Points of General Relativistic Shock Wave Interaction are Regularity Singularities where Spacetime is Not Locally Flat

Moritz Reintjes

(Joint work with Blake Temple.)
HYP2012 - Padova
28 June, 2012

Part I

Intuitive Introduction

What are shock waves?

What are shock waves?

Shock waves are discontinuities evolving in time.

Intuitive Introduction

FullImage_2005122151636_846.jpg (JPEG-Grafik, 500 × 326 Pixel) http://www.teamdroid.com/img-2/

Where do shock waves appear?

Where do shock waves appear?

- Shock waves form in fluids and gases, governed by compressible Euler equations.

Where do shock waves appear?

- Shock waves form in fluids and gases, governed by compressible Euler equations.
- In General Relativity (GR), shock waves can be present in the matter content of spacetime.

Einstein equations:

"spacetime curvature $\simeq$ matter content"

Einstein equations:

"spacetime curvature $\simeq$ matter content"

- In GR, shock waves are discontinuities in matter content.

Einstein equations:

"spacetime curvature $\simeq$ matter content"

- In GR, shock waves are discontinuities in matter content.
- Spacetime curvature is determined by *metric tensor*.

Einstein equations:

"spacetime curvature $\simeq$ matter content"

- In GR, shock waves are discontinuities in matter content.
- Spacetime curvature is determined by *metric tensor*.

$\longrightarrow$ How do shock waves effect the metric tensor?

In some coordinates, the **metric is only** $C^{0,1}$, (Lipschitz continuous).

In some coordinates, the **metric is only** $C^{0,1}$, (Lipschitz continuous).

- That's counterintuitive: Einstein equations are 2^{nd} order system

In some coordinates, the **metric is only** $C^{0,1}$, (Lipschitz continuous).

- That's counterintuitive: Einstein equations are 2^{nd} order system
  ($\rightarrow$ would expect $C^{1,1}$ metric regularity).

In some coordinates, the **metric is only** $C^{0,1}$, (Lipschitz continuous).

- That's counterintuitive: Einstein equations are 2^{nd} order system
  ($\rightarrow$ would expect $C^{1,1}$ metric regularity).
- Moreover, many fundamental features of spacetime require a $C^{1,1}$ metric regularity.

In some coordinates, the **metric is only** $C^{0,1}$, (Lipschitz continuous).

- That's counterintuitive: Einstein equations are 2nd order system
  ($\rightarrow$ would expect $C^{1,1}$ metric regularity).
- Moreover, many fundamental features of spacetime require a $C^{1,1}$ metric regularity.

Question:

Can we raise the metric regularity to $C^{1,1}$ by transforming to different coordinates?

Question:

Can we raise the metric regularity to $C^{1,1}$ by transforming to different coordinates?

Question:

Can we raise the metric regularity to $C^{1,1}$ by transforming to different coordinates?

- Across a single shock wave: **Yes!** (Israel, 1966)

Question:

Can we raise the metric regularity to $C^{1,1}$ by transforming to different coordinates?

- Across a single shock wave: **Yes!** (Israel, 1966)
- At point of shock wave interaction:

Question:

Can we raise the metric regularity to $C^{1,1}$ by transforming to different coordinates?

- Across a single shock wave: **Yes!** (Israel, 1966)
- At point of shock wave interaction: **No!** (R. and Temple, 2011)

Question:

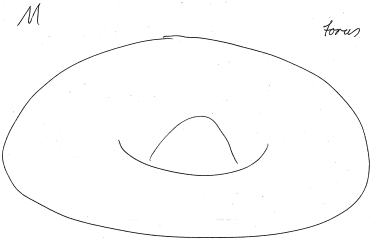
Can we raise the metric regularity to $C^{1,1}$ by transforming to different coordinates?

- Across a single shock wave: **Yes!** (Israel, 1966)
- At point of shock wave interaction: **No!** (R. and Temple, 2011) $\longrightarrow$ "Regularity Singularity"

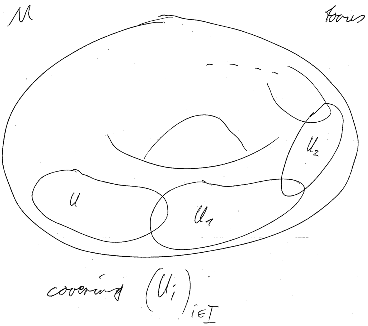
Part II

Background: Shock Waves in General Relativity

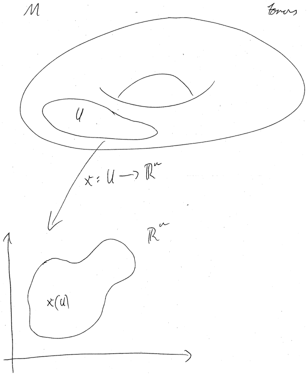
A manifold M is a Hausdorff-space locally diffeomorphic to $\mathbb{R}^n$:



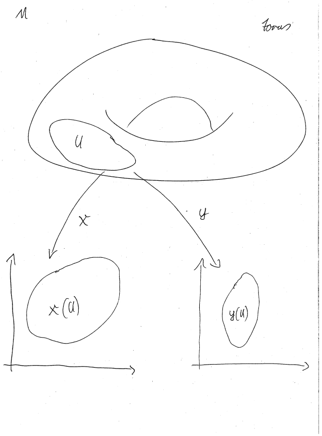
A manifold M is a Hausdorff-space locally diffeomorphic to $\mathbb{R}^n$:



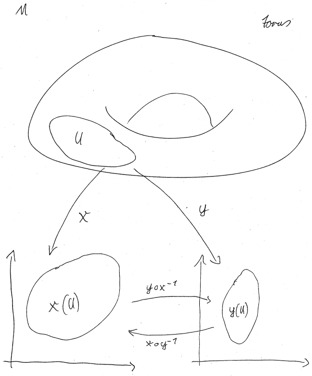
A manifold M is a Hausdorff-space locally diffeomorphic to $\mathbb{R}^n$:



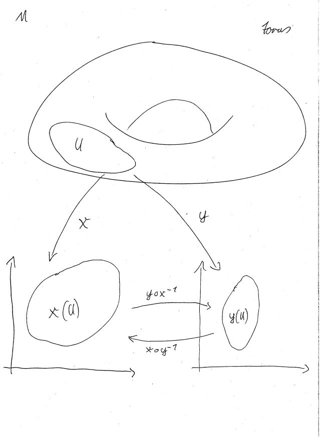
A manifold M is a Hausdorff-space locally diffeomorphic to $\mathbb{R}^n$:



A manifold M is a Hausdorff-space locally diffeomorphic to $\mathbb{R}^n$:

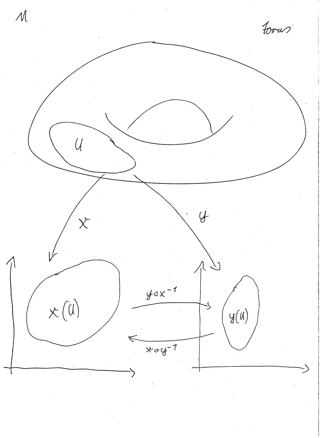


A manifold M is a Hausdorff-space locally diffeomorphic to $\mathbb{R}^n$:



- $x \circ y^{-1}$ and $y \circ x^{-1}$ C^k-differentiable; called "change of coordinates"

A manifold M is a Hausdorff-space locally diffeomorphic to $\mathbb{R}^n$:



- $x \circ y^{-1}$ and $y \circ x^{-1}$ C^k-differentiable; called "change of coordinates"
- collection of all such mappings and domains, (x, U), is called a C^k-atlas

All geometrical information about M (e.g., angles, curvature,...) is captured in the *Lorentz-metric* tensor, g.

All geometrical information about M (e.g., angles, curvature,...) is captured in the *Lorentz-metric* tensor, g.

In coord's x^j, $\qquad g = g_{ij} dx^i dx^j$

All geometrical information about M (e.g., angles, curvature,...) is captured in the *Lorentz-metric* tensor, g.

$$\text{In coord's } x^j, \qquad g = g_{ij}dx^i dx^j := \sum_{i,j=1}^{n} g_{ij}dx^i dx^j.$$

- Convention: sum over repeated indices, ($n = \dim(M)$).

All geometrical information about M (e.g., angles, curvature,...) is captured in the *Lorentz-metric* tensor, g.

$$\text{In coord's } x^j, \qquad g = g_{ij}dx^i dx^j := \sum_{i,j=1}^{n} g_{ij}dx^i dx^j .$$

- Convention: sum over repeated indices, ($n = \dim(M)$).
- Roughly: think of dx^j as dual vector to *j-th* coordinate vector in $x(U)$.

All geometrical information about M (e.g., angles, curvature,...) is captured in the *Lorentz-metric* tensor, g.

$$\text{In coord's } x^j, \qquad g = g_{ij} dx^i dx^j := \sum_{i,j=1}^{n} g_{ij} dx^i dx^j .$$

- Convention: sum over repeated indices, ($n = \dim(M)$).
- Roughly: think of dx^j as dual vector to *j-th* coordinate vector in $x(U)$.
- Pointwise, $(g_{ij})_{1\leq i,j\leq n}$ is a symmetric matrix which has signature (-+++)

All geometrical information about M (e.g., angles, curvature,...) is captured in the *Lorentz-metric* tensor, g.

$$\text{In coord's } x^j, \qquad g = g_{ij} dx^i dx^j := \sum_{i,j=1}^{n} g_{ij} dx^i dx^j .$$

- Convention: sum over repeated indices, ($n = \dim(M)$).
- Roughly: think of dx^j as dual vector to *j-th* coordinate vector in $x(U)$.
- Pointwise, $(g_{ij})_{1 \leq i,j \leq n}$ is a symmetric matrix which has signature (-+++)
- In new coord's, $g(x) = g_{\mu\nu}(y) dy^\nu dy^\nu$, the metric components transform as

$$\boxed{g_{ij}(x) = J^\mu_i J^\nu_j g_{\mu\nu}(y(x)),}$$

where $J^\mu_j := \frac{\partial x^\mu \circ y^{-1}}{\partial y^j}$ denotes the Jacobian.

Spacetime is a 4-D manifold with a Lorentz-metric
($\rightarrow$ "Equivalence Principle").

At the heart of GR are the *Einstein equations*:

“spacetime curvature $\simeq$ energy/matter content”

At the heart of GR are the *Einstein equations*:

"spacetime curvature $\simeq$ energy/matter content"

- Energy and matter content of spacetime is described by the *energy-momentum-tensor* $T^{\mu\nu}$, which depends on type of matter-fields considered.

