Points of General Relativistic Shock Wave Interaction are Regularity Singularities where Spacetime is Not Locally Flat

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(Joint work with Blake Temple.)
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Part I

Intuitive Introduction
What are shock waves?
What are shock waves?

Shock waves are discontinuities evolving in time.
Intuitive Introduction

M. Reintjes

GR Shock Interaction are Regularity Singularities
Where do shock waves appear?
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- Shock waves form in fluids and gases, governed by compressible Euler equations.
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- Shock waves form in fluids and gases, governed by compressible Euler equations.
- In General Relativity (GR), shock waves can be present in the matter content of spacetime.
Einstein equations:

“spacetime curvature $\simeq$ matter content”
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- In GR, shock waves are discontinuities in matter content.
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- In GR, shock waves are discontinuities in matter content.
- Spacetime curvature is determined by \textit{metric tensor}.
Einstein equations:

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- In GR, shock waves are discontinuities in matter content.
- Spacetime curvature is determined by metric tensor.

→ How do shock waves effect the metric tensor?
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- Moreover, many fundamental features of spacetime require a $C^{1,1}$ metric regularity.
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- Moreover, many fundamental features of spacetime require a $C^{1,1}$ metric regularity.

Question:
Can we raise the metric regularity to $C^{1,1}$ by transforming to different coordinates?
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Across a single shock wave: Yes! (Israel, 1966)

At point of shock wave interaction: No! (R. and Temple, 2011)

→ “Regularity Singularity”
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Part II

Background: Shock Waves in General Relativity
A manifold $M$ is a Hausdorff-space locally diffeomorphic to $\mathbb{R}^n$.
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$M$ 

$\text{covering } (U_i)_{i \in I}$ 

$U$, $U_1$, $U_2$
A manifold $M$ is a Hausdorff-space locally diffeomorphic to $\mathbb{R}^n$: 

\[ x : U \rightarrow \mathbb{R}^n \]
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\[ x \circ y^{-1} \text{ and } y \circ x^{-1} \] 

are called "change of coordinates". The collection of all such mappings and domains, $(x, U)$, is called a $C^k$-atlas.
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- collection of all such mappings and domains, $(x, U)$, is called a $C^k$-atlas
All geometrical information about $M$ (e.g., angles, curvature,...) is captured in the *Lorentz-metric* tensor, $g$. 

\[ g = g_{ij} dx^i dx^j := \sum_{i, j=1}^{n} g_{ij} dx^i dx^j. \]

Convention: sum over repeated indices, ($n = \dim(M)$).

Roughly: think of $dx^j$ as dual vector to $j$-th coordinate vector in $x$ ($U$). 

Pointwise, ($g_{ij}$)$_{1 \leq i, j \leq n}$ is a symmetric matrix which has signature (-+++). 

In new coord's, $g(x) = g_{\mu\nu}(y) dy^\nu dy^\nu$, the metric components transform as $g_{ij}(x) = J^\mu_i J^\nu_j g_{\mu\nu}(y(x))$, where $J^\mu_j := \partial x^\mu \circ y^{-1} \partial y^j$ denotes the Jacobian.
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All geometrical information about \( M \) (e.g., angles, curvature,...) is captured in the *Lorentz-metric* tensor, \( g \).

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In coord’s $x^j$, \[ g = g_{ij} \, dx^i \, dx^j := \sum_{i,j=1}^{n} g_{ij} \, dx^i \, dx^j. \]

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- Pointwise, \((g_{ij})_{1 \leq i,j \leq n}\) is a symmetric matrix which has signature \((-++++)\)
- In new coord’s, \( g(x) = g_{\mu\nu}(y)dy^\nu dy^\nu \), the metric components transform as

\[
    g_{ij}(x) = J_i^\mu J_j^\nu g_{\mu\nu}(y(x)),
\]

where \( J_i^\mu := \frac{\partial x^\mu \circ y^{-1}}{\partial y^j} \) denotes the Jacobian.
Spacetime is a 4-D manifold with a Lorentz-metric (→ “Equivalence Principle”).
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- Spacetime curvature is described by Einstein tensor, $G^{\mu\nu}$, via “measuring failure of 2\textsuperscript{nd} order (covariant) derivatives to commute”.
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- \( G^{\mu\nu} \) comprises entirely of the metric tensor, \( g_{\mu\nu} \), and its first and second derivatives.
- \( G \) is the unique (modulo a constant) curvature tensor being divergence-free, \( \text{div} \, G = 0 \), thus imposing conservation of energy in the Einstein equations.
A *single* shock wave being present in the Einstein equations is characterized by:

\[ T_{\mu\nu} \] is discontinuous across a hypersurface \( \Sigma \) and \( C_0 \) elsewhere. Across \( \Sigma \), the Rankine Hugoniot jump conditions hold, that is,

\[
\left[ T_{\mu\nu} \right] N^\nu = 0,
\]

where \( N^\nu \) normal to \( \Sigma \),

\[
[u] := u_L - u_R \text{ denotes the jump in } u \text{ across } \Sigma,
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\( u_L/R \) denotes the left/right limit of \( u \) to \( \Sigma \).

