On the attainable set for Temple class systems with characteristic boundary

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$$\begin{cases} \partial_t u - \partial_x f(u) = h(t,x) & t > 0\,,\ x \in [a,b] \\ \\ u(0,x) = u_o(x) & x \in [a,b] \\ \text{in } x = a \text{ “bdry cond” } \alpha_o(t) & \\ \text{in } x = b \text{ “bdry cond” } \alpha_1(t) & \end{cases}$$

$u = u(t,x) \in \mathbb{R}^n$ “conserved” quantities
$f \colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ smooth flux, Ω unbounded
Df is strictly hyperbolic (i.e. real and distinct eigenvalues)

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$h(t,x) \in \mathbf{H} \subset \mathbf{L}^1_{loc}(\,]0,+\infty[\,\times\mathbb{R},\mathbb{R}^n)$ source control
$\alpha_j = \alpha_j(t) \in \mathcal{A} \subset \mathbf{L}^\infty([0,+\infty[,\ldots)$ bdry control on $x = a, b$

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For any choice of u_o, is it possible to find a triple $(h, \alpha_o, \alpha_1)_{u_o}$ so that the **entropy** solution reaches in finite time a given u_1? and how to characterize the attainable maps?

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\begin{aligned}
-C_o < \lambda_1(u) &< -C_1 < \lambda_2(u) < \dots \\
&< -C_{p-1} < \lambda_p(u) < C_p < \dots \\
&< \lambda_{n-1}(u) < C_{n-1} < \lambda_n(u) < C_n
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for all $u \in \Omega$ and suitable $C_j \in \mathbb{R}^+$

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+ some technical assumptions (more on these later)

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OUR GOAL: entropy solutions for systems, **(C)** bdry, source controls.

Theorem. Assume that **(H)** holds. Fix

- any initial state $u_o \in \mathbf{BV}([a,b],\mathbb{R}^n)$
- any final state $u_1 \in \mathbf{BV}([a,b],\mathbb{R}^n)$ with the additional assumption

$$u_{1,j}(y) - u_{1,j}(x) \leq K(y-x)\,, \qquad \forall\, y \geq x \;\; \forall\, j \in \{1,\ldots,n\}$$

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Then, there exists T large enough and controls (h,α_o,α_1) with $h(\cdot) \in \mathbf{C}^1$ and $h(0) = h(T) = 0$ so that the system has an entropy soln. defined on $[0,T]$ and such a solution satisfies

$$u(0,x) = u_o(x) \qquad\qquad u(T,x) = u_1(x)$$

provided that. . .

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(T1) Temple systems are *rich systems* (or semi–hamiltonian), i.e. there exist *potentials* $w \mapsto M_j(w)$ such that for $i \neq j$

$$\partial_i M_j = \frac{1}{\lambda_i - \lambda_j} \partial_i \lambda_j$$

with $\partial_i \phi = D\phi \cdot r_i$ for every scalar function $w \mapsto \phi(w)$.

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We require that for each $j \in \{1, \dots, n\}$

$$\partial_j \lambda_j(w) \cdot \exp M_j(w) \to 0 \qquad \text{as } |w| \to \infty$$

Since $e^{M_j} \approx \|Dw_j\|_\infty^{-1}$, **(T1)** roughly means that $D\lambda_j \cdot r_j \to 0$ faster than $|Dw_j|$.

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(T2) For the "vertical" characteristic family p, there holds

$$\lim_{|w|\to\infty} \frac{\lambda_p(w)}{\partial_p \lambda_p(w)} = +\infty$$

In particular, if $|w|$ is large enough, we always obtain a strictly positive speed $\lambda_p(w)$.

