta}(V(\xi_0), V(\xi))(V(\xi) - V(\xi_0))$ is Lipschitz at ξ_0 due to separation of the eigenvalues. Together this yields a subsequence on which V has a convergent difference quotient $\Rightarrow\Leftarrow$. Thus $\lambda^{\alpha}(V(\xi)) = \xi$ at at most one point $\implies$ at most one contact in each degenerate sector. □

Shocks

The next important result is that shocks must have a neighborhood on either side on which V is constant. The size of the neighborhood is lower bounded proportional to shock strength. We will need to invoke a Lax condition (which can be derived by using the implicit function theorem to construct the Hugoniot locus, and using the entropy/entropy-flux pair to examine admissibility). For a backward sector, it states that if we have subsequences as before such that $V(\xi_k^{\pm}) \to V^{\pm}$,

$$\lambda(V^-) < \xi < \lambda(V^+).$$

Shocks must have a constant neighborhood

The argument for a backward (that is, $x < 0$) sector is as follows.

Theorem

For any ξ_0 at which a shock occurs, there are $\sigma^+(\xi_0) > \xi_0$ (maximal) and $\sigma^-(\xi_0) < \xi_0$ (minimal) so that V is constant on $(\sigma^-(\xi_0), \xi_0), (\xi_0, \sigma^+(\xi_0))$. Moreover,

$$\sigma^-(\xi_0) \leq \xi_0 - \delta_L J(V;\xi_0),$$
$$\sigma^+(\xi_0) \geq \xi_0 + \delta_L J(V;\xi_0),$$

for some $\delta_L > 0$ independent of V.

Remainder of main result

Once it is known that the shock set is discrete, it can be shown that the sum of the jumps must be finite - so that the jump part of V is of bounded variation. Similarly, the continuous part can also be shown to be of bounded variation.

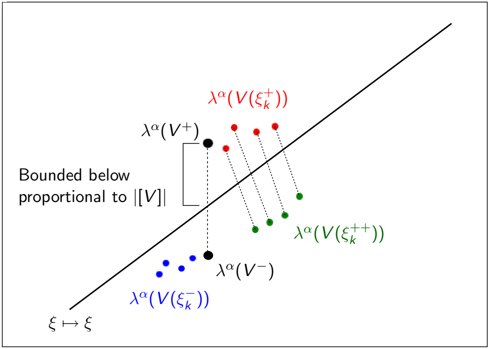
The Lax condition shows that at most one shock or rarefaction can occur in a forward $(x > 0)$ genuinely nonlinear sector. This proves the uniqueness of forward Riemann solutions in L^∞. Uniqueness need not hold backward in time - examples with infinitely many shocks or infinitely many shocks interspersed with compression waves can be constructed in backward sectors, so our result is optimal. However, two simple waves cannot occur consecutively in a backward sector.

Ongoing work

The case of full Euler is non-strictly hyperbolic - the linearly degenerate eigenvalue has multiplicity two, corresponding to shear waves and entropy jumps. The analysis has been extended to the case of eigenvalues with constant multiplicity (hence degenerate) on $\mathcal{P}_\epsilon$. The proof of at most one contact is more complicated in this case, as the Saks theorem is no longer relevant. However, a result [Elling 2011] can give a subsequence on which V is Holder-$1/p_\alpha$ (where p_α is the multiplicity of the eigenvalue), and with more work the single contact result holds true.

Efforts to extend BV regularity to Euler flow with large L^∞ solutions are in progress.

Thank you!!!



Pick $\{\xi_k^\pm\}$ with $\xi_k^+ \searrow \xi_0$. Suppose there is no $\delta_0^+ > 0$ such that $\lambda^\alpha(V(\xi)) > \xi$ for all $\xi \in (\xi_0, \xi_0 + \delta_0^+)$. Pick ξ_k^{++} converging to ξ_0 with $\lambda^\alpha(V(\xi_k^{++})) \leq \xi_k^{++}$ and $\xi_k^+ < \xi_k^{++}$ (take a subsequence of ξ_k^+ if necessary). Suppose $V^{++} \neq V^+$. Then the backward Lax condition, for the pair of sequences $\{\xi_k^+\}, \{\xi_k^{++}\}$, implies $\lambda^\alpha(V^+) < \xi_0 < \lambda^\alpha(V^{++})$, $\Rightarrow\Leftarrow$. If $V^{++} = V^+$, then $\lambda^\alpha(V^{++}) = \lambda^\alpha(V^+) > \xi_0$, $\Rightarrow\Leftarrow$.

Proof of Theorem 3

Sketch of proof.

Therefore, there exists $\delta_0^+ > 0$ such that $\lambda^\alpha(V(\xi)) > \xi$ for $\xi \in (\xi_0, \xi_0 + \delta_0^+)$. Analogously, there exists a $\delta_0^- > 0$ such that $\lambda^\alpha(V(\xi)) < \xi$ for $\xi \in (\xi_0 - \delta_0^-, \xi_0)$.
Suppose there were another discontinuity at $\xi_1 \in (\xi_0, \xi_0 + \delta_0^+)$.
Then, perform the same argument for the shock at ξ_1 to find an $\eta \in (\xi_0 + \delta_0^+) \cap (\xi_1 - \delta_1^-, \xi_1)$. Then,

$$\lambda^\alpha(V(\eta)) \overset{\eta\in(\xi_0,\xi_0+\delta_0^+)}{>} \eta \overset{\eta\in(\xi_1-\delta_1^-,\xi_1)}{>} \lambda^\alpha(V(\eta)),$$

$\Rightarrow\Leftarrow$. Therefore V is continuous on $(\xi_0, \xi_0 + \delta_0^+)$. By definition of δ_0^+ and Theorem 1, V must be constant on $(\xi_0, \xi_0 + \delta_0^+)$. □