Adaptive measure-valued coupling of non-linear hyperbolic PDEs

Florent Renac[1]

Joint work with
Claude Marmignon[1] and Frédéric Coquel[2]

[1]ONERA The French Aerospace Lab (France)
[2]CNRS - CMAP, Ecole Polytechnique (France)

HYP2012
June 25-29, 2012
Padova

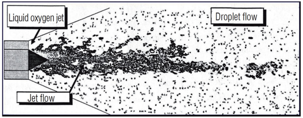
Numerical simulation of systems with multiscale phenomena

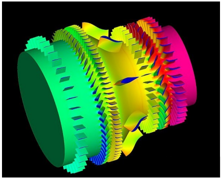
Context:
- complex systems decomposed into sub-systems simulated by specific softwares with different modelings
- need to couple problems corresponding to distinct modeling levels

Examples: multiphase flows, turbulent flows, porous/non-porous media, kinetic/hydrodynamic representations, etc.

Aim:
- simulating the whole system with an efficient and reliable coupling method [Godlewski et al. '04 '05, Ambroso et al. '08]
- coupling hyperbolic systems of conservation laws with non-homogeneous closure laws (i.e. depending on space variables)

Application:
- modelling with adaptive simplified closure laws
- application to incomplete EOS [Menikoff & Plohr '89]





Outline

1. Model problem
   - Euler equations
   - Non-homogeneous equation of state
2. Numerical method
   - Coupling procedure
   - Relaxation method
   - Discretization method
3. Numerical experiments
   - Simplified EOS
   - Space time adaptive EOS
4. Concluding remarks

ONERA
THE FRENCH AEROSPACE LAB

Outline

Euler equations with heterogeneity in space

$$\begin{aligned} \partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}, x) &= 0, \qquad \forall x \in \mathcal{D}(t), \quad t > 0 \\ \mathbf{u}(x, 0) &= \mathbf{u}_0(x), \quad \forall x \in \mathbb{R} \end{aligned}$$

- Time-dependant space domain:

$$\mathcal{D}(t) = \bigcup_{i=1}^{n(t)} (a_{i-1}(t), a_i(t)) \quad \text{with} \quad a_0(t) = -\infty, \quad a_{n(t)}(t) = +\infty, \quad t > 0$$

- Conservative variables:

$$\mathbf{u} = (\rho, \rho u, \rho E)^\top$$

- Heterogeneous physical fluxes in x through pressure definition:

$$\mathbf{f}(\mathbf{u}, x) = \begin{pmatrix} \rho u \\ \rho u^2 + p(\mathbf{u}, x) \\ (\rho E + p(\mathbf{u}, x))\, u \end{pmatrix}$$

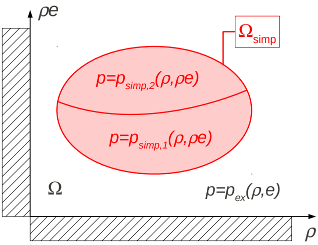
- Set of admissible states:

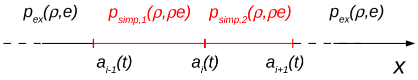
$$\Omega = \left\{ \mathbf{u} \in \mathbb{R}^3 : \ \rho > 0, \ u \in \mathbb{R}, \ e = E - \frac{u^2}{2} > 0 \right\}$$

ONERA THE FRENCH AEROSPACE LAB

Non-homogeneous equation of state (EOS)

$$p(\mathbf{u}, x) = \begin{cases} p_{ex}(\rho, e) & \text{if } \mathbf{u}(x,t) \in \Omega \backslash \Omega_{simp} \\ p_{simp}(\rho, \rho e) & \text{if } \mathbf{u}(x,t) \in \Omega_{simp} \end{cases}$$





EOS:
- exact (but complex) EOS $p_{ex}(\rho, e)$
- incomplete (but simple) EOS $p_{simp}(\rho, \rho e)$

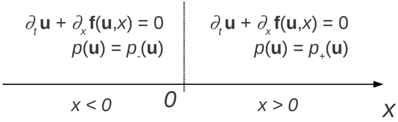
ONERA THE FRENCH AEROSPACE LAB

Outline

Coupling procedure

Consider the model problem

$$p(\mathbf{u},x) = \begin{cases} p_-(\mathbf{u}) & \text{if } x < 0 \\ p_+(\mathbf{u}) & \text{if } x > 0 \end{cases}$$



Coupling prescribed by a vector-valued Dirac measure concentrated at $x = 0$:

$$\begin{aligned} \partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u},x) &= \boldsymbol{\mathcal{M}}\big(\mathbf{u}(0^+,t),\mathbf{u}(0^-,t)\big)\delta_{x=0}, \quad \forall x \in \mathbb{R}, \quad t > 0 \\ \mathbf{u}(x,0) &= \mathbf{u}_0(x), \quad \forall x \in \mathbb{R} \end{aligned}$$

Coupling procedure (cont.)