Theorem. Assume that **(H)**, **(T1)** and **(T2)** hold. Fix

- any initial state $u_o \in \mathbf{BV}([a,b],\mathbb{R}^n)$
- any final state $u_1 \in \mathbf{BV}([a,b],\mathbb{R}^n)$ with the additional assumption

$$u_{1,j}(y) - u_{1,j}(x) \leq K(y-x)\,, \qquad \forall\, y \geq x \;\; \forall\, j \in \{1,\ldots,n\}$$

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Sketch of the proof

We rewrite the system in Riemann coordinates

$$\partial_t w_j + \lambda_j(w)\partial_x w_j = g_j(t) \qquad t > 0\,,\ x \in [a, b]$$

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Three steps in the construction

1. - from w_o to a constant state

2. - from any constant state to any constant state

3. - from a constant state to w_1

We perform each step separately in the cases **(NC)** bdry and **(C)** bdry, to better stress the underlying ideas

Step 1: *from* w_o *to a constant state*

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In the case of **(NC)** bdry: no control needed.

Let w be the solution to our problem on $\mathbb{R}$ with initial datum $w(0,x) = w_o(x)\chi_{[a,b]}(x)$.

Then, at time $T_1 > \max_{j=1,\dots,n} \sup_{u\in\Omega} \frac{b-a}{|\lambda_j(u)|}$,

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In the case of **(C)** bdry: use $g_p(\cdot)$ on a small interval $[0,\tau]$ so that

$$\lambda_p\left(w_o + \int_0^\tau g_p(s)\,ds\right) > 0$$

and then proceed as above

Bdry controls $\alpha_o, \alpha_1 =$ trace of w on $x = a, b$

Step 2: *from any constant state* c_o *to any constant state* c_1, *with*

$$\lambda_p(c_o) \cdot \lambda_p(c_1) \neq 0$$

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Fix any time $T_2 > 0$ and just use

$$g(t) = (c_1 - c_o) \cdot \rho(t)$$

for a scalar function $\rho(t) \in \mathbf{C}_c^1([0, T_2], \mathbb{R})$ with $\int_0^{T_2} \rho = 1$

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Step 3: *from a constant state to* w_1

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3.a Revert time: Perform the change

$$v(t,x) = w(T-t, a+b-x)$$

and study the forward (in time) Cauchy problem

$$\partial_t v_j + \lambda_j(v)\partial_x v_j = g_j(t) \qquad t > 0\,,\ x \in [a,b] \qquad (*)$$

with initial datum $v(0,x) = w_1(a+b-x)$ satisfying

$$v_j(0,y) - v_j(0,x) \geq -K(y-x)\,, \qquad \forall\, y \geq x \ \ \forall\, j \in \{1,\ldots,n\}$$

$(*)$ cannot be reduced to **Step 1** due to entropy admissibility condition!

3.b Oleinik–type cond $\Rightarrow$ the soln to $(*)$ with $g \equiv 0$ is $\mathbf{C}^1$ at time 0^+

Indeed

- jumps in $v(0,x)$ generate rarefaction waves
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Hence, we look for T and g so that

- the solution $v(t,\cdot)$ to $(*)$ remains in $\mathbf{C}^1$ for all $t \in [0,T]$
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3.c An upper bound on $\partial_x v_j$ is easy to find (Oleinik–type estimates for the solution v to $(*)$)

3.d Search for a lower bound to $z_j = \partial_x v_j$. By differentiating $(*)$

$$\partial_t z_j + \lambda_j(v)\partial_x z_j = -\partial_x \lambda_j(v) z_j = -\partial_j \lambda_j(v)\, z_j^2 - \sum_{k \neq j} \partial_k \lambda_j(v)\, z_k z_j$$

with $\partial_k \lambda_j = D\lambda_j \cdot r_k = \frac{\partial \lambda_j}{\partial w_k}$

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Temple system $\Rightarrow$ there exists a scalar potential M_j such that

$$-\sum_{k \neq j} \partial_k \lambda_j(v)\, z_k = \sum_{k \neq j} (\lambda_j - \lambda_k) \partial_k M_j \cdot z_k = \dots$$

$$= \frac{d}{dt} M_j(v(t, x_j(t))) - \sum_{k=1}^{d} \partial_k M_j(v(t, x_j(t)))\, g_k$$

where $x_j(\cdot)$ is the j–th characteristic curve, i.e. the solution of

$$\dot{x}_j = \lambda_j(v(t, x_j(t))) \qquad x_j(0) = x_o$$

which allows to reformulate our equations for $z_j(t, x_j(t))$ in the form

$$\frac{d}{dt}\, z_j = -\partial_j \lambda_j(v) z_j^2 + \left(\frac{d}{dt}\, M_j - \sum_{k=1}^{d} \partial_k M_j\, g_k \right) z_j$$

