A posteriori estimates from approximate solutions of the Euler or Navier-Stokes equations

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(joint work with Carlo Morosi, Politecnico di Milano, carlo.morosi@polimi.it)

Introduction

- I consider the incompressible Euler or Navier-Stokes (NS) Cauchy problem

$$\frac{\partial u}{\partial t} = \nu \Delta u + \mathcal{P}(u,u) + f, \qquad \mathrm{div} u = 0, \qquad u(x,0) = u_0(x)$$

$u = u(x,t)$; $x = (x_1, ..., x_d) \in \mathbf{T}^d := (\mathbf{R}/2\pi\mathbf{Z})^d$ (space-periodic bdry cond.); Δ the Laplacian; $\mathcal{P}$ the bilinear map such that, for $v, w : \mathbf{T}^d \to \mathbf{R}^d$,

$$\mathcal{P}(v,w) := -\mathfrak{L}(v \bullet \mathrm{grad} w)$$

with $\mathfrak{L}$ the Leray projection onto the space of divergence free vector fields; $f = f(x,t)$ the (Leray projected) density of external forces; Viscosity $\nu = 0$ in the Euler case; $\nu \in (0, +\infty)$ in the NS case.

- The aim is to derive fully quantitative estimates on the exact solution u analyzing *a posteriori* any approximate solution u_a (in particular: estimates on the interval of existence of u , that could be $[0, +\infty)$, and on the distance between u and u_a).
- Concrete examples will be given in space dimension $d = 3$.

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Functional setting on the torus $\mathbf{T}^d := (\mathbf{R}/2\pi\mathbf{Z})^d$

- We use the Fourier basis $e_k(x) := (2\pi)^{-d/2} e^{ik \bullet x}$ $(x \in \mathbf{T}^d, k \in \mathbf{Z}^d)$. .
- A vector field v on $\mathbf{T}^d$ means a vector distribution $v \in \mathbb{D}'(\mathbf{T}^d, \mathbf{R}^d) \equiv \mathbb{D}'$. This has Fourier coefficients $v_k = \overline{v_{-k}} \in \mathbf{C}^d$, such that $v = \sum_{k \in \mathbf{Z}^d} v_k e_k$
- The space of zero mean vector fields is

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$$\mathfrak{L} : \mathbb{D}' \to \mathbb{D}'_\Sigma \,, \qquad v \mapsto \mathfrak{L}v \text{ such that } (\mathfrak{L}v)_k = \mathfrak{L}_k v_k \text{ for } k \in \mathbf{Z}^d \,,$$

  $\mathfrak{L}_k$ the projection of $\mathbf{C}^d$ onto the orthogonal complement $k^\perp$.
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- For $n \in \mathbf{R}$, we consider the n-th Sobolev space of zero mean vector fields

$$\mathbb{H}^n_0 := \{v \in \mathbb{D}'_0 \mid \sqrt{-\Delta}^n v \in \mathbb{L}^2\} ;$$

its inner product and norm are

$$\langle v|w\rangle_n := \langle \sqrt{-\Delta}^n v | \sqrt{-\Delta}^n w\rangle_{L^2} = \sum_{k \in \mathbf{Z}^d_0} |k|^{2n}\, \overline{v_k} \bullet w_k \ , \quad \|v\|_n := \sqrt{\langle v|v\rangle_n} \ .$$

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Some inequalities for $\mathcal{P}(v, w) := -\mathfrak{L}(v \bullet \text{grad} w)$ (on $\mathbf{T}^d$).

- **Basic inequality**: $\|\mathcal{P}(v,w)\|_n \leqslant K_n \|v\|_n \|w\|_{n+1}$ for $v \in \mathbb{H}^n_{\Sigma 0}$, $w \in \mathbb{H}^{n+1}_{\Sigma 0}$ and $n > d/2$ (for a suitable constant $K_n > 0$).
- **Kato inequality**: $|\langle \mathcal{P}(v,w) | w \rangle_n| \leqslant G_n \|v\|_n \|w\|^2_n$ for $v \in \mathbb{H}^n_{\Sigma 0}$, $w \in \mathbb{H}^{n+1}_{\Sigma 0}$ and $n > d/2 + 1$ (for a suitable constant $G_n > 0$).
- **Proposition** (Morosi and P: arXiv:1007.4412v2 [math.AP], 2010; Commun. Pure Appl. Anal., 2012). One can take

$$K_n = \sqrt{\sup_{k \in \mathbf{Z}^d_0} \frac{|k|^{2n}}{(2\pi)^d} \mathcal{K}_n(k)}\ , \qquad \mathcal{K}_n(k) := \sum_{h \in \mathbf{Z}^d_0, h \neq k} \frac{|h \wedge k|^2}{|h|^{2n+2} |k-h|^{2n+2}}\ ;$$