Coupling condition

$$\mathcal{M}(\mathbf{u}(0^+,t),\mathbf{u}(0^-,t)) = \mathbf{f}\left(\mathbf{u}(0^+,t),0^+\right) - \mathbf{f}\left(\mathbf{u}(0^-,t),0^-\right), \quad \forall t > 0$$

Example

conservative coupling (continuity of physical fluxes):

$$\mathbf{f}\left(\mathbf{u}(0^+,t),0^+\right) = \mathbf{f}\left(\mathbf{u}(0^-,t),0^-\right)$$

thus

$$\mathcal{M} \equiv 0$$

$\mathbf{v}$-state coupling (continuity of physical variables $\mathbf{v} = \gamma^{-1}(\mathbf{u})$):

$$\gamma_+^{-1}(\mathbf{u}(0^+,t)) = \gamma_-^{-1}(\mathbf{u}(0^-,t))$$

e.g. for $\mathbf{v} = (\rho, u, p)^\top$ we get

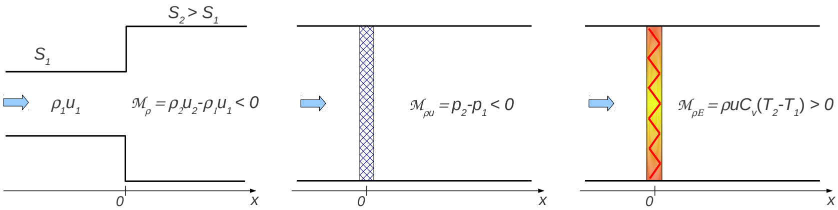
$$\mathcal{M} = (0, 0, \rho e(0^+,t)u(0^+,t) - \rho e(0^-,t)u(0^-,t))^\top$$

[Godlewski et al. '05, Ambroso et al. '08a. '07, Galié '09]

ONERA THE FRENCH AEROSPACE LAB

Non-conservative framework

$$\boldsymbol{\mathcal{M}} = \begin{pmatrix} \mathcal{M}_\rho \\ \mathcal{M}_{\rho \boldsymbol{u}} \\ \mathcal{M}_{\rho \boldsymbol{E}} \end{pmatrix}$$



(a) sudden pipe expansion (b) pressure loss (c) heat transfert

Relaxation method

- Approximate the Euler eq. with a Suliciu's relaxation procedure ($\lambda \gg 1$):

$$\begin{aligned} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}, x) &= \mathcal{M}\delta_{x=0} + \lambda \mathcal{R}(\mathbf{U}, x), \quad \forall x \in \mathbb{R}, \quad t > 0 \\ \mathbf{U}(x, 0) &= \mathbf{U}_0(x), \quad \forall x \in \mathbb{R} \end{aligned}$$

with

$$\mathbf{U} = (\rho, \rho u, \rho E, \rho\pi)^\top$$

and

$$\mathbf{F}(\mathbf{U}, x) = \begin{pmatrix} \rho u \\ \rho u^2 + \pi \\ (\rho E + \pi) u \\ (\rho\pi + a^2) u \end{pmatrix}$$

and

$$\mathcal{R}(\mathbf{U}, x) = \big(0, 0, 0, \rho(p(\mathbf{u}, x) - \pi)\big)^\top$$

- Advantage: the system has 4 LD fields $\{u - a\tau,\ u,\ u,\ u + a\tau\}$
- Whitham condition:

$$a^2 > \max\{-(\partial_\tau p)_s : \text{ for all } \tau, s \text{ under consideration}\}$$

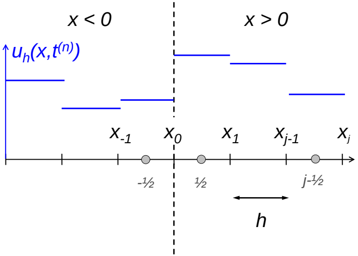
[Jin & Xin '95, Suliciu '98, Ambroso et al. '08b]

Discretization method

Numerical solution

- Let $\bigcup_{j\in\mathbb{Z}} [x_{j-1}, x_j)$ a partition of the domain $\mathbb{R}$ with $x_j = jh$ and $h > 0$
- Look for a piecewise constant solution $\mathbf{U}_h$ at time step $t^{(n)} = n\Delta t$:

$$\mathbf{U}_h(x, t^{(n)}) = \mathbf{U}^{(n)}_{j-\frac{1}{2}}, \quad \forall x \in (x_{j-1}, x_j), \quad n > 0$$