At the heart of GR are the *Einstein equations*:

"spacetime curvature $\simeq$ energy/matter content"

- Energy and matter content of spacetime is described by the *energy-momentum-tensor* $T^{\mu\nu}$, which depends on type of matter-fields considered.
- Spacetime curvature is described by Einstein tensor, $G^{\mu\nu}$, via "measuring failure of 2^{nd} order (covariant) derivatives to commute".

At the heart of GR are the *Einstein equations*:

$$\boxed{G^{\mu\nu} = \kappa T^{\mu\nu}}$$

- Energy and matter content of spacetime is described by the *energy-momentum-tensor* $T^{\mu\nu}$, which depends on type of matter-fields considered.
- Spacetime curvature is described by Einstein tensor, $G^{\mu\nu}$, via "measuring failure of 2^{nd} order (covariant) derivatives to commute".

At the heart of GR are the *Einstein equations*:

$$\boxed{G^{\mu\nu} = \kappa T^{\mu\nu}}$$

- Energy and matter content of spacetime is described by the *energy-momentum-tensor* $T^{\mu\nu}$, which depends on type of matter-fields considered.
- Spacetime curvature is described by Einstein tensor, $G^{\mu\nu}$, via "measuring failure of 2^{nd} order (covariant) derivatives to commute".

- $G^{\mu\nu}$ comprises entirely of the metric tensor, $g_{\mu\nu}$, and its first and second derivatives.

At the heart of GR are the *Einstein equations*:

$$\boxed{G^{\mu\nu} = \kappa T^{\mu\nu}}$$

- Energy and matter content of spacetime is described by the *energy-momentum-tensor* $T^{\mu\nu}$, which depends on type of matter-fields considered.
- Spacetime curvature is described by Einstein tensor, $G^{\mu\nu}$, via "measuring failure of 2^{nd} order (covariant) derivatives to commute".

- $G^{\mu\nu}$ comprises entirely of the metric tensor, $g_{\mu\nu}$, and its first and second derivatives.
- G is the unique (modulo a constant) curvature tensor being divergence-free, $\textbf{div}G = 0$, thus imposing conservation of energy in the Einstein equations.

A *single* shock wave being present in the Einstein equations is characterized by:

A *single* shock wave being present in the Einstein equations is characterized by:

- $T^{\mu\nu}$ is discontinuous across a hypersurface Σ and C^0 elsewhere.

A *single* shock wave being present in the Einstein equations is characterized by:

- $T^{\mu\nu}$ is discontinuous across a hypersurface Σ and C^0 elsewhere.
- Across Σ, the Rankine Hugoniot jump conditions hold, that is,

$$\boxed{[T^{\mu\nu}]N_\nu = 0,}$$

A *single* shock wave being present in the Einstein equations is characterized by:

- $T^{\mu\nu}$ is discontinuous across a hypersurface Σ and C^0 elsewhere.
- Across Σ, the Rankine Hugoniot jump conditions hold, that is,

$$\boxed{[T^{\mu\nu}]N_\nu = 0,}$$

  where

  - N^ν normal to Σ,
  - $[u] := u_L - u_R$ denotes the jump in u across Σ
  - $u_{L/R}$ denotes the left/right limit of u to Σ.

A *single* shock wave being present in the Einstein equations is characterized by:

- $T^{\mu\nu}$ is discontinuous across a hypersurface Σ and C^0 elsewhere.
- Across Σ, the Rankine Hugoniot jump conditions hold, that is,

$$\boxed{[T^{\mu\nu}]N_\nu = 0,}$$

  where
  - N^ν normal to Σ,
  - $[u] := u_L - u_R$ denotes the jump in u across Σ
  - $u_{L/R}$ denotes the left/right limit of u to Σ.
- Einstein equations, $G^{\mu\nu} = \kappa T^{\mu\nu}$, hold strongly off Σ.

Remark:

The (probably) most important setting for shock waves in GR are the *Coupled Einstein Euler equations*,

Remark:

The (probably) most important setting for shock waves in GR are the *Coupled Einstein Euler equations*,

$$\begin{aligned} G^{\mu\nu} &= \kappa T^{\mu\nu}, \\ \operatorname{div} T &= 0, \end{aligned}$$

Remark:

The (probably) most important setting for shock waves in GR are the *Coupled Einstein Euler equations*,

$$\begin{aligned} G^{\mu\nu} &= \kappa T^{\mu\nu}, \\ \operatorname{div} T &= 0, \end{aligned}$$

where $\quad T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} \quad$ (*perfect fluid*),

Remark:

The (probably) most important setting for shock waves in GR are the *Coupled Einstein Euler equations*,

$$\begin{aligned} G^{\mu\nu} &= \kappa T^{\mu\nu}, \\ \operatorname{div} T &= 0, \end{aligned}$$

where $\quad T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu} \quad$ (*perfect fluid*),

- u tangent to fluid flow,
- ρ energy-density,
- $p = p(\rho)$ pressure.

Remark:

The (probably) most important setting for shock waves in GR are the *Coupled Einstein Euler equations*,

$$\begin{aligned} G^{\mu\nu} &= \kappa T^{\mu\nu}, \\ \operatorname{div} T &= 0, \end{aligned}$$

where $T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}$ (*perfect fluid*),

- u tangent to fluid flow,
- ρ energy-density,
- $p = p(\rho)$ pressure.

Shock waves can form in the relativistic Euler equations, $\operatorname{div} T = 0$, out of smooth initial data.

Part III

The Question of the Metric Regularity

Let's illustrate the effect of shock waves on the metric regularity:

Let's illustrate the effect of shock waves on the metric regularity:

- Choose Standard Schwarzschild Coordinates,
  that is, coord's where the metric reads
  $ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + r^2\left(d\vartheta^2 + \sin^2(\vartheta)d\varphi^2\right),$

Let's illustrate the effect of shock waves on the metric regularity:

- Choose Standard Schwarzschild Coordinates, that is, coord's where the metric reads
  $$ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + r^2\left(d\vartheta^2 + \sin^2(\vartheta)d\varphi^2\right),$$
- Then, the first Einstein equation reads
  $$\frac{\partial B}{\partial r} + B\frac{B-1}{r} \; = \; \kappa A B^2 r\, T^{00}.$$

Let's illustrate the effect of shock waves on the metric regularity:

- Choose Standard Schwarzschild Coordinates, that is, coord's where the metric reads
  $ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + r^2\left(d\vartheta^2 + \sin^2(\vartheta)d\varphi^2\right),$
- Then, the first Einstein equation reads
  $\frac{\partial B}{\partial r} + B\frac{B-1}{r} = \kappa AB^2 r\, T^{00}.$
- Now, let a shock waves be present in $T^{\mu\nu}$

Let's illustrate the effect of shock waves on the metric regularity:

- Choose Standard Schwarzschild Coordinates,
  that is, coord's where the metric reads
  $$ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + r^2\left(d\vartheta^2 + \sin^2(\vartheta)d\varphi^2\right),$$
- Then, the first Einstein equation reads
  $\frac{\partial B}{\partial r} + B\frac{B-1}{r} \;=\; \kappa A B^2 r\, T^{00}.$
- Now, let a shock waves be present in $T^{\mu\nu}$
  $\Longrightarrow \quad T^{00}$ is discontinuous

Let's illustrate the effect of shock waves on the metric regularity:

- Choose Standard Schwarzschild Coordinates, that is, coord's where the metric reads $ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + r^2\left(d\vartheta^2 + \sin^2(\vartheta)d\varphi^2\right),$
- Then, the first Einstein equation reads $\frac{\partial B}{\partial r} + B\frac{B-1}{r} = \kappa AB^2 r\, T^{00}.$
- Now, let a shock waves be present in $T^{\mu\nu}$
  $\Longrightarrow \quad T^{00}$ is discontinuous
  $\Longrightarrow \quad \frac{\partial B}{\partial r}$ is discontinuous

Let's illustrate the effect of shock waves on the metric regularity:

- Choose Standard Schwarzschild Coordinates,
  that is, coord's where the metric reads
  $$ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + r^2\left(d\vartheta^2 + \sin^2(\vartheta)d\varphi^2\right),$$
- Then, the first Einstein equation reads
  $\frac{\partial B}{\partial r} + B\frac{B-1}{r} = \kappa A B^2 r\, T^{00}.$
- Now, let a shock waves be present in $T^{\mu\nu}$
  $\Longrightarrow \quad T^{00}$ is discontinuous
  $\Longrightarrow \quad \frac{\partial B}{\partial r}$ is discontinuous
  $\longrightarrow \quad B \in C^{0,1} \setminus C^1$

Conclusion:

- In some coordinates, the Einstein equations contain first order differential equations.

Conclusion:

- In some coordinates, the Einstein equations contain first order differential equations.
- Thus, if shock waves are present in $T^{\mu\nu}$, the metric can only be in $C^{0,1}$, but not in C^1.

Conclusion:

- In some coordinates, the Einstein equations contain first order differential equations.
- Thus, if shock waves are present in $T^{\mu\nu}$, the metric can only be in $C^{0,1}$, but not in C^1.

Central Question:

Do there exist coordinates x^j such that the metric in the new coordinates, g_{ij}, is in $C^{1,1}$?

What's interesting about $C^{0,1}$ versus $C^{1,1}$ metric regularity?

What's interesting about $C^{0,1}$ versus $C^{1,1}$ metric regularity?

- $C^{1,1}$-regularity is crucial to define curvature tensors, G_{ij}, R_{ij},..., in a classical (non-distributional) sense

What's interesting about $C^{0,1}$ versus $C^{1,1}$ metric regularity?

- $C^{1,1}$-regularity is crucial to define curvature tensors, G_{ij}, R_{ij},..., in a classical (non-distributional) sense
- $C^{1,1}$-regularity is required for the Einstein equations to hold strongly.

What's interesting about $C^{0,1}$ versus $C^{1,1}$ metric regularity?

- $C^{1,1}$-regularity is crucial to define curvature tensors, G_{ij}, R_{ij},..., in a classical (non-distributional) sense
- $C^{1,1}$-regularity is required for the Einstein equations to hold strongly.
- If one cannot smooth the metric to $C^{1,1}$, it cannot be locally Minkowski!

What's interesting about $C^{0,1}$ versus $C^{1,1}$ metric regularity?

- $C^{1,1}$-regularity is crucial to define curvature tensors, G_{ij}, R_{ij},..., in a classical (non-distributional) sense
- $C^{1,1}$-regularity is required for the Einstein equations to hold strongly.
- If one cannot smooth the metric to $C^{1,1}$, it cannot be locally Minkowski!
  ($\rightarrow$ No observer in free-fall?!)

What's interesting about $C^{0,1}$ versus $C^{1,1}$ metric regularity?