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($h \wedge k$ the exterior product; it is the usual vector product, if $d = 3$). This implies, e.g., that one can take

$$K_2 = 0.335\ ,\ K_3 = 0.323\ ,\ K_4 = 0.441\ ,\ K_5 = 0.510\ ,\ K_{10} = 2.88\ ;$$

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Cauchy problem for Euler/NS equations (on $\mathbf{T}^d$).

- Choose $n > d/2 + 1$, $\nu \geqslant 0$, a forcing $f \in C([0, +\infty), \mathbb{H}^n_{\Sigma 0})$, and a datum $u_0 \in \mathbb{H}^{n+2}_{\Sigma 0}$. The corresponding Cauchy problem is:

$$\textit{Find } u \in C([0,T), \mathbb{H}^{n+2}_{\Sigma 0}) \cap C^1([0,T), \mathbb{H}^n_{\Sigma 0}) \quad \text{such that}$$

$$\frac{du}{dt} = \nu \Delta u + \mathcal{P}(u,u) + f \ , \qquad u(0) = u_0 \ .$$

Here are some basic facts (T. Kato, Lecture Notes in Mathematics **448**,1975; J. T.Beale, T. Kato, A. Majda, Commun. Math. Phys.,1984):

- There is a unique maximal (:= unextendable) solution u of domain $[0, T)$, with $T = T(u_0) \in (0, +\infty]$.
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Cauchy problem for Euler/NS equations (on $\mathbf{T}^d$).

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Approximate solutions for the Euler/NS Cauchy problem.

- An approximate solution of the Euler/NS Cauchy problem means any map $u_a \in C([0, T_a), \mathbb{H}^{n+2}_{\Sigma 0}) \cap C^1([0, T_a), \mathbb{H}^n_{\Sigma 0})$ (with $T_a \in (0, +\infty]$).
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The theory of approximate solutions. A basic lemma

- From now on, we consider for the Euler/NS Cauchy problem:
  - the maximal (exact) solution $u \in C([0,T), \mathbb{H}^{n+2}_{\Sigma 0}) \cap C^1([0,T), \mathbb{H}^n_{\Sigma 0})$;
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$$w(t) := u(t) - u_a(t) , \qquad 0 \leqslant t < \min(T, T_a)$$

and its norm $\|w(t)\|_n$ (a *continuous* function of t, possibly non derivable when $w(t) = 0$). This fulfills

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where $(d^+\|w\|_n/dt)(t) := \limsup_{h\to 0^+}(\|w(t+h)\|_n - \|w(t)\|_n)/h$ is the right, upper Dini derivative .

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The theory of approximate solutions. Control inequalities

- Again, we consider for the Euler/NS Cauchy problem:
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- **Proposition** (Morosi and P, *Nonlinear Analysis*, 2012). Consider a function $\mathcal{R}_n \in C([0, T_c), [0, +\infty))$, with $T_c \in (0, T_a]$; assume this fulfills the *control inequalities*

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$$T \geqslant T_c ; \qquad \|u(t) - u_a(t)\|_n \leqslant \mathcal{R}_n(t) \qquad \text{for } 0 \leqslant t < T_c .$$

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and, passing to $u = u_a + w$,

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- I recall that $e_k(x) := (2\pi)^{-d/2} e^{ik\bullet x}$ for all $k \in \mathbf{Z}^d$. We choose a set of modes

$$G \subset \mathbf{Z}_0^d\ , \quad G \text{ finite and such that} k \in G \Leftrightarrow -k \in G\ ;$$

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($G + G := \{p + q |\ p, q \in G\}$. Recall that $\mathfrak{L}_k$ projects onto $k^\perp$ and γ_h are the Fourier coefficients of u_G , for $h \in G$; $\gamma_h := 0$ if $h \in \mathbf{Z}_0^d \setminus G$.)