Einstein equations, \( G_{\mu\nu} = \kappa T_{\mu\nu} \), hold strongly off \( \Sigma \).
A *single* shock wave being present in the Einstein equations is characterized by:

- $T^{\mu\nu}$ is discontinuous across a hypersurface $\Sigma$ and $C^0$ elsewhere.

Across $\Sigma$, the Rankine Hugoniot jump conditions hold, that is,

$$\left[T^{\mu\nu}\right]_N = 0,$$

where $N^\nu$ normal to $\Sigma$, $[u] := u_L - u_R$ denotes the jump in $u$ across $\Sigma$, $u_L / R$ denotes the left/right limit of $u$ to $\Sigma$. The Einstein equations, $G^{\mu\nu} = \kappa T^{\mu\nu}$, hold strongly off $\Sigma$. 

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- Einstein equations, $G^{\mu\nu} = \kappa T^{\mu\nu}$, hold strongly off $\Sigma$. 
Remark:
The (probably) most important setting for shock waves in GR are the *Coupled Einstein Euler equations*,

\[ G_{\mu\nu} = \kappa T_{\mu\nu}, \]

\[ \text{div} T = 0, \]

where \( T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \) (perfect fluid), \( u \) tangent to fluid flow, \( \rho \) energy-density, \( p = p(\rho) \) pressure.

Shock waves can form in the relativistic Euler equations, \( \text{div} T = 0 \), out of smooth initial data.
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Part III

The Question of the Metric Regularity
Let’s illustrate the effect of shock waves on the metric regularity:
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- Choose Standard Schwarzschild Coordinates, that is, coord’s where the metric reads

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- Then, the first Einstein equation reads
  \[ \frac{\partial B}{\partial r} + B \frac{B - 1}{r} = \kappa A B^2 r \, T^{00}. \]

- Now, let a shock waves be present in \( T^{\mu\nu} \)
  \[ \implies T^{00} \text{ is discontinuous} \]
  \[ \implies \frac{\partial B}{\partial r} \text{ is discontinuous} \]
  \[ \rightarrow B \in C^{0,1} \setminus C^1 \]
Conclusion:

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Conclusion:

- In some coordinates, the Einstein equations contain first order differential equations.
- Thus, if shock waves are present in $T^{\mu\nu}$, the metric can only be in $C^{0,1}$, but not in $C^1$.

Central Question:

Do there exist coordinates $x^j$ such that the metric in the new coordinates, $g_{ij}$, is in $C^{1,1}$?
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What’s interesting about $C^{0,1}$ versus $C^{1,1}$ metric regularity?

- $C^{1,1}$-regularity is crucial to define curvature tensors, $G_{ij}$, $R_{ij}$, ..., in a classical (non-distributional) sense.
- $C^{1,1}$-regularity is required for the Einstein equations to hold strongly.

If one cannot smooth the metric to $C^{1,1}$, it cannot be locally Minkowski! ($\rightarrow$ No observer in free-fall?!)

$C^{1,1}$ is a quite common assumption in GR, e.g., $C^{1,1}$-regularity is required in Singularity Theorems (of Penrose, Hawking and Ellis).
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- If one cannot smooth the metric to $C^{1,1}$, it cannot be locally Minkowski! ($\rightarrow$ No observer in free-fall?!) 
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Part IV

The Metric Regularity Across a Single Shock Surface
“Israel’s Theorem“
(based on Israel 1966) (see also: Smoller and Temple 1994)

Suppose:

(M,g) a (Riemann) manifold with a
$C^1$, 1-atlas

$g_{\mu\nu}$ is $C^0$, 1 across a single smooth surface $\Sigma$,
($g_{\mu\nu}$ solves the Einstein equations strongly away from $\Sigma$)

Then the following is equivalent:

There exist coordinates $x^\alpha$ such that $g^{\alpha\beta} \in C^1$, 1,
(w.r.t. partial differentiation in $x^\alpha$).

The RH jump conditions, $[T_{\mu\nu}]_{N\nu} = 0$, hold on $\Sigma$ and
$T_{ij}$ is in $L^\infty$.