- $C^{1,1}$-regularity is crucial to define curvature tensors, G_{ij}, R_{ij},..., in a classical (non-distributional) sense
- $C^{1,1}$-regularity is required for the Einstein equations to hold strongly.
- If one cannot smooth the metric to $C^{1,1}$, it cannot be locally Minkowski!
  ($\rightarrow$ No observer in free-fall?!)
- $C^{1,1}$ is a quite common assumption in GR, e.g., $C^{1,1}$ regularity is required in Singularity Theorems (of Penrose, Hawking and Ellis).

Part IV

The Metric Regularity Across a Single Shock Surface

"Israel's Theorem"
(based on Israel 1966) (see also: Smoller and Temple 1994)

"Israel's Theorem"
(based on Israel 1966) (see also: Smoller and Temple 1994)

Suppose:

- (M,g) a (Riemann) manifold with a $C^{1,1}$-atlas
- $g_{\mu\nu}$ is $C^{0,1}$ across a single smooth surface Σ,
- ($g_{\mu\nu}$ solves the Einstein equations strongly away from Σ)

"Israel's Theorem" (based on Israel 1966) (see also: Smoller and Temple 1994)

Suppose:

- (M,g) a (Riemann) manifold with a $C^{1,1}$-atlas
- $g_{\mu\nu}$ is $C^{0,1}$ across a single smooth surface Σ,
- ($g_{\mu\nu}$ solves the Einstein equations strongly away from Σ)

Then the following is equivalent:

“Israel's Theorem“
(based on Israel 1966) (see also: Smoller and Temple 1994)

Suppose:

- (M,g) a (Riemann) manifold with a $C^{1,1}$-atlas
- $g_{\mu\nu}$ is $C^{0,1}$ across a single smooth surface Σ,
- ($g_{\mu\nu}$ solves the Einstein equations strongly away from Σ)

Then the following is equivalent:

- There exist coordinates x^α such that $g_{\alpha\beta} \in C^{1,1}$, (w.r.t. partial differentiation in x^α).

“Israel's Theorem“
(based on Israel 1966) (see also: Smoller and Temple 1994)

Suppose:

- (M,g) a (Riemann) manifold with a $C^{1,1}$-atlas
- $g_{\mu\nu}$ is $C^{0,1}$ across a single smooth surface Σ,
- ($g_{\mu\nu}$ solves the Einstein equations strongly away from Σ)

Then the following is equivalent:

- There exist coordinates x^α such that $g_{\alpha\beta} \in C^{1,1}$, (w.r.t. partial differentiation in x^α).
- The RH jump conditions, $[T^{\mu\nu}]N_\nu = 0$, hold on Σ and T^{ij} is in L^∞.

"Israel's Theorem"
(based on Israel 1966) (see also: Smoller and Temple 1994)

Suppose:

- (M,g) a (Riemann) manifold with a $C^{1,1}$-atlas
- $g_{\mu\nu}$ is $C^{0,1}$ across a single smooth surface Σ,
- ($g_{\mu\nu}$ solves the Einstein equations strongly away from Σ)

Then the following is equivalent:

- There exist coordinates x^α such that $g_{\alpha\beta} \in C^{1,1}$, (w.r.t. partial differentiation in x^α).
- The RH jump conditions, $[T^{\mu\nu}]N_\nu = 0$, hold on Σ and T^{ij} is in L^∞.

Lesson: Across a **single** shock one can always lift metric regularity to $C^{1,1}$!

Part V

The Metric Regularity at Points of Shock Wave Interaction

Israel's Theorem addresses the metric regularity across a **single** shock surface only.

Israel's Theorem addresses the metric regularity across a **single** shock surface only.

However, shock waves can **interact**.

Israel's Theorem addresses the metric regularity across a **single** shock surface only.

However, shock waves can **interact**.

Can one still lift the metric regularity if two shock waves interact?

Israel's Theorem addresses the metric regularity across a **single** shock surface only.

However, shock waves can **interact**.

Can one still lift the metric regularity if two shock waves interact?

NO, one cannot!

Israel's Theorem addresses the metric regularity across a **single** shock surface only.

However, shock waves can **interact**.

Can one still lift the metric regularity if two shock waves interact?

NO, one cannot!
(R. & Temple, 2011)

Before we state our theorem, let me introduce the shock wave interaction we consider:

Points of Regular Shock Wave Interaction in SSC:

Points of Regular Shock Wave Interaction in SSC:

Assumption on spacetime:

- Suppose spacetime M is **spherically symmetric**.

Points of Regular Shock Wave Interaction in SSC:

Assumption on spacetime:

- Suppose spacetime M is **spherically symmetric**.
- Assume Standard Schwarzschild Coordinates (=:SSC) exists around a point $p \in M$,

Points of Regular Shock Wave Interaction in SSC:

Assumption on spacetime:

- Suppose spacetime M is **spherically symmetric**.
- Assume Standard Schwarzschild Coordinates (=:SSC) exists around a point $p \in M$, that is, coord's $(t, r, \vartheta, \varphi)$ where the metric reads

$$g = -A(t,r)dt^2 + B(t,r)dr^2 + r^2 d\Omega^2,$$

with $d\Omega^2 := d\vartheta^2 + \sin^2(\vartheta)d\varphi^2$.

Points of Regular Shock Wave Interaction in SSC:

Assumption on spacetime:

- Suppose spacetime M is **spherically symmetric**.
- Assume Standard Schwarzschild Coordinates (=:SSC) exists around a point $p \in M$, that is, coord's $(t, r, \vartheta, \varphi)$ where the metric reads

$$g = -A(t,r)dt^2 + B(t,r)dr^2 + r^2 d\Omega^2,$$

with $d\Omega^2 := d\vartheta^2 + \sin^2(\vartheta)d\varphi^2$.

Remark: Spherical symmetry, though being restrictive, includes many important spacetimes:

Points of Regular Shock Wave Interaction in SSC:

Assumption on spacetime:

- Suppose spacetime M is **spherically symmetric**.
- Assume Standard Schwarzschild Coordinates (=:SSC) exists around a point $p \in M$, that is, coord's $(t, r, \vartheta, \varphi)$ where the metric reads

$$g = -A(t,r)dt^2 + B(t,r)dr^2 + r^2 d\Omega^2,$$

with $d\Omega^2 := d\vartheta^2 + \sin^2(\vartheta)d\varphi^2$.

Remark: Spherical symmetry, though being restrictive, includes many important spacetimes:

- Schwarzschild spacetime (outside of black hole or star),

Points of Regular Shock Wave Interaction in SSC:

Assumption on spacetime:

- Suppose spacetime M is **spherically symmetric**.
- Assume Standard Schwarzschild Coordinates (=:SSC) exists around a point $p \in M$, that is, coord's $(t, r, \vartheta, \varphi)$ where the metric reads

$$g \;=\; -A(t,r)dt^2 + B(t,r)dr^2 + r^2 d\Omega^2,$$

  with $d\Omega^2 := d\vartheta^2 + \sin^2(\vartheta)d\varphi^2$.

Remark: Spherical symmetry, though being restrictive, includes many important spacetimes:

- Schwarzschild spacetime (outside of black hole or star),
- Oppenheimer-Tolman spacetime (inside of gaseous star),

Points of Regular Shock Wave Interaction in SSC:

Assumption on spacetime:

- Suppose spacetime M is **spherically symmetric**.
- Assume Standard Schwarzschild Coordinates (=:SSC) exists around a point $p \in M$, that is, coord's $(t, r, \vartheta, \varphi)$ where the metric reads

$$g = -A(t,r)dt^2 + B(t,r)dr^2 + r^2 d\Omega^2,$$

with $d\Omega^2 := d\vartheta^2 + \sin^2(\vartheta)d\varphi^2$.

Remark: Spherical symmetry, though being restrictive, includes many important spacetimes:

- Schwarzschild spacetime (outside of black hole or star),
- Oppenheimer-Tolman spacetime (inside of gaseous star),
- Friedman-Robertson-Walker spacetime (cosmology).

Assumption on shock waves:

Assumption on shock waves:

- Assume shock waves are **radial**, that is, the shock surfaces, Σ_1 and Σ_2, are 2-spheres evolving in time.

Assumption on shock waves:

- Assume shock waves are **radial**, that is, the shock surfaces, Σ_1 and Σ_2, are 2-spheres evolving in time.
- More precisely, Σ_i can be parameterized as

$$\Sigma_i(t, \vartheta, \varphi) = (t, x_i(t), \vartheta, \varphi)\ , \qquad x_i(t) > 0, \quad (i = 1, 2).$$

Assumption on shock waves:

- Assume shock waves are **radial**, that is, the shock surfaces, Σ_1 and Σ_2, are 2-spheres evolving in time.
- More precisely, Σ_i can be parameterized as

$$\Sigma_i(t,\vartheta,\varphi) = (t, x_i(t), \vartheta, \varphi)\ , \qquad x_i(t) > 0, \quad (i = 1, 2).$$

- Note: $\Sigma_i(t)$ is a 2-sphere with radius $x_i(t)$ and center $r = 0$.

Assumption on shock waves:

- Assume shock waves are **radial**, that is, the shock surfaces, Σ_1 and Σ_2, are 2-spheres evolving in time.
- More precisely, Σ_i can be parameterized as

$$\Sigma_i(t, \vartheta, \varphi) = (t, x_i(t), \vartheta, \varphi)\ , \qquad x_i(t) > 0, \quad (i = 1, 2).$$

- Note: $\Sigma_i(t)$ is a 2-sphere with radius $x_i(t)$ and center $r = 0$.
- Instead of Σ_i it suffices to consider curves $\gamma_i(t) = (t, x_i(t))$, (so-called "shock curves").

Definition

$p \in M$ is a “point of regular shock wave interaction in SSC” if

Definition

$p \in M$ is a "point of regular shock wave interaction in SSC" if

- $\gamma_i(t) = (t, x_i(t))$, $i = 1, 2$, are smooth timelike curves defined on $t \in (-\epsilon, 0)$.

Definition

$p \in M$ is a "point of regular shock wave interaction in SSC" if

- $\gamma_i(t) = (t, x_i(t))$, $i = 1, 2$, are smooth timelike curves defined on $t \in (-\epsilon, 0)$.
- γ_1 and γ_2 intersect in $p = \gamma_1(0) = \gamma_2(0)$.

Definition

$p \in M$ is a "point of regular shock wave interaction in SSC" if

- $\gamma_i(t) = (t, x_i(t))$, $i = 1, 2$, are smooth timelike curves defined on $t \in (-\epsilon, 0)$.
- γ_1 and γ_2 intersect in $p = \gamma_1(0) = \gamma_2(0)$.
- The SSC-metric, $g_{\mu\nu}$, is only $C^{0,1}$ across each shock curve and C^2 off and along them.