- In order to apply the control inequalities , one can compute directly the norms

$$\delta_n := \| u_0 - u_G(0) \|_n$$

and, after solving the Galerkin equations for the coefficients $\gamma_k(t)$,

$$\epsilon_n(t) := \| err(u_G)(t) \|_n \ , \qquad \mathcal{D}_m(t) := \| u_G(t) \|_m \ (m = n, n+1)$$

(Alternatively, the are known methods to estimate $\| u_0 - u_G(0) \|_n$ and $\| err(u_G)(t) \|_n$ via the norms of $u_0, u_G(t), f(t)$ and the radius $|G|$.)

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$(G + G := \{p + q |\ p, q \in G\}$. Recall that $\mathfrak{L}_k$ projects onto $k^\perp$ and γ_h are the Fourier coefficients of u_G , for $h \in G$; $\gamma_h := 0$ if $h \in \mathbf{Z}_0^d \setminus G$.)

- In order to apply the control inequalities , one can compute directly the norms

$$\delta_n := \|u_0 - u_G(0)\|_n$$

and, after solving the Galerkin equations for the coefficients $\gamma_k(t)$,

$$\epsilon_n(t) := \|err(u_G)(t)\|_n \ , \qquad \mathcal{D}_m(t) := \|u_G(t)\|_m \ (m = n, n+1)$$

(Alternatively, the are known methods to estimate $\|u_0 - u_G(0)\|_n$ and $\|err(u_G)(t)\|_n$ via the norms of $u_0, u_G(t), f(t)$ and the radius $|G|$.)

An application. Galerkin approximate solutions

- The datum error of u_G is $u_0 - u_G(0) = (1 - E^G)u_0 = \sum_{k \in \mathbf{Z}_0^d \setminus G} u_{0k}$;
- The differential error $err(u_G) := du_G/dt - \nu\Delta u_G - \mathcal{P}(u_G, u_G)$ is given by

$$err(u_G) = -(1 - E^G)\mathcal{P}(u_G, u_G) - (1 - E^G)f \ ;$$

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Example. Galerkin method for the BNW datum

- The space dimension is $d = 3$. There is no external forcing: $f = 0$.
- The initial datum of Behr, Nečas and Wu (from now on: BNW datum) is

$$u_0(x_1, x_2, x_3) :=$$

$$2\big(\cos(x_1+x_2)+\cos(x_1+x_3), -\cos(x_1+x_2)+\cos(x_2+x_3), -\cos(x_1+x_3)-\cos(x_2+x_3)\big)$$

In terms of Fourier modes:

$$u_0 = \sum_{k=\pm a, \pm b, \pm c} u_{0k} e_k \ , \qquad a := (1,1,0), \quad b := (1,0,1), \quad c := (0,1,1) \ ;$$

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- In [Morosi and P, *Nonlinear Analysis*, 2012] the Galerkin solution u_G has been computed numerically for the BNW datum and several values of ν , choosing a set of 150 modes

$$G := \{\pm(1,1,0), \pm(1,0,1), \pm(0,1,1), ..., \pm(3,3,-2), \pm(3,3,2)\} .$$

- Then, the theory of approximate solutions has been applied to u_G in the framework of the spaces $\mathbb{H}^n_{\Sigma 0}, \mathbb{H}^{n+1}_{\Sigma 0}.\mathbb{H}^{n+2}_{\Sigma 0}$ with $n = 3$.
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- The final step has been the analysis of the control Cauchy problem

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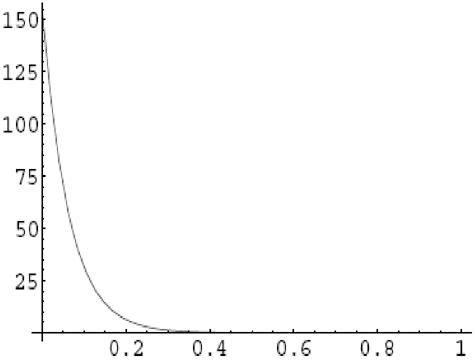
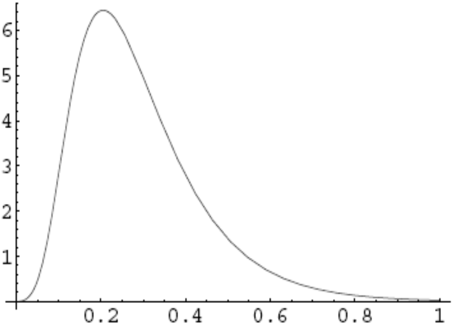
Example. Galerkin method for the BNW datum

Here are some results from the numerical computation of u_G and $\mathcal{R}_3$.