Lesson:
Across a single shock one can always lift metric regularity to $C^1$, 1!
“Israel’s Theorem“
(based on Israel 1966) (see also: Smoller and Temple 1994)

Suppose:

- $(M, g)$ a (Riemann) manifold with a $C^{1,1}$-atlas
- $g_{\mu\nu}$ is $C^{0,1}$ across a single smooth surface $\Sigma$,
- $(g_{\mu\nu}$ solves the Einstein equations strongly away from $\Sigma$)
“Israel’s Theorem“
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Suppose:
- \((M,g)\) a (Riemann) manifold with a \(C^{1,1}\)-atlas
- \(g_{\mu\nu}\) is \(C^{0,1}\) across a single smooth surface \(\Sigma\),
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Then the following is equivalent:

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\text{There exist coordinates } x^\alpha \text{ such that } g_{\alpha\beta} \in C^{1,1}, \quad \text{(w.r.t. partial differentiation in } x^\alpha)\]

\[
\text{The RH jump conditions, } [T_{\mu\nu}]_{\nu} = 0, \text{ hold on } \Sigma \text{ and } T_{ij} \text{ is in } L^\infty.
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Lesson:
Across a single shock one can always lift metric regularity to \(C^{1,1}\)!
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Suppose:

- $(M,g)$ a (Riemann) manifold with a $C^{1,1}$-atlas
- $g_{\mu\nu}$ is $C^{0,1}$ across a single smooth surface $\Sigma$,
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Then the following is equivalent:

- There exist coordinates $x^\alpha$ such that $g_{\alpha\beta} \in C^{1,1}$,
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  hold on \(\Sigma\) and \(T^i_j\) is in \(L^\infty\).

Lesson: Across a single shock one can always lift metric regularity to \(C^{1,1}\)!
Part V

The Metric Regularity at Points of Shock Wave Interaction
Israel’s Theorem addresses the metric regularity across a single shock surface only.

However, shock waves can interact. Can one still lift the metric regularity if two shock waves interact? **NO, one cannot!** (R. & Temple, 2011)

Before we state our theorem, let me introduce the shock wave interaction we consider:

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However, shock waves can **interact**.

Can one still lift the metric regularity if two shock waves interact?

**NO, one cannot!**

(R. & Temple, 2011)

Before we state our theorem, let me introduce the shock wave interaction we consider:
Points of Regular Shock Wave Interaction in SSC:

Assumption on spacetime:
Suppose spacetime $M$ is spherically symmetric. Assume Standard Schwarzschild Coordinates (=:SSC) exists around a point $p \in M$, that is, coord's $(t, r, \vartheta, \phi)$ where the metric reads $g = -A(t, r)dt^2 + B(t, r)dr^2 + r^2d\Omega^2$, with $d\Omega^2 := d\vartheta^2 + \sin^2(\vartheta)d\phi^2$.

Remark: Spherical symmetry, though being restrictive, includes many important spacetimes: Schwarzschild spacetime (outside of black hole or star), Oppenheimer-Tolman spacetime (inside of gaseous star), Friedman-Robertson-Walker spacetime (cosmology).
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- Suppose spacetime $M$ is *spherically symmetric*.
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Remark: Spherical symmetry, though being restrictive, includes many important spacetimes:

- Schwarzschild spacetime (outside of black hole or star),
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Assumption on shock waves:

Assume shock waves are radial, that is, the shock surfaces, $\Sigma_1$ and $\Sigma_2$, are 2-spheres evolving in time. More precisely, $\Sigma_i(t, \vartheta, \phi) = (t, x_i(t), \vartheta, \phi)$, $x_i(t) > 0$, $(i = 1, 2)$.

Note: $\Sigma_i(t)$ is a 2-sphere with radius $x_i(t)$ and center $r = 0$.

Instead of $\Sigma_i$ it suffices to consider curves $\gamma_i(t) = (t, x_i(t))$, (so-called “shock curves”).
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Definition

$p \in M$ is a “point of regular shock wave interaction in SSC” if

\[ \gamma_i(t) = \left( t, x_i(t) \right), \quad i = 1, 2, \]

are smooth timelike curves defined on $t \in (-\epsilon, 0)$. $\gamma_1$ and $\gamma_2$ intersect in $p = \gamma_1(0) = \gamma_2(0)$.

The SSC-metric, $g_{\mu\nu}$, is only $C^0, 1$ across each shock curve and $C^2$ off and along them. (Einstein equations hold strongly off $\Sigma_i$.) Rankine Hugoniot conditions, $[T_{\mu\nu}]_{i} (N_i)_{\nu} = 0$, hold across each $\gamma_i$, (for $i = 1, 2$), and in the limit to $t \rightarrow 0$.

Shocks interact with distinct speeds, $\dot{x}_1(0) \neq \dot{x}_2(0)$. So, $p$ is a 2-sphere with radius $x_1(0) = x_2(0)$ and center $r = 0$.