Discretization method away from the coupling interface

2-step algorithm from $t^{(n)}$ to $t^{(n+1)}$

1. solve the relaxation system with $\lambda = 0$:
   - Euler method for the time discretization
   - Finite volume method for the space discretization:

$$\begin{aligned}
\mathbf{U}^{(n+1)}_{j-\frac{1}{2}} &= \mathbf{U}^{(n)}_{j-\frac{1}{2}} - \frac{\Delta t}{h}\Big(\mathbf{g}_-(\mathbf{U}^{(n)}_{j-\frac{1}{2}}, \mathbf{U}^{(n)}_{j+\frac{1}{2}}) - \mathbf{g}_-(\mathbf{U}^{(n)}_{j-\frac{3}{2}}, \mathbf{U}^{(n)}_{j-\frac{1}{2}})\Big), \quad j < 0 \\
\mathbf{U}^{(n+1)}_{j+\frac{1}{2}} &= \mathbf{U}^{(n)}_{j+\frac{1}{2}} - \frac{\Delta t}{h}\Big(\mathbf{g}_+(\mathbf{U}^{(n)}_{j+\frac{1}{2}}, \mathbf{U}^{(n)}_{j+\frac{3}{2}}) - \mathbf{g}_+(\mathbf{U}^{(n)}_{j-\frac{1}{2}}, \mathbf{U}^{(n)}_{j+\frac{1}{2}})\Big), \quad j > 0
\end{aligned}$$

   - Godunov method:

$$\mathbf{g}_\pm(\cdot,\cdot) = \mathbf{F}^\pm\big(\mathcal{W}(0,\cdot,\cdot)\big)$$

   with $\mathcal{W}$ solution of a local Riemann problem

2. projection of the solution onto the equilibrium state for $\lambda \to \infty$:

$$\lim_{\lambda\to\infty} \pi^{(n+1)} - p^{(n+1)} = 0$$

ONERA
THE FRENCH AEROSPACE LAB

Discretization method at a coupling interface

We consider $\mathcal{W}(\frac{x}{t-t^{(n)}};\mathbf{U}^{(n)}_{-\frac{1}{2}},\mathbf{U}^{(n)}_{+\frac{1}{2}},\boldsymbol{\mathcal{M}}^{(n)})$ solution of the Riemann problem:

$$\begin{aligned} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U},x) &= \boldsymbol{\mathcal{M}}^{(n)}, \quad \forall x \in (x_{-\frac{1}{2}}, x_{+\frac{1}{2}}), \quad t \in (t^{(n)}, t^{(n+1)}] \\ \mathbf{U}(x,t^{(n)}) &= \begin{cases} \mathbf{U}^{(n)}_{-\frac{1}{2}} & x < 0 \\ \mathbf{U}^{(n)}_{+\frac{1}{2}} & x > 0 \end{cases} \end{aligned}$$

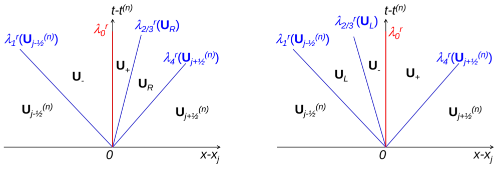
Coupling condition at discrete level:

$$\boldsymbol{\mathcal{M}}^{(n)} = \mathbf{g}_+(\mathbf{U}^{(n)}_{-\frac{1}{2}}, \mathbf{U}^{(n)}_{\frac{1}{2}}) - \mathbf{g}_-(\mathbf{U}^{(n)}_{-\frac{1}{2}}, \mathbf{U}^{(n)}_{\frac{1}{2}})$$

ONERA THE FRENCH AEROSPACE LAB

Discretization method at a coupling interface (cont.)

Structure of the solution:



(a) $\lambda_2^r > 0$ (b) $\lambda_2^r < 0$

Galié et al. (2009) proved existence and unicity of self-similar solution for $p = p(\rho)$ and $\boldsymbol{\mathcal{M}}^{(n)}$ in a suitable domain with $\mathcal{M}_\rho = 0$.