Definition

$p \in M$ is a "point of regular shock wave interaction in SSC" if

- $\gamma_i(t) = (t, x_i(t))$, $i = 1, 2$, are smooth timelike curves defined on $t \in (-\epsilon, 0)$.
- γ_1 and γ_2 intersect in $p = \gamma_1(0) = \gamma_2(0)$.
- The SSC-metric, $g_{\mu\nu}$, is only $C^{0,1}$ across each shock curve and C^2 off and along them.
- (Einstein equations hold strongly off Σ_i.)

Definition

$p \in M$ is a "point of regular shock wave interaction in SSC" if

- $\gamma_i(t) = (t, x_i(t))$, $i = 1, 2$, are smooth timelike curves defined on $t \in (-\epsilon, 0)$.
- γ_1 and γ_2 intersect in $p = \gamma_1(0) = \gamma_2(0)$.
- The SSC-metric, $g_{\mu\nu}$, is only $C^{0,1}$ across each shock curve and C^2 off and along them.
- (Einstein equations hold strongly off Σ_i.)
- Rankine Hugoniot conditions, $[T^{\mu\nu}]_i (N_i)_\nu = 0$, hold across each γ_i, (for $i = 1, 2$), and in the limit to $t \nearrow 0$.

Definition

$p \in M$ is a "point of regular shock wave interaction in SSC" if

- $\gamma_i(t) = (t, x_i(t))$, $i = 1, 2$, are smooth timelike curves defined on $t \in (-\epsilon, 0)$.
- γ_1 and γ_2 intersect in $p = \gamma_1(0) = \gamma_2(0)$.
- The SSC-metric, $g_{\mu\nu}$, is only $C^{0,1}$ across each shock curve and C^2 off and along them.
- (Einstein equations hold strongly off Σ_i.)
- Rankine Hugoniot conditions, $[T^{\mu\nu}]_i (N_i)_\nu = 0$, hold across each γ_i, (for $i = 1, 2$), and in the limit to $t \nearrow 0$.
- Shocks interact with distinct speeds, $\dot{x}_1(0) \neq \dot{x}_2(0)$.

Definition

$p \in M$ is a "point of regular shock wave interaction in SSC" if

- $\gamma_i(t) = (t, x_i(t))$, $i = 1, 2$, are smooth timelike curves defined on $t \in (-\epsilon, 0)$.
- γ_1 and γ_2 intersect in $p = \gamma_1(0) = \gamma_2(0)$.
- The SSC-metric, $g_{\mu\nu}$, is only $C^{0,1}$ across each shock curve and C^2 off and along them.
- (Einstein equations hold strongly off Σ_i.)
- Rankine Hugoniot conditions, $[T^{\mu\nu}]_i (N_i)_\nu = 0$, hold across each γ_i, (for $i = 1, 2$), and in the limit to $t \nearrow 0$.
- Shocks interact with distinct speeds, $\dot{x}_1(0) \neq \dot{x}_2(0)$.

- So, p is 2-sphere with radius $x_1(0) = x_2(0)$ and center $r = 0$.

Definition

$p \in M$ is a "point of regular shock wave interaction in SSC" if

- $\gamma_i(t) = (t, x_i(t))$, $i = 1, 2$, are smooth timelike curves defined on $t \in (-\epsilon, 0)$.
- γ_1 and γ_2 intersect in $p = \gamma_1(0) = \gamma_2(0)$.
- The SSC-metric, $g_{\mu\nu}$, is only $C^{0,1}$ across each shock curve and C^2 off and along them.
- (Einstein equations hold strongly off Σ_i.)
- Rankine Hugoniot conditions, $[T^{\mu\nu}]_i (N_i)_\nu = 0$, hold across each γ_i, (for $i = 1, 2$), and in the limit to $t \nearrow 0$.
- Shocks interact with distinct speeds, $\dot{x}_1(0) \neq \dot{x}_2(0)$.

- So, p is 2-sphere with radius $x_1(0) = x_2(0)$ and center $r = 0$.
- We expect this structure to be generic, for radial shock waves in spherical symmetry!

- Vogler simulated such a shock wave interaction with the above structure (2011).

- Vogler simulated such a shock wave interaction with the above structure (2011).
- Existence before and after interaction was established by Groah and Temple (2005).

Let's state our main theorem:

Theorem 1, (R. and Temple, 2011)

Assume p is "a point of regular shock wave interaction in SSC". Then: $\nexists$ $C^{1,1}$ coordinate transformation, defined in a neighborhood of p, such that both holds:

- The metric components are C^1 functions of the new coordinates.
- The metric has a nonzero determinant at p.

Theorem 1, (R. and Temple, 2011)

Assume p is "a point of regular shock wave interaction in SSC".
Then: $\nexists$ $C^{1,1}$ coordinate transformation, defined in a neighborhood of p, such that both holds:

- The metric components are C^1 functions of the new coordinates.
- The metric has a nonzero determinant at p.

Remark:

- Theorem 1, asserts a trade off between
  - a (non-removable) lack of C^1 metric-regularity
  - a vanishing metric determinant

Theorem 1, (R. and Temple, 2011)

Assume p is "a point of regular shock wave interaction in SSC". Then: $\nexists$ $C^{1,1}$ coordinate transformation, defined in a neighborhood of p, such that both holds:

- The metric components are C^1 functions of the new coordinates.
- The metric has a nonzero determinant at p.

Remark:

- Theorem 1, asserts a trade off between
  - a (non-removable) lack of C^1 metric-regularity
  - a vanishing metric determinant
- This is our motivation for calling points of shock wave interaction "*Regularity Singularities*".

Theorem 1, (R. and Temple, 2011)

Assume p is "a point of regular shock wave interaction in SSC". Then: $\nexists$ $C^{1,1}$ coordinate transformation, defined in a neighborhood of p, such that both holds:

- The metric components are C^1 functions of the new coordinates.
- The metric has a nonzero determinant at p.

<u>Remark:</u>

We only require two shock waves to be present before (or after) the interaction.

Theorem 1, (R. and Temple, 2011)

Assume p is "a point of regular shock wave interaction in SSC". Then: $\nexists$ $C^{1,1}$ coordinate transformation, defined in a neighborhood of p, such that both holds:

- The metric components are C^1 functions of the new coordinates.
- The metric has a nonzero determinant at p.

Remark:
We only require two shock waves to be present before (or after) the interaction.
$\Longrightarrow$ We address many physical shock wave interaction, e.g.:

Theorem 1, (R. and Temple, 2011)

Assume p is "a point of regular shock wave interaction in SSC". Then: $\nexists$ $C^{1,1}$ coordinate transformation, defined in a neighborhood of p, such that both holds:

- The metric components are C^1 functions of the new coordinates.
- The metric has a nonzero determinant at p.

Remark:

We only require two shock waves to be present before (or after) the interaction.

$\Longrightarrow$ We address many physical shock wave interaction, e.g.:

- two shock waves come in; two shock waves go out
- two shock waves come in; one shock and one rarefaction wave go out
- two compression waves come in; two shock waves go out

Theorem 1, (R. and Temple, 2011)

Assume p is "a point of regular shock wave interaction in SSC". Then: $\nexists$ $C^{1,1}$ coordinate transformation, defined in a neighborhood of p, such that both holds:

- The metric components are C^1 functions of the new coordinates.
- The metric has a nonzero determinant at p.

Remark on Proof:

The main step is to prove the result for a smaller atlas first,

Theorem 1, (R. and Temple, 2011)

Assume p is "a point of regular shock wave interaction in SSC". Then: $\nexists$ $C^{1,1}$ coordinate transformation, defined in a neighborhood of p, such that both holds:

- The metric components are C^1 functions of the new coordinates.
- The metric has a nonzero determinant at p.

Remark on Proof:

The main step is to prove the result for a smaller atlas first, namely, the atlas consisting of "**coordinate transformations of the** (t, r)**-plane**", i.e., transformations which keep the SSC angular variables fixed.

Theorem 1, (R. and Temple, 2011)

Assume p is "a point of regular shock wave interaction in SSC". Then: $\nexists$ $C^{1,1}$ coordinate transformation, defined in a neighborhood of p, such that both holds:

- The metric components are C^1 functions of the new coordinates.
- The metric has a nonzero determinant at p.

Remark on Proof:

The main step is to prove the result for a smaller atlas first, namely, the atlas consisting of "**coordinate transformations of the** (t, r)**-plane**", i.e., transformations which keep the SSC angular variables fixed. (My presentation is restricted to this part.)

Part VI

The Proof of Theorem 1

Outline of the proof (Thm 2):

Outline of the proof (Thm 2):

(i) Assume J^{μ}_{α} is the Jacobian of a coordinate transformation smoothing the metric from $C^{0,1}$ to C^1.

Outline of the proof (Thm 2):

(i) Assume J^μ_α is the Jacobian of a coordinate transformation smoothing the metric from $C^{0,1}$ to C^1. We derive a condition J^μ_α **must** meet at each shock curve.

Outline of the proof (Thm 2):

(i) Assume J^μ_α is the Jacobian of a coordinate transformation smoothing the metric from $C^{0,1}$ to C^1. We derive a condition J^μ_α **must** meet at each shock curve.

(ii) We characterize all J^μ_α, (defined on a neighborhood of shocks), satisfying that condition, by deriving an explicit form to represent them in.

Outline of the proof (Thm 2):

(i) Assume J^μ_α is the Jacobian of a coordinate transformation smoothing the metric from $C^{0,1}$ to C^1. We derive a condition J^μ_α **must** meet at each shock curve.

(ii) We characterize all J^μ_α, (defined on a neighborhood of shocks), satisfying that condition, by deriving an explicit form to represent them in.

(iii) Now, J^μ_α is integrable to coordinates,

Outline of the proof (Thm 2):

(i) Assume J^μ_α is the Jacobian of a coordinate transformation smoothing the metric from $C^{0,1}$ to C^1. We derive a condition J^μ_α **must** meet at each shock curve.

(ii) We characterize all J^μ_α, (defined on a neighborhood of shocks), satisfying that condition, by deriving an explicit form to represent them in.

(iii) Now, J^μ_α is integrable to coordinates,

$$\implies \quad J^\mu_{\alpha,\beta} = J^\mu_{\beta,\alpha}.$$

Outline of the proof (Thm 2):

(i) Assume J^μ_α is the Jacobian of a coordinate transformation smoothing the metric from $C^{0,1}$ to C^1. We derive a condition J^μ_α **must** meet at each shock curve.

(ii) We characterize all J^μ_α, (defined on a neighborhood of shocks), satisfying that condition, by deriving an explicit form to represent them in.