We expect this structure to be generic, for radial shock waves in spherical symmetry!
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M. Reintjes  
GR Shock Interaction are Regularity Singularities
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- Existence before and after interaction was established by Groah and Temple (2005).
Let’s state our main theorem:
Theorem 1, (R. and Temple, 2011)

Assume \( p \) is “a point of regular shock wave interaction in SSC”. Then: \( \not\exists \ C^{1,1} \) coordinate transformation, defined in a neighborhood of \( p \), such that both holds:

- The metric components are \( C^1 \) functions of the new coordinates.
- The metric has a nonzero determinant at \( p \).

Remark: Theorem 1, asserts a trade off between a (non-removable) lack of \( C^1 \) metric-regularity and a vanishing metric determinant. This is our motivation for calling points of shock wave interaction “Regularity Singularities.”
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We only require two shock waves to be present before (or after) the interaction.

We address many physical shock wave interaction, e.g.:

- two shock waves come in; two shock waves go out
- two shock waves come in; one shock and one rarefaction wave go out
- two compression waves come in; two shock waves go out
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The main step is to prove the result for a smaller atlas first, namely, the atlas consisting of “coordinate transformations of the $(t, r)$-plane”, i.e., transformations which keep the SSC angular variables fixed. (My presentation is restricted to this part.)
Part VI

The Proof of Theorem 1
Outline of the proof (Thm 2):

(i) Assume $J \mu \alpha$ is the Jacobian of a coordinate transformation smoothing the metric from $C^0, 1$ to $C^1$.

We derive a condition $J \mu \alpha$ must meet at each shock curve.

(ii) We characterize all $J \mu \alpha$, (defined on a neighborhood of shocks), satisfying that condition, by deriving an explicit form to represent them in.

(iii) Now, $J \mu \alpha$ is integrable to coordinates, $\Rightarrow J \mu \alpha, \beta = J \mu \beta, \alpha$.

Taking limit to point of interaction $p$ of above equation yields $\text{Det}(g_{\alpha \beta}(p)) = 0$. 

M. Reintjes

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$$
\implies J_{\alpha,\beta}^\mu = J_{\beta,\alpha}^\mu.
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\text{Det} \left( g_{\alpha\beta}(p) \right) = 0.
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Proof:

Step (i):
Assume (for contradiction) there exist coordinates $x^\alpha$, such that the transformed metric, $g_{\alpha\beta} = J^\mu\alpha J^\nu\beta g_{\mu\nu}$, (1) is in $C^1$, where $J^\mu\alpha = \partial_x^\mu \partial_x^\alpha$ (Jacobian) and $g_{\mu\nu}$ metric in SSC.

(Indices $\mu, \nu, \sigma$ refer to SSC and $\alpha, \beta, \gamma$ to new coords.)

Now, $g_{\alpha\beta}$ being in $C^1$ implies that, for all $\alpha, \beta, \gamma \in \{0, \ldots, 3\}$,

\[ [g_{\alpha\beta,\gamma}]^i = 0 \]

(2)

$\cdot_i$ jump across the shock curve $\gamma_i$ if $\gamma_i := \partial f / \partial x^\gamma$ denotes differentiation w.r.t. new coords $x^\alpha$.

Thus, differentiating the RHS of (1) and taking the jump leads to

\[ [J^\mu\alpha,\gamma]_i J^\nu\beta g_{\mu\nu} + [J^\nu\beta,\gamma]_i J^\mu\alpha g_{\mu\nu} + J^\mu\alpha J^\nu\beta [g_{\mu\nu,\gamma}]_i = 0 \].
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$$ g_{\alpha\beta} = J^\mu_\alpha J^\nu_\beta g_{\mu\nu}, \quad (1) $$

is in $C^1$, where $J^\mu_\alpha = \frac{\partial x^\mu}{\partial x^\alpha}$ (Jacobian) and $g_{\mu\nu}$ metric in SSC. (Indices $\mu, \nu, \sigma$ refer to SSC and $\alpha, \beta, \gamma$ to new coords.)

- Now, $g_{\alpha\beta}$ being in $C^1$ implies that, for all $\alpha, \beta, \gamma \in \{0, \ldots, 3\}$,

$$ [g_{\alpha\beta,\gamma}]_i = 0. \quad (2) $$

- $[.]_i$ jump across the shock curve $\gamma_i$

- $f,\gamma := \frac{\partial f}{\partial x^\gamma}$ denotes differentiation w.r.t. new coords $x^\alpha$.