Extended to $\mathcal{M}_\rho \neq 0$ in the present study

Outline

Shock tube problem

- Riemann problem with shock tube conditions:

$$(\rho, u, p)^\top = \begin{cases} (0.4, 1, 1)^\top, & \text{for } x \le 0 \\ (0.8, -0.5, 1)^\top, & \text{for } x > 0 \end{cases}$$

- for a gas with an (exact) van der Waals EOS with constant heats

$$(p_{ex} + a\rho^2)(1 - b\rho) = \rho r T$$

with $a = 3,\ b = \frac{1}{3},\ \frac{r}{C_v} = 0.329$

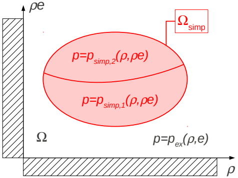
- The simplified EOS is a stiffened equation of Grüneisen type:

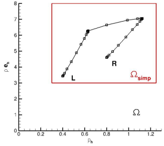
$$p_{simp}(\rho, \rho e) = (\gamma - 1)\rho e + c_{ref}^2(\rho - \rho_{ref})$$

[Callen '88, Menikoff & Plohr '89, Quartapelle et al. '03]

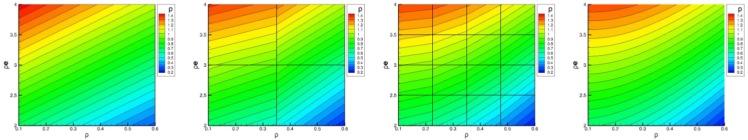
Simplified EOS via diadic hierarchy of Ω_{simp}

- The domain Ω_{simp} is defined s.t. the simplified EOS is valid everywhere:





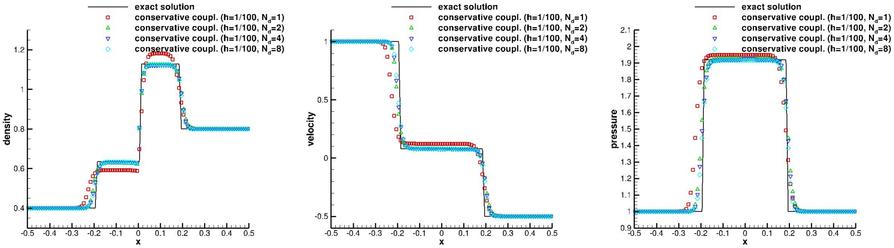
- Use diadic partition of Ω_{simp} into N_d^2 subdomains:



(a) $N_d = 1$ (b) $N_d = 2$ (c) $N_d = 4$ (d) exact EOS

Conservative coupling ($\mathcal{M} \equiv 0$)

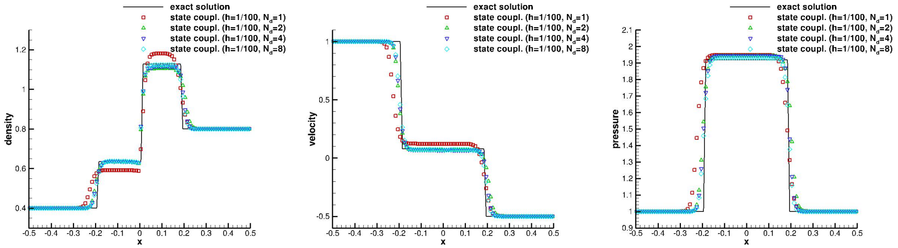
N_d^2 subdomains in Ω_{simp}, $t = 0.13$ and $h = 1/100$



(a) density (b) velocity (c) pressure

(ρ, u, p)-state coupling

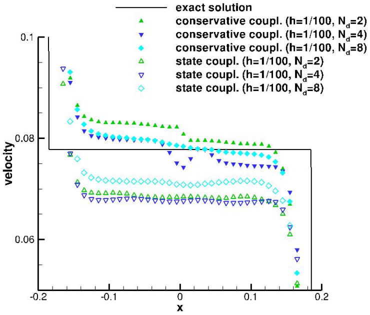
N_d^2 subdomains in Ω_{simp}, $t = 0.13$ and $h = 1/100$

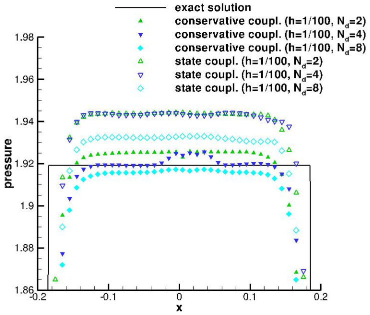


(a) density (b) velocity (c) pressure

Comparison of coupling methods

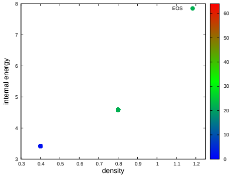
N_d^2 subdomains in Ω_{simp}, $t = 0.13$ and $h = 1/100$