(iii) Now, J^μ_α is integrable to coordinates,

$$\implies \quad J^\mu_{\alpha,\beta} = J^\mu_{\beta,\alpha}.$$

Taking limit to point of interaction p of above equation yields

$$Det\left(g_{\alpha\beta}(p)\right) = 0.$$

Proof:

Proof:Step (i):

- Assume (for contradiction) there exist coordinates x^α, such that the transformed metric,

$$g_{\alpha\beta} = J^\mu_\alpha J^\nu_\beta g_{\mu\nu}, \tag{1}$$

is in C^1,

Proof:Step (i):

- Assume (for contradiction) there exist coordinates x^α, such that the transformed metric,

$$g_{\alpha\beta} = J^\mu_\alpha J^\nu_\beta g_{\mu\nu}, \qquad (1)$$

is in C^1, where $J^\mu_\alpha = \frac{\partial x^\mu}{\partial x^\alpha}$ (Jacobian) and $g_{\mu\nu}$ metric in SSC.

Proof:Step (i):

- Assume (for contradiction) there exist coordinates x^α, such that the transformed metric,

$$g_{\alpha\beta} = J^\mu_\alpha J^\nu_\beta g_{\mu\nu}, \tag{1}$$

is in C^1, where $J^\mu_\alpha = \frac{\partial x^\mu}{\partial x^\alpha}$ (Jacobian) and $g_{\mu\nu}$ metric in SSC. (Indices μ, ν, σ refer to SSC and α, β, γ to new coords.)

Proof: Step (i):

- Assume (for contradiction) there exist coordinates x^α, such that the transformed metric,

$$g_{\alpha\beta} = J^\mu_\alpha J^\nu_\beta g_{\mu\nu}, \tag{1}$$

is in C^1, where $J^\mu_\alpha = \frac{\partial x^\mu}{\partial x^\alpha}$ (Jacobian) and $g_{\mu\nu}$ metric in SSC. (Indices μ, ν, σ refer to SSC and α, β, γ to new coords.)

- Now, $g_{\alpha\beta}$ being in C^1 implies that, for all $\alpha, \beta, \gamma \in \{0, ..., 3\}$,

$$[g_{\alpha\beta,\gamma}]_i = 0. \tag{2}$$

Proof:Step (i):

- Assume (for contradiction) there exist coordinates x^α, such that the transformed metric,

$$g_{\alpha\beta} = J^\mu_\alpha J^\nu_\beta g_{\mu\nu}, \tag{1}$$

  is in C^1, where $J^\mu_\alpha = \frac{\partial x^\mu}{\partial x^\alpha}$ (Jacobian) and $g_{\mu\nu}$ metric in SSC. (Indices μ, ν, σ refer to SSC and α, β, γ to new coords.)
- Now, $g_{\alpha\beta}$ being in C^1 implies that, for all $\alpha, \beta, \gamma \in \{0, ..., 3\}$,

$$[g_{\alpha\beta,\gamma}]_i = 0. \tag{2}$$

  - $[\cdot]_i$ jump across the shock curve γ_i
  - $f_{,\gamma} := \frac{\partial f}{\partial x^\gamma}$ denotes differentiation w.r.t. new coords x^α.

Proof:Step (i):

- Assume (for contradiction) there exist coordinates x^α, such that the transformed metric,

$$g_{\alpha\beta} = J^\mu_\alpha J^\nu_\beta g_{\mu\nu}, \tag{1}$$

is in C^1, where $J^\mu_\alpha = \frac{\partial x^\mu}{\partial x^\alpha}$ (Jacobian) and $g_{\mu\nu}$ metric in SSC. (Indices μ, ν, σ refer to SSC and α, β, γ to new coords.)

- Now, $g_{\alpha\beta}$ being in C^1 implies that, for all $\alpha, \beta, \gamma \in \{0, ..., 3\}$,

$$[g_{\alpha\beta,\gamma}]_i = 0. \tag{2}$$

  - $[\cdot]_i$ jump across the shock curve γ_i
  - $f_{,\gamma} := \frac{\partial f}{\partial x^\gamma}$ denotes differentiation w.r.t. new coords x^α.

- Thus, differentiating the RHS of (1) and taking the jump leads to

$$[J^\mu_{\alpha,\gamma}]_i J^\nu_\beta g_{\mu\nu} + [J^\nu_{\beta,\gamma}]_i J^\mu_\alpha g_{\mu\nu} + J^\mu_\alpha J^\nu_\beta [g_{\mu\nu,\gamma}]_i = 0.$$

- $$[J^\mu_{\alpha,\gamma}]_i J^\nu_\beta g_{\mu\nu} + [J^\nu_{\beta,\gamma}]_i J^\mu_\alpha g_{\mu\nu} + J^\mu_\alpha J^\nu_\beta [g_{\mu\nu,\gamma}]_i = 0, \qquad (3)$$

  is a necessary condition for smoothing the metric.

- $$[J^{\mu}_{\alpha,\gamma}]_i J^{\nu}_{\beta} g_{\mu\nu} + [J^{\nu}_{\beta,\gamma}]_i J^{\mu}_{\alpha} g_{\mu\nu} + J^{\mu}_{\alpha} J^{\nu}_{\beta} [g_{\mu\nu,\gamma}]_i = 0, \qquad (3)$$

  is a necessary condition for smoothing the metric.
- (3) is linear in $[J^{\mu}_{\alpha,\gamma}]_i$.

- $$[J^\mu_{\alpha,\gamma}]_i J^\nu_\beta g_{\mu\nu} + [J^\nu_{\beta,\gamma}]_i J^\mu_\alpha g_{\mu\nu} + J^\mu_\alpha J^\nu_\beta [g_{\mu\nu,\gamma}]_i = 0, \qquad (3)$$

  is a necessary condition for smoothing the metric.
- (3) is linear in $[J^\mu_{\alpha,\gamma}]_i$.
- Equation (3) simplifies significantly once we
  - substitute the explicit form of the SSC metric, $g_{\mu\nu}$,
  - use our assumption, that the coord transfo only acts on the (t, r)-plane.

- $$[J^\mu_{\alpha,\gamma}]_i J^\nu_\beta g_{\mu\nu} + [J^\nu_{\beta,\gamma}]_i J^\mu_\alpha g_{\mu\nu} + J^\mu_\alpha J^\nu_\beta [g_{\mu\nu,\gamma}]_i = 0, \qquad (3)$$

  is a necessary condition for smoothing the metric.
- (3) is linear in $[J^\mu_{\alpha,\gamma}]_i$.
- Equation (3) simplifies significantly once we
  - substitute the explicit form of the SSC metric, $g_{\mu\nu}$,
  - use our assumption, that the coord transfo only acts on the (t, r)-plane.
- By assumption, J^μ_α satisfies the integrability condition, $J^\mu_{\alpha,\beta} = J^\mu_{\beta,\alpha}$,

- $$[J^{\mu}_{\alpha,\gamma}]_i J^{\nu}_{\beta} g_{\mu\nu} + [J^{\nu}_{\beta,\gamma}]_i J^{\mu}_{\alpha} g_{\mu\nu} + J^{\mu}_{\alpha} J^{\nu}_{\beta} [g_{\mu\nu,\gamma}]_i = 0, \qquad (3)$$

  is a necessary condition for smoothing the metric.
- (3) is linear in $[J^{\mu}_{\alpha,\gamma}]_i$.
- Equation (3) simplifies significantly once we
  - substitute the explicit form of the SSC metric, $g_{\mu\nu}$,
  - use our assumption, that the coord transfo only acts on the (t, r)-plane.
- By assumption, J^{μ}_{α} satisfies the integrability condition, $J^{\mu}_{\alpha,\beta} = J^{\mu}_{\beta,\alpha}$, which implies

$$[J^{\mu}_{\alpha,\beta}]_i = [J^{\mu}_{\beta,\alpha}]_i \ . \qquad (4)$$

- $$[J^\mu_{\alpha,\gamma}]_i J^\nu_\beta g_{\mu\nu} + [J^\nu_{\beta,\gamma}]_i J^\mu_\alpha g_{\mu\nu} + J^\mu_\alpha J^\nu_\beta [g_{\mu\nu,\gamma}]_i = 0, \qquad (3)$$

  is a necessary condition for smoothing the metric.
- (3) is linear in $[J^\mu_{\alpha,\gamma}]_i$.
- Equation (3) simplifies significantly once we
  - substitute the explicit form of the SSC metric, $g_{\mu\nu}$,
  - use our assumption, that the coord transfo only acts on the (t, r)-plane.
- By assumption, J^μ_α satisfies the integrability condition, $J^\mu_{\alpha,\beta} = J^\mu_{\beta,\alpha}$, which implies

$$[J^\mu_{\alpha,\beta}]_i = [J^\mu_{\beta,\alpha}]_i \ . \qquad (4)$$

- A long computation shows that the unique solution, $[J^\mu_{\alpha,\gamma}]_i$, of (3) together with (4) is given by:

- A long computation shows that the unique solution, $[J^\mu_{\alpha,\gamma}]_i$, of (3) together with (4) is given by:

$$\begin{aligned}
[J^t_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A}J^t_0 + \frac{[A_r]_i}{A}J^r_0\right); \quad [J^t_{0,r}]_i = -\frac{1}{2}\left(\frac{[A_r]_i}{A}J^t_0 + \frac{[B_t]_i}{A}J^r_0\right)\\
[J^t_{1,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A}J^t_1 + \frac{[A_r]_i}{A}J^r_1\right); \quad [J^t_{1,r}]_i = -\frac{1}{2}\left(\frac{[A_r]_i}{A}J^t_1 + \frac{[B_t]_i}{A}J^r_1\right)\\
[J^r_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_r]_i}{B}J^t_0 + \frac{[B_t]_i}{B}J^r_0\right); \quad [J^r_{0,r}]_i = -\frac{1}{2}\left(\frac{[B_t]_i}{B}J^t_0 + \frac{[B_r]_i}{B}J^r_0\right)\\
[J^r_{1,t}]_i &= -\frac{1}{2}\left(\frac{[A_r]_i}{B}J^t_1 + \frac{[B_t]_i}{B}J^r_1\right); \quad [J^r_{1,r}]_i = -\frac{1}{2}\left(\frac{[B_t]_i}{B}J^t_1 + \frac{[B_r]_i}{B}J^r_1\right). \quad (5)
\end{aligned}$$

- A long computation shows that the unique solution, $[J^{\mu}_{\alpha,\gamma}]_i$, of (3) together with (4) is given by:

$$
\begin{aligned}
[J^t_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A}J^t_0 + \frac{[A_r]_i}{A}J^r_0\right); \qquad [J^t_{0,r}]_i = -\frac{1}{2}\left(\frac{[A_r]_i}{A}J^t_0 + \frac{[B_t]_i}{A}J^r_0\right)\\
[J^t_{1,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A}J^t_1 + \frac{[A_r]_i}{A}J^r_1\right); \qquad [J^t_{1,r}]_i = -\frac{1}{2}\left(\frac{[A_r]_i}{A}J^t_1 + \frac{[B_t]_i}{A}J^r_1\right)\\
[J^r_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_r]_i}{B}J^t_0 + \frac{[B_t]_i}{B}J^r_0\right); \qquad [J^r_{0,r}]_i = -\frac{1}{2}\left(\frac{[B_t]_i}{B}J^t_0 + \frac{[B_r]_i}{B}J^r_0\right)\\
[J^r_{1,t}]_i &= -\frac{1}{2}\left(\frac{[A_r]_i}{B}J^t_1 + \frac{[B_t]_i}{B}J^r_1\right); \qquad [J^r_{1,r}]_i = -\frac{1}{2}\left(\frac{[B_t]_i}{B}J^t_1 + \frac{[B_r]_i}{B}J^r_1\right).(5)
\end{aligned}
$$

- Notation:
  - $A_t := \frac{\partial A}{\partial t}$, ...