- Thus, differentiating the RHS of (1) and taking the jump leads to

$$ [J^\mu_\alpha,\gamma]_i J^\nu_\beta g_{\mu\nu} + [J^\nu_\beta,\gamma]_i J^\mu_\alpha g_{\mu\nu} + J^\mu_\alpha J^\nu_\beta [g_{\mu\nu,\gamma}]_i = 0. $$
\[
\left[ J^\mu_{\alpha,\gamma} \right]_i J^\nu_\beta g_{\mu\nu} + \left[ J^\nu_{\beta,\gamma} \right]_i J^\mu_\alpha g_{\mu\nu} + J^\mu_\alpha J^\nu_\beta [g_{\mu\nu,\gamma}]_i = 0, \quad (3)
\]

is a necessary condition for smoothing the metric.
\[
\left[J^\mu_{\alpha,\gamma}\right]_i J^\nu_\beta g_{\mu\nu} + \left[J^\nu_{\beta,\gamma}\right]_i J^\mu_\alpha g_{\mu\nu} + J^\mu_\alpha J^\nu_\beta [g_{\mu\nu,\gamma}]_i = 0, \tag{3}
\]

is a necessary condition for smoothing the metric.

(3) is linear in \([J^\mu_{\alpha,\gamma}]_i\).
\[
\left[J^\mu_{\alpha, \gamma}\right]_i J^\nu_{\beta} g_{\mu \nu} + \left[J^\nu_{\beta, \gamma}\right]_i J^\mu_{\alpha} g_{\mu \nu} + J^\mu_{\alpha} J^\nu_{\beta} \left[g_{\mu \nu, \gamma}\right]_i = 0, \quad (3)
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(3) is linear in \( [J^\mu_{\alpha, \gamma}]_i \).

Equation (3) simplifies significantly once we
- substitute the explicit form of the SSC metric, \( g_{\mu \nu} \),
- use our assumption, that the coord transfo only acts on the \((t, r)\)-plane.
\[
\left[ J^\mu_{\alpha, \gamma} \right]_i J^\nu_\beta g_{\mu \nu} + \left[ J^\nu_\beta, \gamma \right]_i J^\mu_\alpha g_{\mu \nu} + J^\mu_\alpha J^\nu_\beta \left[ g_{\mu \nu}, \gamma \right]_i = 0, \tag{3}
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By assumption, \(J^\mu_\alpha\) satisfies the integrability condition,

\[
J^\mu_{\alpha, \beta} = J^\mu_{\beta, \alpha},
\]

are Regularity Singularities
\[
[J^{\mu}_{\alpha,\gamma}]_i J^{\nu}_{\beta} g_{\mu\nu} + [J^{\nu}_{\beta,\gamma}]_i J^{\mu}_{\alpha} g_{\mu\nu} + J^{\mu}_{\alpha} J^{\nu}_{\beta} [g_{\mu\nu,\gamma}]_i = 0, \quad (3)
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By assumption, \(J^{\mu}_{\alpha}\) satisfies the integrability condition, \(J^{\mu}_{\alpha,\beta} = J^{\mu}_{\beta,\alpha}\), which implies

\[
[J^{\mu}_{\alpha,\beta}]_i = [J^{\mu}_{\beta,\alpha}]_i. \quad (4)
\]
\begin{equation}
\left[J^\mu_{\alpha,\gamma}\right]_i J^\nu_\beta g_{\mu\nu} + \left[J^\nu_{\beta,\gamma}\right]_i J^\mu_\alpha g_{\mu\nu} + J^\mu_\alpha J^\nu_\beta \left[g_{\mu\nu,\gamma}\right]_i = 0, \tag{3}
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Equation (3) simplifies significantly once we

- substitute the explicit form of the SSC metric, \(g_{\mu\nu}\),
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By assumption, \(J^\mu_{\alpha}\) satisfies the integrability condition,
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\begin{equation}
\left[J^\mu_{\alpha,\beta}\right]_i = \left[J^\mu_{\beta,\alpha}\right]_i. \tag{4}
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A long computation shows that the unique solution, \([J^\mu_{\alpha,\gamma}]_i\), of (3) together with (4) is given by:
A long computation shows that the unique solution, \( [J_{\mu,\alpha,\gamma}]_i \), of (3) together with (4) is given by:

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\begin{align*}
[J_{0, t}]_i &= -\frac{1}{2} \left( \frac{[A_t]_i}{A} J_{0, t} + \frac{[A_r]_i}{A} J_{0, r} \right); & [J_{0, r}]_i &= -\frac{1}{2} \left( \frac{[A_r]_i}{A} J_{0, t} + \frac{[B_t]_i}{A} J_{0, r} \right) \\
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\end{align*}
\]

(5)