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(Riccati) $\Downarrow$

$$\frac{1}{z_j(T)} < e^{-N_j(T)} \left(-\frac{1}{K} + \int_0^T \partial_j \lambda_j(v)\, e^{N_j(t)}\, dt \right)$$

with $v = v(t, x_j(t))$ and

$$N_j(t) = M_j(v(t, x_j(t))) - \int_0^t \sum_{k=1}^{d} \partial_k M_j(v(s, x_i(s)))\, g_k(s)\, ds\,,$$

(NC) boundary: Fix $T_3 > \max_j \sup_{u\in\Omega} \frac{b-a}{|\lambda_j(u)|}$ and choose $C > 0$ s.t.

$$|v| > C \qquad \Longrightarrow \qquad \partial_j \lambda_j(v) \cdot e^{M_j(v)} < \frac{1}{4KT_3}$$ by **(T1)**

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Strategy: choose $\tau > 0$ small enough and use g_j only on $[0, \tau]$ so that

$$|v_j(\tau, x_j(\tau))| = \left| v_j(0, x_o) + \int_0^\tau g_j \right| \geq C$$

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Hence, for $t \in [0, T_3]$

$$\begin{aligned} \frac{1}{z_j(t)} &< e^{-N_j(t)} \left(-\frac{1}{K} + \int_0^\tau \partial_j\lambda_j(v)\, e^{N_j(s)}\, ds \right. \\ &\qquad \left. + \int_\tau^t \partial_j\lambda_j(v)\, e^{M_j(s)}\, ds \right) \\ &< e^{-N_j(t)} \left(-\frac{1}{K} + \left\{ \begin{matrix} \text{small for} \\ \text{small } \tau \end{matrix} \right\} + \left\{ \begin{matrix} \text{small for} \\ \text{large } |v| \end{matrix} \right\} \right) < 0 \end{aligned}$$

(C) bdry: $\lambda_p \sim 0 \Longrightarrow$ very large T_3! Choose $C > 0$ s.t.

$$|v| > C \qquad \Longrightarrow \qquad \frac{\inf\limits_{B_v} \lambda_p}{\sup\limits_{B_v} \partial_p \lambda_p} > 8K(b-a) \qquad \text{by } \textbf{(T2)}$$

with $B_v = B(v, \|w_1\|_\infty)$ and take $T_3 = \frac{1}{4K \sup_{B_C} \partial_p \lambda_p}$

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Up to possibly reduce τ, we can choose g_p so that $\int_0^\tau g_p = C$ and

$$\frac{1}{z_p(t)} < -\frac{e^{-N_p(t)}}{2K} < 0 \qquad \forall\, t \in [0, T_3]$$

in a similar fashion to **(NC)**

In addition, we also obtain

$$\begin{aligned} x_p(T_3) - x_p(0) &= \int_0^\tau \lambda_p(v) + \int_\tau^{T_3} \lambda_p(v) \\ &\geq -\sup_\Omega \lambda_p \tau + \frac{\inf_{B_C} \lambda_p}{4K \sup_{B_C} \partial_p \lambda_p} \\ &> -(b-a) + 2(b-a) \end{aligned}$$

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$$\Downarrow$$

- the solution $v(t,x)$ is still $\mathbf{C}^1$ at $t = T_3$
- all $x_p(\cdot)$ cross $[a,b]$ before time $T_3 \Longrightarrow v$ is constant at $t = T_3$

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- unbounded eigenvalues

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- no uniformly separated eigenvalues
- sources depending on x (possibly localized)

Thanks for Your Attention!