- A long computation shows that the unique solution, $[J^{\mu}_{\alpha,\gamma}]_i$, of (3) together with (4) is given by:

$$
\begin{aligned}
[J^t_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A}J^t_0 + \frac{[A_r]_i}{A}J^r_0\right); \quad & [J^t_{0,r}]_i = -\frac{1}{2}\left(\frac{[A_r]_i}{A}J^t_0 + \frac{[B_t]_i}{A}J^r_0\right)\\
[J^t_{1,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A}J^t_1 + \frac{[A_r]_i}{A}J^r_1\right); \quad & [J^t_{1,r}]_i = -\frac{1}{2}\left(\frac{[A_r]_i}{A}J^t_1 + \frac{[B_t]_i}{A}J^r_1\right)\\
[J^r_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_r]_i}{B}J^t_0 + \frac{[B_t]_i}{B}J^r_0\right); \quad & [J^r_{0,r}]_i = -\frac{1}{2}\left(\frac{[B_t]_i}{B}J^t_0 + \frac{[B_r]_i}{B}J^r_0\right)\\
[J^r_{1,t}]_i &= -\frac{1}{2}\left(\frac{[A_r]_i}{B}J^t_1 + \frac{[B_t]_i}{B}J^r_1\right); \quad & [J^r_{1,r}]_i = -\frac{1}{2}\left(\frac{[B_t]_i}{B}J^t_1 + \frac{[B_r]_i}{B}J^r_1\right) .(5)
\end{aligned}
$$

- Notation:
  - $A_t := \frac{\partial A}{\partial t}$, ...
  - $\mu, \nu \in \{t, r\}$ and $\alpha, \beta \in \{0, 1\}$

- A long computation shows that the unique solution, $[J^{\mu}_{\alpha,\gamma}]_i$, of (3) together with (4) is given by:

$$
\begin{aligned}
[J^t_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A}J^t_0 + \frac{[A_r]_i}{A}J^r_0\right); \quad [J^t_{0,r}]_i = -\frac{1}{2}\left(\frac{[A_r]_i}{A}J^t_0 + \frac{[B_t]_i}{A}J^r_0\right)\\
[J^t_{1,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A}J^t_1 + \frac{[A_r]_i}{A}J^r_1\right); \quad [J^t_{1,r}]_i = -\frac{1}{2}\left(\frac{[A_r]_i}{A}J^t_1 + \frac{[B_t]_i}{A}J^r_1\right)\\
[J^r_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_r]_i}{B}J^t_0 + \frac{[B_t]_i}{B}J^r_0\right); \quad [J^r_{0,r}]_i = -\frac{1}{2}\left(\frac{[B_t]_i}{B}J^t_0 + \frac{[B_r]_i}{B}J^r_0\right)\\
[J^r_{1,t}]_i &= -\frac{1}{2}\left(\frac{[A_r]_i}{B}J^t_1 + \frac{[B_t]_i}{B}J^r_1\right); \quad [J^r_{1,r}]_i = -\frac{1}{2}\left(\frac{[B_t]_i}{B}J^t_1 + \frac{[B_r]_i}{B}J^r_1\right).
\end{aligned} \tag{5}
$$

- Notation:
  - $A_t := \frac{\partial A}{\partial t}$, ...
  - $\mu, \nu \in \{t, r\}$ and $\alpha, \beta \in \{0, 1\}$
  - J^t_0 denotes the $\mu = t$ and $\alpha = 0$ component of the Jacobian J^{μ}_{α}
- (5) is a necessary condition on $[J^{\mu}_{\alpha,\gamma}]_i$ for smoothing the metric to C^1.

- A long computation shows that the unique solution, $[J^{\mu}_{\alpha,\gamma}]_i$, of (3) together with (4) is given by:

$$
\begin{aligned}
[J^t_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A}J^t_0 + \frac{[A_r]_i}{A}J^r_0\right); \qquad [J^t_{0,r}]_i = -\frac{1}{2}\left(\frac{[A_r]_i}{A}J^t_0 + \frac{[B_t]_i}{A}J^r_0\right)\\
[J^t_{1,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A}J^t_1 + \frac{[A_r]_i}{A}J^r_1\right); \qquad [J^t_{1,r}]_i = -\frac{1}{2}\left(\frac{[A_r]_i}{A}J^t_1 + \frac{[B_t]_i}{A}J^r_1\right)\\
[J^r_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_r]_i}{B}J^t_0 + \frac{[B_t]_i}{B}J^r_0\right); \qquad [J^r_{0,r}]_i = -\frac{1}{2}\left(\frac{[B_t]_i}{B}J^t_0 + \frac{[B_r]_i}{B}J^r_0\right)\\
[J^r_{1,t}]_i &= -\frac{1}{2}\left(\frac{[A_r]_i}{B}J^t_1 + \frac{[B_t]_i}{B}J^r_1\right); \qquad [J^r_{1,r}]_i = -\frac{1}{2}\left(\frac{[B_t]_i}{B}J^t_1 + \frac{[B_r]_i}{B}J^r_1\right).(5)
\end{aligned}
$$

- Notation:
  - $A_t := \frac{\partial A}{\partial t}$, ...
  - $\mu, \nu \in \{t, r\}$ and $\alpha, \beta \in \{0, 1\}$
  - J^t_0 denotes the $\mu = t$ and $\alpha = 0$ component of the Jacobian J^{μ}_{α}
- (5) is a necessary condition on $[J^{\mu}_{\alpha,\gamma}]_i$ for smoothing the metric to C^1.

  It's only defined on the shock curves!

Step (ii):

- Next, we characterize all $C^{0,1}$-functions, defined on some open neighborhood $\mathcal{N}$ of p, that meet (5).

Step (ii):

- Next, we characterize all $C^{0,1}$-functions, defined on some open neighborhood $\mathcal{N}$ of p, that meet (5).
- To understand how this is done, we illustrate the procedure for J_0^t.

- By (5), the jump of the derivatives of J_0^t across $\gamma_i(t) = (t, x_i(t))$ should satisfy

$$\begin{aligned} [J^t_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A}J^t_0 + \frac{[A_r]_i}{A}J^r_0\right) \\ \text{and} \quad [J^t_{0,r}]_i &= -\frac{1}{2}\left(\frac{[A_r]_i}{A}J^t_0 + \frac{[B_t]_i}{A}J^r_0\right). \end{aligned} \tag{6}$$

- By (5), the jump of the derivatives of J_0^t across $\gamma_i(t) = (t, x_i(t))$ should satisfy

$$\begin{aligned} [J^t_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A}J^t_0 + \frac{[A_r]_i}{A}J^r_0\right) \\ \text{and} \quad [J^t_{0,r}]_i &= -\frac{1}{2}\left(\frac{[A_r]_i}{A}J^t_0 + \frac{[B_t]_i}{A}J^r_0\right). \end{aligned} \tag{6}$$

- Introduce $\quad J_0^t(t,r) = \sum_i \alpha_i(t)\,|x_i(t) - r| + \Phi(t,r)$,

- By (5), the jump of the derivatives of J_0^t across $\gamma_i(t) = (t, x_i(t))$ should satisfy

$$\begin{aligned} [J^t_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A} J_0^t + \frac{[A_r]_i}{A} J_0^r\right) \\ \text{and} \quad [J^t_{0,r}]_i &= -\frac{1}{2}\left(\frac{[A_r]_i}{A} J_0^t + \frac{[B_t]_i}{A} J_0^r\right). \end{aligned} \tag{6}$$

- Introduce $\quad J_0^t(t,r) = \sum_i \alpha_i(t)\,|x_i(t) - r| + \Phi(t,r)$,
  - Φ some function C^1 across γ_i.
  - $\alpha_i(t) := \frac{1}{4A\circ\gamma_i(t)}\left([A_r]_i\, J_0^t \circ \gamma_i(t) + [B_t]_i\, J_0^r \circ \gamma_i(t)\right),$

- By (5), the jump of the derivatives of J_0^t across $\gamma_i(t) = (t, x_i(t))$ should satisfy

$$
\begin{aligned}
[J^t_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A}J_0^t + \frac{[A_r]_i}{A}J_0^r\right) \\
\text{and} \quad [J^t_{0,r}]_i &= -\frac{1}{2}\left(\frac{[A_r]_i}{A}J_0^t + \frac{[B_t]_i}{A}J_0^r\right).
\end{aligned}
\tag{6}
$$

- Introduce $\quad J_0^t(t,r) = \sum_i \alpha_i(t)\,|x_i(t) - r| + \Phi(t,r)\ ,$
    - Φ some function C^1 across γ_i.
    - $\alpha_i(t) := \frac{1}{4A\circ\gamma_i(t)}\left([A_r]_i\; J_0^t \circ \gamma_i(t) + [B_t]_i\; J_0^r \circ \gamma_i(t)\right),$
- $J_0^t(t,r)$ satisfies (6), since:

- By (5), the jump of the derivatives of J_0^t across $\gamma_i(t) = (t, x_i(t))$ should satisfy

$$\begin{aligned} [J^t_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A} J_0^t + \frac{[A_r]_i}{A} J_0^r\right) \\ \text{and} \quad [J^t_{0,r}]_i &= -\frac{1}{2}\left(\frac{[A_r]_i}{A} J_0^t + \frac{[B_t]_i}{A} J_0^r\right). \end{aligned} \tag{6}$$

- Introduce $\quad J_0^t(t,r) = \sum_i \alpha_i(t)\,|x_i(t) - r| + \Phi(t,r)$,
  - Φ some function C^1 across γ_i.
  - $\alpha_i(t) := \frac{1}{4A\circ\gamma_i(t)}\left([A_r]_i\, J_0^t \circ \gamma_i(t) + [B_t]_i\, J_0^r \circ \gamma_i(t)\right)$,
- $J_0^t(t,r)$ satisfies (6), since:
  - The value of $[J^t_{0,r}]_i$ follows from:

- By (5), the jump of the derivatives of J_0^t across $\gamma_i(t) = (t, x_i(t))$ should satisfy

$$\begin{aligned} [J^t_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A}J_0^t + \frac{[A_r]_i}{A}J_0^r\right) \\ \text{and} \quad [J^t_{0,r}]_i &= -\frac{1}{2}\left(\frac{[A_r]_i}{A}J_0^t + \frac{[B_t]_i}{A}J_0^r\right). \end{aligned} \tag{6}$$

- Introduce $J_0^t(t,r) = \sum_i \alpha_i(t)\,|x_i(t) - r| + \Phi(t,r)$,
  - Φ some function C^1 across γ_i.
  - $\alpha_i(t) := \frac{1}{4A\circ\gamma_i(t)}\left([A_r]_i\; J_0^t\circ\gamma_i(t) + [B_t]_i\; J_0^r\circ\gamma_i(t)\right)$,
- $J_0^t(t,r)$ satisfies (6), since:
  - The value of $[J^t_{0,r}]_i$ follows from:
    - $\frac{d}{dX}|X| = H(X)$, for the Heaviside function H,
    - and $[H(x_i(t) - r)]_j \;=\; 2\delta_{ij}$.