- **Notation:**
  - \(A_t := \frac{\partial A}{\partial t}, \ldots\)
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J^r_{0,i} &= -\frac{1}{2} \left( \frac{[A_r]_i}{B} J^t_0 + \frac{[B_t]_i}{B} J^r_0 \right) ; \\
J^r_{1,i} &= -\frac{1}{2} \left( \frac{[B_t]_i}{B} J^t_1 + \frac{[B_r]_i}{B} J^r_1 \right) .
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**Notation:**
- \(A_t := \frac{\partial A}{\partial t}, \ldots\)
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- $A_t := \frac{\partial A}{\partial t}$, ...
- $\mu, \nu \in \{t, r\}$ and $\alpha, \beta \in \{0, 1\}$
- $J^t_0$ denotes the $\mu = t$ and $\alpha = 0$ component of the Jacobian $J^\mu_\alpha$

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It's only defined on the shock curves!
Step (ii):

Next, we characterize all $C^{0,1}$-functions, defined on some open neighborhood $\mathcal{N}$ of $p$, that meet (5).
Step (ii):

- Next, we characterize all $C^{0,1}$-functions, defined on some open neighborhood $\mathcal{N}$ of $p$, that meet (5).
- To understand how this is done, we illustrate the procedure for $J^t_0$. 
By (5), the jump of the derivatives of $J^t_0$ across $\gamma_i(t) = (t, x_i(t))$ should satisfy

$$[J^t_0, t]_i = -\frac{1}{2} \left( \frac{[A_t]_i}{A} J^t_0 + \frac{[A_r]_i}{A} J^r_0 \right)$$

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(6)

Introduce $J^t_0(t, r) = \sum_i \alpha_i(t) |x_i(t) - r| + \Phi(t, r)$,
By (5), the jump of the derivatives of $J_0^t$ across $\gamma_i(t) = (t, x_i(t))$ should satisfy

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(6)

Introduce $J_0^t(t, r) = \sum_i \alpha_i(t) |x_i(t) - r| + \Phi(t, r),$ where

- $\Phi$ some function $C^1$ across $\gamma_i$.
- $\alpha_i(t) := \frac{1}{4A \circ \gamma_i(t)} ([A_r]_i J_0^t \circ \gamma_i(t) + [B_t]_i J_0^r \circ \gamma_i(t)),$
By (5), the jump of the derivatives of $J^t_0$ across $\gamma_i(t) = (t, x_i(t))$ should satisfy

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$J_t^0(t, r)$ satisfies (6), since:
- The value of $[J_{0,r}]_i$ follows from:
By (5), the jump of the derivatives of $J_0^t$ across $\gamma_i(t) = (t, x_i(t))$ should satisfy

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$$J_0^t(t, r) = \sum_i \alpha_i(t) |x_i(t) - r| + \Phi(t, r),$$

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$J_0^t(t, r)$ satisfies (6), since:

- The value of $[J_{0,r}^t]_i$ follows from:
  - $\frac{d}{dX}|X| = H(X)$, for the Heaviside function $H$,
  - and $[H(x_i(t) - r)]_j = 2\delta_{ij}$. 

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By (5), the jump of the derivatives of \( J^t_0 \) across \( \gamma_i(t) = (t, x_i(t)) \) should satisfy

\[
[J^t_0, t]_i = -\frac{1}{2} \left( \frac{[A_t]_i}{A} J^t_0 + \frac{[A_r]_i}{A} J^r_0 \right)
\]

and

\[
[J^t_0, r]_i = -\frac{1}{2} \left( \frac{[A_r]_i}{A} J^t_0 + \frac{[B_t]_i}{A} J^r_0 \right).
\] (6)

Introduce \( J^t_0(t, r) = \sum_i \alpha_i(t) |x_i(t) - r| + \Phi(t, r) \),

\[\Phi \text{ some function } C^1 \text{ across } \gamma_i.\]

\[\alpha_i(t) := \frac{1}{4 A \circ \gamma_i(t)} ([A_r]_i J^t_0 \circ \gamma_i(t) + [B_t]_i J^r_0 \circ \gamma_i(t)) ,\]

\( J^t_0(t, r) \) satisfies (6), since:

- The value of \( [J^t_0, r]_i \) follows from:
  - \( \frac{d}{dX} |X| = H(X) \), for the Heaviside function \( H \),
  - and \( [H(x_i(t) - r)]_j = 2 \delta_{ij}. \)

- The required value of \( [J^t_0, t]_i \) follows from the identities:
  - \( [A_r]_i = -\dot{x}_i[B_t]_i \), (by RH jump condition and Einstein eqns).
By (5), the jump of the derivatives of $J^t_0$ across $\gamma_i(t) = (t, x_i(t))$ should satisfy