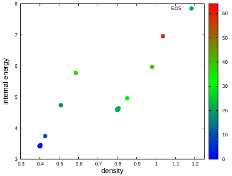


Space time adaptive EOS

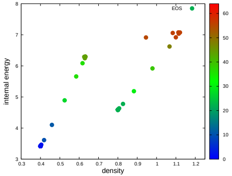
Conservative coupling ($\mathcal{M} \equiv 0$): computation with $N_d^2 = 64$ different EOS ($h = 1/100$)



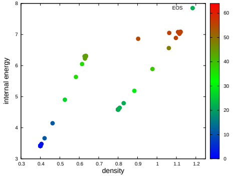
(a) $t = 0$



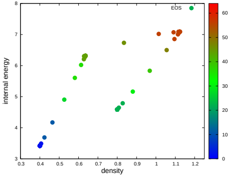
(b) $t = 0.01$



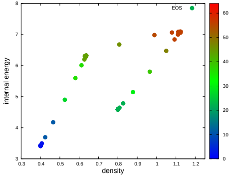
(c) $t = 0.05$



(d) $t = 0.07$



(e) $t = 0.11$



(f) $t = 0.13$

ONERA
THE FRENCH AEROSPACE LAB

Outline

ONERA
THE FRENCH AEROSPACE LAB

Concluding remarks

Adaptive coupling allows to:

- use simplified EOS instead of real gas EOS with complex thermodynamic laws
- reproduce the correct behaviour of real gas dynamics
- use different coupling mehods (source term measure offers a general framework)

Future work:

- use mixed coupling methods according to physical requirements
- minimization of convex non-linear cost functions built on the mass $\mathcal{M}$ at the interface

ONERA
THE FRENCH AEROSPACE LAB

Thank you for your attention

Bibliography

[1] A. Ambroso, C. Chalons, F. Coquel, E. Godlewski, F. Lagoutière, P.-A. Raviart and N. Seguin, Coupling of general Lagrangian systems. Math. Comp., 77 (2008), 909–941.

[2] A. Ambroso, C. Chalons, F. Coquel, E. Godlewski and P.-A. Raviart, Relaxation methods and coupling procedures. Int. J. Numer. Meth. Fluids, 56 (2008), 1123–1129.

[3] A. Ambroso, J.-M. Hérard and O. Hurisse, A Method to Couple HEM and HRM Two-Phase Flow Models. Computers and Fluids, 38 (2009), 738–756.

[4] B. Boutin, Etude mathématique et numérique d'équations hyperboliques non-linéaires : couplage de modèles et chocs non classiques. Thèse de doctorat, Univ. Pierre et Marie Curie, Paris 6, Nov. 2009.

[5] H. B. Callen, Thermodynamics and an introduction to thermostatistics, second edition, John Wiley & Sons, 1988.

[6] T. Galié, Couplage interfacial de modèles en dynamique des fluides. Application aux écoulements diphasiques. Thèse de doctorat, Univ. Pierre et Marie Curie, Paris 6, Mars 2009.

[7] E. Godlewski and P.-A. Raviart, The numerical Interface Coupling of Nonlinear Hyperbolic Systems of Conservation Laws. I. The Scalar Case. Numer. Math., 97 (2004), 81–130.

[8] E. Godlewski, K.-C. Le Thanh and P.-A. Raviart, The numerical Interface Coupling of Nonlinear Hyperbolic Systems of Conservation Laws. II. The case of systems. M2AN, 39 (2005), 649–6920.

[9] S. Jin and Z. Xin, The Relaxation Schemes for Systems of Conservation Laws in Arbitrary Space Dimension, Comm. Pure. Appl. Math., 48 (1995), pp. 235–276.

[10] A. Murrone and P. Villedieu, Numerical modeling of dispersed two-phase flows. The Onera Journal Aerospace Lab, 2 (2011).

[11] R. Menikoff and B. J. Plohr, The Riemann problem for fluid flow of real materials, Rev. Mod. Phy., 61 (1989), 75–130.

[12] L. Quartapelle, L. Castelletti, A. Guardone and G. Quaranta, Solution of the Riemann problem of classical gasdynamics, J. Comput. Phys., 190 (2003), 118–140.

[13] J. U. Schluter, X. Wu, E. V. D. Weide, S. Hahn, J. Alonso and H. Pitsch, Multi-Code Simulations : a Generalized Coupling Approach. AIAA paper 2005-4997, 2005.

[14] I. Suliciu, On the thermodynamics of fluids with relaxation and phase transitions. Fluids with relaxation, Int. J. Engag. Sci., 36 (1998), pp. 921–947.

ONERA
THE FRENCH AEROSPACE LAB