- By (5), the jump of the derivatives of J_0^t across $\gamma_i(t) = (t, x_i(t))$ should satisfy

$$\begin{aligned} [J^t_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A} J^t_0 + \frac{[A_r]_i}{A} J^r_0\right) \\ \text{and} \quad [J^t_{0,r}]_i &= -\frac{1}{2}\left(\frac{[A_r]_i}{A} J^t_0 + \frac{[B_t]_i}{A} J^r_0\right). \end{aligned} \tag{6}$$

- Introduce $\quad J_0^t(t,r) = \sum_i \alpha_i(t)\,|x_i(t) - r| + \Phi(t,r)$,
  - Φ some function C^1 across γ_i.
  - $\alpha_i(t) := \frac{1}{4A\circ\gamma_i(t)}\left([A_r]_i\; J_0^t \circ \gamma_i(t) + [B_t]_i\; J_0^r \circ \gamma_i(t)\right)$,
- $J_0^t(t,r)$ satisfies (6), since:
  - The value of $[J^t_{0,r}]_i$ follows from:
    - $\frac{d}{dX}|X| = H(X)$, for the Heaviside function H,
    - and $[H(x_i(t) - r)]_j = 2\delta_{ij}$.
  - The required value of $[J^t_{0,t}]_i$ follows from the identities:
    - $[A_r]_i = -\dot{x}_i [B_t]_i$, (by RH jump condition and Einstein eqns).

- By (5), the jump of the derivatives of J_0^t across $\gamma_i(t) = (t, x_i(t))$ should satisfy

$$\begin{aligned} [J^t_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A}J_0^t + \frac{[A_r]_i}{A}J_0^r\right) \\ \text{and} \quad [J^t_{0,r}]_i &= -\frac{1}{2}\left(\frac{[A_r]_i}{A}J_0^t + \frac{[B_t]_i}{A}J_0^r\right). \end{aligned} \tag{6}$$

- Introduce $\quad J_0^t(t,r) = \sum_i \alpha_i(t)\,|x_i(t) - r| + \Phi(t,r)\,,$
  - Φ some function C^1 across γ_i.
  - $\alpha_i(t) \;:=\; \frac{1}{4A\circ\gamma_i(t)}\left([A_r]_i\; J_0^t\circ\gamma_i(t) + [B_t]_i\; J_0^r\circ\gamma_i(t)\right),$
- $J_0^t(t,r)$ satisfies (6), since:
  - The value of $[J^t_{0,r}]_i$ follows from:
    - $\frac{d}{dX}|X| = H(X)$, for the Heaviside function H,
    - and $[H(x_i(t) - r)]_j \;=\; 2\delta_{ij}$.
  - The required value of $[J^t_{0,t}]_i$ follows from the identities:
    - $[A_r]_i = -\dot{x}_i[B_t]_i$, (by RH jump condition and Einstein eqns).
    - $\dot{x}_i[A_r]_i = -[A_t]_i$, (by smoothness of $g_{\mu\nu}$ along shocks).

- In fact, all functions that meet (5) are of the above form,

$$J_0^t(t,r) = \sum_i \alpha_i(t)\,|x_i(t) - r| + \Phi(t,r)\ ,$$

- In fact, all functions that meet (5) are of the above form,

$$J_0^t(t,r) = \sum_i \alpha_i(t)\,|x_i(t) - r| + \Phi(t,r)\ ,$$

since $J_0^t(t,r) - \sum_i \alpha_i(t)\,|x_i(t) - r|$ is a function C^1 across γ_i.

- In fact, all functions that meet (5) are of the above form,

$$J_0^t(t,r) = \sum_i \alpha_i(t)\,|x_i(t) - r| + \Phi(t,r)\ ,$$

since $J_0^t(t,r) - \sum_i \alpha_i(t)\,|x_i(t) - r|$ is a function C^1 across γ_i.

- In summary, we obtain the following Lemma:

Lemma

If the RH jump condition hold, then there exists a set of functions $J^\mu_\alpha \in C^{0,1}(\mathcal{N} \cap \overline{\mathbb{R}^2_-})$ that satisfies the smoothing condition (5) on $\gamma_i \cap \mathcal{N}$, $(i = 1, 2)$. All such J^μ_α assume the canonical form

$$
\begin{aligned}
J^t_0(t,r) &= \sum_i \alpha_i(t)\,|x_i(t) - r| + \Phi(t,r)\,, \quad & \alpha_i(t) &= \frac{[A_r]_i\;\phi_i(t) + [B_t]_i\;\omega_i(t)}{4A \circ \gamma_i(t)}\,,\\
J^t_1(t,r) &= \sum_i \beta_i(t)\,|x_i(t) - r| + N(t,r)\,, \quad & \beta_i(t) &= \frac{[A_r]_i\;\nu_i(t) + [B_t]_i\;\zeta_i(t)}{4A \circ \gamma_i(t)}\,,\\
J^r_0(t,r) &= \sum_i \delta_i(t)\,|x_i(t) - r| + \Omega(t,r)\,, \quad & \delta_i(t) &= \frac{[B_t]_i\;\phi_i(t) + [B_r]_i\;\omega_i(t)}{4B \circ \gamma_i(t)}\,,\\
J^r_1(t,r) &= \sum_i \epsilon_i(t)\,|x_i(t) - r| + Z(t,r)\,, \quad & \epsilon_i(t) &= \frac{[B_t]_i\;\nu_i(t) + [B_r]_i\;\zeta_i(t)}{4B \circ \gamma_i(t)}\,, \quad (7)
\end{aligned}
$$

where

$$\phi_i = \Phi \circ \gamma_i, \quad \omega_i = \Omega \circ \gamma_i, \quad \zeta_i = Z \circ \gamma_i, \quad \nu_i = N \circ \gamma_i\,, \tag{8}$$

and $\Phi, \Omega, Z, N \in C^{0,1}(\mathcal{N} \cap \overline{\mathbb{R}^2_-})$ have matching derivatives on each shock curve $\gamma_i(t)$,

$$[U_r]_i = 0 = [U_t]_i\,, \tag{9}$$

for $U = \Phi, \Omega, Z, N$, $t \in (-\epsilon, 0)$.

Step (iii):

- By assumption, the Jacobian are integrable to coordinates and thus satisfy $J^{\mu}_{\alpha,\beta} = J^{\mu}_{\beta,\alpha}$,

Step (iii):

- By assumption, the Jacobian are integrable to coordinates and thus satisfy $J^{\mu}_{\alpha,\beta} = J^{\mu}_{\beta,\alpha}$, which is equivalent to

$$J^{\mu}_{\alpha,\nu} J^{\nu}_{\beta} = J^{\mu}_{\beta,\nu} J^{\nu}_{\alpha} \ .$$

Step (iii):

- By assumption, the Jacobian are integrable to coordinates and thus satisfy $J^\mu_{\alpha,\beta} = J^\mu_{\beta,\alpha}$, which is equivalent to

$$J^\mu_{\alpha,\nu} J^\nu_\beta = J^\mu_{\beta,\nu} J^\nu_\alpha \; .$$

- Moreover, the Jacobian must assume the canonical form (7).

Step (iii):

- By assumption, the Jacobian are integrable to coordinates and thus satisfy $J^{\mu}_{\alpha,\beta} = J^{\mu}_{\beta,\alpha}$, which is equivalent to

$$J^{\mu}_{\alpha,\nu} J^{\nu}_{\beta} = J^{\mu}_{\beta,\nu} J^{\nu}_{\alpha} \ .$$

- Moreover, the Jacobian must assume the canonical form (7).
- Substituting the canonical form (7) into the above integrability condition and taking the jump across any of the shocks, (WLOG across γ_1),

Step (iii):

- By assumption, the Jacobian are integrable to coordinates and thus satisfy $J^\mu_{\alpha,\beta} = J^\mu_{\beta,\alpha}$, which is equivalent to

$$J^\mu_{\alpha,\nu} J^\nu_\beta = J^\mu_{\beta,\nu} J^\nu_\alpha \ .$$

- Moreover, the Jacobian must assume the canonical form (7).
- Substituting the canonical form (7) into the above integrability condition and taking the jump across any of the shocks, (WLOG across γ_1), implies that for all $t < 0$,

$$\delta_1(t)\dot{x}_1(t)\beta_2(t) - \epsilon_1(t)\dot{x}_1(t)\alpha_2(t) + \epsilon_1(t)\delta_2(t) - \delta_1(t)\epsilon_2(t) = 0.$$

- Taking the limit $t \to 0^+$ of

$$\delta_1(t)\dot{x}_1(t)\beta_2(t) - \epsilon_1(t)\dot{x}_1(t)\alpha_2(t) + \epsilon_1(t)\delta_2(t) - \delta_1(t)\epsilon_2(t) = 0,$$

gives

$$\frac{1}{4B}\left(\frac{\dot{x}_1\dot{x}_2}{A} + \frac{1}{B}\right)[B_r]_1[B_r]_2\left(\dot{x}_1 - \dot{x}_2\right)\left(\phi_0\zeta_0 - \nu_0\omega_0\right) = 0, \tag{10}$$

- Taking the limit $t \to 0^+$ of

$$\delta_1(t)\dot{x}_1(t)\beta_2(t) - \epsilon_1(t)\dot{x}_1(t)\alpha_2(t) + \epsilon_1(t)\delta_2(t) - \delta_1(t)\epsilon_2(t) = 0,$$

gives

$$\frac{1}{4B}\left(\frac{\dot{x}_1\dot{x}_2}{A} + \frac{1}{B}\right)[B_r]_1[B_r]_2\left(\dot{x}_1 - \dot{x}_2\right)\left(\phi_0\zeta_0 - \nu_0\omega_0\right) = 0, \tag{10}$$

where

  - $\phi_0 = \lim_{t \to 0^+} \phi_1(t) = \lim_{t \to 0^+} \phi_2(t)$
  - $\phi_i(t) := \Phi \circ \gamma_i(t)$
  - $\zeta_0, ..., \omega_0$ defined analogously.