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$$J^t_0(t, r) = \sum_i \alpha_i(t) |x_i(t) - r| + \Phi(t, r),$$

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  - $[A_r]_i = -\dot{x}_i [B_t]_i$, (by RH jump condition and Einstein eqns).
  - $\dot{x}_i [A_r]_i = -[A_t]_i$, (by smoothness of $g_{\mu\nu}$ along shocks).
In fact, all functions that meet (5) are of the above form,

\[ J^t_0(t, r) = \sum_i \alpha_i(t) |x_i(t) - r| + \Phi(t, r), \]

since \( J^t_0(t, r) - \sum_i \alpha_i(t) |x_i(t) - r| \) is a function \( C_1 \) across \( \gamma \).
In fact, all functions that meet (5) are of the above form,

\[ J_0^t(t, r) = \sum_i \alpha_i(t)|x_i(t) - r| + \Phi(t, r), \]

since \( J_0^t(t, r) - \sum_i \alpha_i(t)|x_i(t) - r| \) is a function \( C^1 \) across \( \gamma_i \).
In fact, all functions that meet (5) are of the above form,

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since \( J_0^t(t, r) - \sum_i \alpha_i(t) |x_i(t) - r| \) is a function \( C^1 \) across \( \gamma_i \).

In summary, we obtain the following Lemma:
Lemma

If the RH jump condition hold, then there exists a set of functions \( J_{\alpha}^t \in C^{0,1}(N \cap \mathbb{R}^2_-) \) that satisfies the smoothing condition (5) on \( \gamma_i \cap N, (i = 1, 2) \). All such \( J_{\alpha}^t \) assume the canonical form

\[
\begin{align*}
J_0^t(t, r) &= \sum_i \alpha_i(t) |x_i(t) - r| + \Phi(t, r), & \alpha_i(t) &= \frac{[A_r]_i \phi_i(t) + [B_t]_i \omega_i(t)}{4A \circ \gamma_i(t)}, \\
J_1^t(t, r) &= \sum_i \beta_i(t) |x_i(t) - r| + N(t, r), & \beta_i(t) &= \frac{[A_r]_i \nu_i(t) + [B_t]_i \zeta_i(t)}{4A \circ \gamma_i(t)}, \\
J_0^r(t, r) &= \sum_i \delta_i(t) |x_i(t) - r| + \Omega(t, r), & \delta_i(t) &= \frac{[B_t]_i \phi_i(t) + [B_r]_i \omega_i(t)}{4B \circ \gamma_i(t)}, \\
J_1^r(t, r) &= \sum_i \epsilon_i(t) |x_i(t) - r| + Z(t, r), & \epsilon_i(t) &= \frac{[B_t]_i \nu_i(t) + [B_r]_i \zeta_i(t)}{4B \circ \gamma_i(t)},
\end{align*}
\]

(7)

where

\[
\phi_i = \Phi \circ \gamma_i, \quad \omega_i = \Omega \circ \gamma_i, \quad \zeta_i = Z \circ \gamma_i, \quad \nu_i = N \circ \gamma_i,
\]

(8)

and \( \Phi, \Omega, Z, N \in C^{0,1}(N \cap \mathbb{R}^2_-) \) have matching derivatives on each shock curve \( \gamma_i(t) \),

\[
[U_r]_i = 0 = [U_t]_i,
\]

(9)

for \( U = \Phi, \Omega, Z, N, t \in (-\epsilon, 0) \).
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- Moreover, the Jacobian must assume the canonical form (7).
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Substituting the canonical form (7) into the above integrability condition and taking the jump across any of the shocks, (WLOG across $\gamma_1$),
Step (iii):

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Moreover, the Jacobian must assume the canonical form (7). Substituting the canonical form (7) into the above integrability condition and taking the jump across any of the shocks, (WLOG across $\gamma_1$), implies that for all $t < 0$,

$$\delta_1(t)\dot{x}_1(t)\beta_2(t) - \epsilon_1(t)\dot{x}_1(t)\alpha_2(t) + \epsilon_1(t)\delta_2(t) - \delta_1(t)\epsilon_2(t) = 0.$$
Taking the limit $t \to 0^+$ of

$$\delta_1(t)\dot{x}_1(t)\beta_2(t) - \epsilon_1(t)\dot{x}_1(t)\alpha_2(t) + \epsilon_1(t)\delta_2(t) - \delta_1(t)\epsilon_2(t) = 0,$$

gives

$$\frac{1}{4B} \left( \frac{\dot{x}_1\dot{x}_2}{A} + \frac{1}{B} \right) [B_r]_1 [B_r]_2 (\dot{x}_1 - \dot{x}_2) (\phi_0\zeta_0 - \nu_0\omega_0) = 0,$$

(10)
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where

- $\phi_0 = \lim_{t \to 0^+} \phi_1(t) = \lim_{t \to 0^+} \phi_2(t)$
- $\phi_i(t) := \Phi \circ \gamma_i(t)$
- $\zeta_0, \ldots, \omega_0$ defined analogously.
In (10), that is,

\[
\frac{1}{4B} \left( \frac{\dot{x}_1 \dot{x}_2}{A} + \frac{1}{B} \right) [B_r]_1 [B_r]_2 (\dot{x}_1 - \dot{x}_2) (\phi_0 \zeta_0 - \nu_0 \omega_0) = 0,
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all factors must be nonzero, except the last one, thus
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However,

\[
\text{Det} \left( J^\mu_{\alpha} \circ \gamma_i(t) \right) = \left( J^t_0 J^r_1 - J^t_1 J^r_0 \right) |_{\gamma_i(t)} = \phi_i(t) \zeta_i(t) - \nu_i(t) \omega_i(t).
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\[
\text{Det} \left( J_{\alpha}^{\mu} \circ \gamma_i(t) \right) = \left( J_0^t J_1^r - J_1^t J_0^r \right) \big|_{\gamma_i(t)} = \phi_i(t) \zeta_i(t) - \nu_i(t) \omega_i(t).
\]

Thus, taking the limit \( t \to 0^+ \) and using (11), yields

\[
\lim_{t \to 0^+} \text{Det} \left( J_{\alpha}^{\mu} \circ \gamma_i(t) \right) = \phi_i(0) \zeta_i(0) - \nu_i(0) \omega_i(0) = \phi_0 \zeta_0 - \nu_0 \omega_0 = 0.
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\]

This completes the proof, since \( g_{\alpha \beta} = J^\mu_{\alpha} J^\nu_{\beta} g_{\mu \nu} \).
So far, we’ve established that there is no coordinate transformation of the \((t, r)\)-plane that smooths the SSC-metric, \(g_{\mu\nu}\), to \(C^1\).

To prove our main Theorem we just need to extend the above result to the full atlas.

Recall our main Theorem:

**Theorem 1, (R. and Temple, 2011)**

Assume \(p\) is “a point of regular shock wave interaction in SSC”. Then: \(\nexists\) \(C^{1,1}\) coordinate transformation, defined in a neighborhood of \(p\), such that both holds:

- The metric components are \(C^1\) functions of the new coordinates.
- The metric has a nonzero determinant at \(p\).
Outline of Proof:

- Assume there exist coordinates, such that the metric in the new coordinates, $g_{\alpha\beta}$, is in $C^1$.
- In general, $g_{\alpha\beta}$ is not of the box-diagonal form,

\[ ds^2 = -A(t, r)dt^2 + B(t, r)dr^2 + 2D(t, r)dtdr + C(t, r)d\Omega^2. \]

(12)

- However, (following the arguments in [Weinberg, *Gravitation and Cosmology*]), there exists a coordinate transformation that takes $g_{\alpha\beta}$ over to a metric of the form (12) and preserves the metric regularity.

- (Remark: A crucial step is to prove a $C^1$ regularity of solutions of Killing’s equation, for a given $C^1$ metric.)

- But (12) is related to our original SSC metric, $g_{\mu\nu}$, by a transformation in the $(t, r)$-plane, contradicting Theorem 2. □
Part VII

Conclusion and Discussion
Conclusion:

At points, \( p \), of regular shock interaction in SSC the gravitational metric suffers a non-removable lack of \( C^1 \) regularity. The Einstein equations cannot hold strongly (only weakly) in any coordinate system. At \( p \), spacetime is not locally flat, that is, there do not exist coordinates \( x^j \), such that the metric satisfies:

\[
g^{ij}(p) = \eta^{ij}, \quad \eta^{ij} = \text{diag}(-1, 1, 1, 1),
\]

where \( g^{ij}, l(p) = 0 \), \( g^{ij}, kl \) are bounded on some neighborhood of \( p \).

In particular, there exist (non-removable) distributional second-order metric derivatives. These distributional derivatives are not hidden by an event horizon. However, all "curvature scalars" remain bounded. (\( \Rightarrow \) No naked singularities!)
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  - $g_{ij}(p) = \eta_{ij}$, where $\eta_{ij} = \text{diag}(-1,1,1,1)$,
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- Having unbounded second order metric derivatives, but no event horizon, regularity singularities might be measurable. What could be such a measurable effect?
- Our Theorem applies to spherically symmetric spacetimes and radial shock waves only. Do regularity singularities persist, if we remove any of our symmetry assumptions?


Thank you for your attention!