- In (10), that is,

$$\frac{1}{4B}\left(\frac{\dot{x}_1\dot{x}_2}{A}+\frac{1}{B}\right)[B_r]_1[B_r]_2\,(\dot{x}_1-\dot{x}_2)\,(\phi_0\zeta_0-\nu_0\omega_0)=0,$$

all factors must be nonzero, except the last one,

- In (10), that is,

$$\frac{1}{4B}\left(\frac{\dot{x}_1\dot{x}_2}{A}+\frac{1}{B}\right)[B_r]_1[B_r]_2\left(\dot{x}_1-\dot{x}_2\right)\left(\phi_0\zeta_0-\nu_0\omega_0\right)=0,$$

all factors must be nonzero, except the last one, thus

$$\phi_0\zeta_0-\nu_0\omega_0=0. \tag{11}$$

- In (10), that is,

$$\frac{1}{4B}\left(\frac{\dot{x}_1\dot{x}_2}{A}+\frac{1}{B}\right)[B_r]_1[B_r]_2\left(\dot{x}_1-\dot{x}_2\right)\left(\phi_0\zeta_0-\nu_0\omega_0\right)=0,$$

  all factors must be nonzero, except the last one, thus

$$\phi_0\zeta_0-\nu_0\omega_0=0. \tag{11}$$

- However,

$$Det\ \left(J^\mu_\alpha\circ\gamma_i(t)\right)=\left(J^t_0J^r_1-J^t_1J^r_0\right)|_{\gamma_i(t)}=\phi_i(t)\zeta_i(t)-\nu_i(t)\omega_i(t).$$

- In (10), that is,

$$\frac{1}{4B}\left(\frac{\dot{x}_1\dot{x}_2}{A}+\frac{1}{B}\right)[B_r]_1[B_r]_2\left(\dot{x}_1-\dot{x}_2\right)\left(\phi_0\zeta_0-\nu_0\omega_0\right)=0,$$

all factors must be nonzero, except the last one, thus

$$\phi_0\zeta_0-\nu_0\omega_0=0. \tag{11}$$

- However,

$$Det\ \left(J^\mu_\alpha\circ\gamma_i(t)\right)=\left(J^t_0J^r_1-J^t_1J^r_0\right)|_{\gamma_i(t)}=\phi_i(t)\zeta_i(t)-\nu_i(t)\omega_i(t).$$

- Thus, taking the limit $t\to 0^+$ and using (11), yields

$$\lim_{t\to 0^+} Det\ \left(J^\mu_\alpha\circ\gamma_i(t)\right)=\phi_i(0)\zeta_i(0)-\nu_i(0)\omega_i(0)=\phi_0\zeta_0-\nu_0\omega_0=0.$$

- In (10), that is,

$$\frac{1}{4B}\left(\frac{\dot{x}_1\dot{x}_2}{A}+\frac{1}{B}\right)[B_r]_1[B_r]_2\left(\dot{x}_1-\dot{x}_2\right)\left(\phi_0\zeta_0-\nu_0\omega_0\right)=0,$$

all factors must be nonzero, except the last one, thus

$$\phi_0\zeta_0-\nu_0\omega_0=0. \tag{11}$$

- However,

$$Det\ \left(J^\mu_\alpha\circ\gamma_i(t)\right)=\left(J^t_0J^r_1-J^t_1J^r_0\right)|_{\gamma_i(t)}=\phi_i(t)\zeta_i(t)-\nu_i(t)\omega_i(t).$$

- Thus, taking the limit $t\to 0^+$ and using (11), yields

$$\lim_{t\to 0^+} Det\ \left(J^\mu_\alpha\circ\gamma_i(t)\right)=\phi_i(0)\zeta_i(0)-\nu_i(0)\omega_i(0)=\phi_0\zeta_0-\nu_0\omega_0=0.$$

- This completes the proof, since $g_{\alpha\beta}=J^\mu_\alpha J^\nu_\beta g_{\mu\nu}$. $\square$

- So far, we've established that there is no coordinate transformation of the (t, r)-plane that smoothes the SSC-metric, $g_{\mu\nu}$, to C^1.
- To prove our main Theorem we just need to extend the above result to the full atlas.
- Recall our main Theorem:

Theorem 1, (R. and Temple, 2011)

Assume p is "a point of regular shock wave interaction in SSC". Then: $\nexists$ $C^{1,1}$ coordinate transformation, defined in a neighborhood of p, such that both holds:

- The metric components are C^1 functions of the new coordinates.
- The metric has a nonzero determinant at p.

Outline of Proof:

- Assume there exist coordinates, such that the metric in the new coordinates, $g_{\alpha\beta}$, is in C^1.
- In general, $g_{\alpha\beta}$ is not of the box-diagonal form,

$$ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + 2D(t,r)dtdr + C(t,r)d\Omega^2 \,. \tag{12}$$

- However, (following the arguments in [Weinberg, *Gravitation and Cosmology*]), there exists a coordinate transformation that takes $g_{\alpha\beta}$ over to a metric of the form (12) and preserves the metric regularity.
- (Remark: A crucial step is to prove a C^1 regularity of solutions of Killing's equation, for a given C^1 metric.)
- But (12) is related to our original SSC metric, $g_{\mu\nu}$, by a transformation in the (t,r)-plane, contradicting Theorem 2. □

Part VII

Conclusion and Discussion

Conclusion:

Conclusion:

- At points, p, of regular shock interaction in SSC the gravitational metric suffers a **non-removable** lack of C^1 regularity.

Conclusion:

- At points, p, of regular shock interaction in SSC the gravitational metric suffers a **non-removable** lack of C^1 regularity.
- The Einstein equations cannot hold strongly (only weakly) in any coordinate system.

Conclusion:

- At points, p, of regular shock interaction in SSC the gravitational metric suffers a **non-removable** lack of C^1 regularity.
- The Einstein equations cannot hold strongly (only weakly) in any coordinate system.
- At p, spacetime is **not locally flat**, that is, there do not exist coordinates x^j, such that the metric satisfies:
  - $g_{ij}(p) = \eta_{ij}$, where $\eta_{ij} = diag(-1, 1, 1, 1)$,
  - $g_{ij,l}(p) = 0$,
  - $g_{ij,kl}$ are bounded on some neighborhood of p.

Conclusion:

- At points, p, of regular shock interaction in SSC the gravitational metric suffers a **non-removable** lack of C^1 regularity.
- The Einstein equations cannot hold strongly (only weakly) in any coordinate system.
- At p, spacetime is **not locally flat**, that is, there do not exist coordinates x^j, such that the metric satisfies:
  - $g_{ij}(p) = \eta_{ij}$, where $\eta_{ij} = diag(-1, 1, 1, 1)$,
  - $g_{ij,l}(p) = 0$,
  - $g_{ij,kl}$ are bounded on some neighborhood of p.
- In particular, there exist (non-removable) distributional second order metric derivatives.

Conclusion:

- At points, p, of regular shock interaction in SSC the gravitational metric suffers a **non-removable** lack of C^1 regularity.
- The Einstein equations cannot hold strongly (only weakly) in any coordinate system.
- At p, spacetime is **not locally flat**, that is, there do not exist coordinates x^j, such that the metric satisfies:
  - $g_{ij}(p) = \eta_{ij}$, where $\eta_{ij} = diag(-1, 1, 1, 1)$,
  - $g_{ij,l}(p) = 0$,
  - $g_{ij,kl}$ are bounded on some neighborhood of p.
- In particular, there exist (non-removable) distributional second order metric derivatives.
- These distributional derivatives are not hidden by an event horizon.
- However, all "curvature scalars" remain bounded.

Conclusion:

- At points, p, of regular shock interaction in SSC the gravitational metric suffers a **non-removable** lack of C^1 regularity.
- The Einstein equations cannot hold strongly (only weakly) in any coordinate system.
- At p, spacetime is **not locally flat**, that is, there do not exist coordinates x^j, such that the metric satisfies:
  - $g_{ij}(p) = \eta_{ij}$, where $\eta_{ij} = diag(-1, 1, 1, 1)$,
  - $g_{ij,l}(p) = 0$,
  - $g_{ij,kl}$ are bounded on some neighborhood of p.
- In particular, there exist (non-removable) distributional second order metric derivatives.
- These distributional derivatives are not hidden by an event horizon.
- However, all "curvature scalars" remain bounded. ($\Rightarrow$ No naked singularities!)

Discussion:

Discussion:

- Having unbounded second order metric derivatives, but no event horizon, regularity singularities might be measurable. What could be such a measurable effect?

Discussion:

- Having unbounded second order metric derivatives, but no event horizon, regularity singularities might be measurable. What could be such a measurable effect?
- Our Theorem applies to spherically symmetric spacetimes and radial shock waves only. Do regularity singularities persist, if we remove any of our symmetry assumptions?

(1) W. Israel, "Singular hypersurfaces and thin shells in general relativity", Il Nuovo Cimento, Vol. XLIV B, N. 1, 1966, pp. 1-14.

(2) J. Smoller and B. Temple, "Shock wave Solutions of the Einstein equations: The Oppenheimer-Snyder model of gravitational collapse extended to the case of non-zero pressure", Archive Rational Mechanics and Analysis, 128 (1994), pp. 249-297, Springer-Verlag 1994.

(3) M. Reintjes, B. Temple, "Points of General Relativistic Shock Wave Interaction are "Regularity Singularities" where Spacetime is Not Locally Flat", arXiv:1105.0798[gr-qc], submitted.

(4) J. Groah and B. Temple, *Shock-Wave Solutions of the Einstein Equations with Perfect Fluid Sources: Existence and Consistency by a Locally Inertial Glimm Scheme*, Memoirs AMS, Vol. 172, Number 813, November 2004, ISSN 0065-9266.

(5) Z. Vogler and B. Temple, "Simulation of General Relativistic Shock Waves by a Locally Inertial Godunov Method featuring Dynamical Time Dilation", Davis Preprint.

Thank you for your attention!