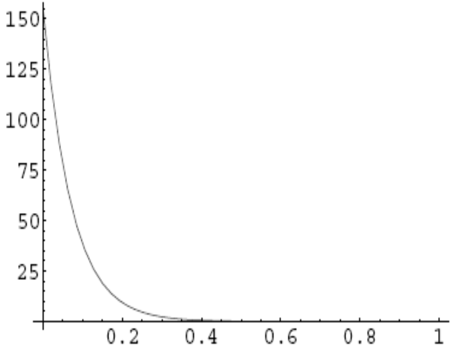
$\nu = \mathbf{8}$. Graph of $\mathcal{D}_3(t) = \|u_G(t)\|_3$.

Example. Galerkin method for the BNW datum



$\boldsymbol{\nu = 8}$. Graph of $\mathcal{R}_3(t)$, that appears to be global.
$\|u(t) - u_G(t)\|_3 \leqslant \mathcal{R}_3(t)$ for all $t \in [0, +\infty)$.

Example. Galerkin method for the BNW datum



$\boldsymbol{\nu = 7}$. Graph of $\mathcal{D}_3(t) = \|u_G(t)\|_3$.

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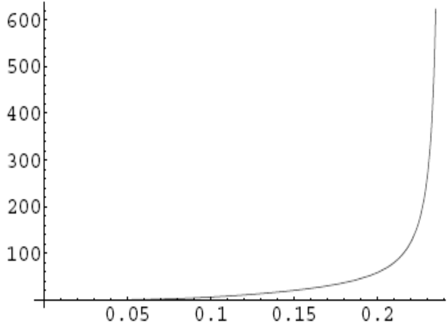

$\boldsymbol{\nu = 7}$. Graph of $\mathcal{R}_3(t)$, that appears to diverge for $t \to T_c = 0.2386\ldots$.
$\|u(t) - u_G(t)\|_3 \le \mathcal{R}_3(t)$ for $t \in [0, T_c)$.

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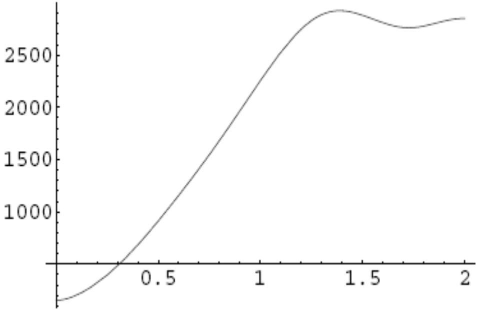

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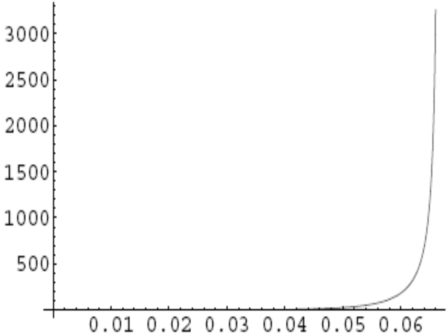

$\boldsymbol{\nu = 0}$. Graph of $\mathcal{R}_3(t)$, that appears to diverge for $t \to T_c = 0.06666...$.
$\|u(t) - u_G(t)\|_3 \le \mathcal{R}_3(t)$ for $t \in [0, T_c)$.

Other approaches to build approximate solutions

- Euler or NS equations have formal power series solutions

$$u(t) = \sum_{j=0}^{+\infty} u_j t^j \quad \text{or} \quad u(t) = \sum_{\ell=0}^{+\infty} \frac{1}{\nu^\ell} U_\ell(\nu t)$$

where u_j, U_ℓ can be computed recursively (see in particular: Brachet et al.,J. Fluid Mech., 1983 and J. Statist. Phys.,1984; Behr, Nečas and Wu, Math. Model. Numer. Anal., 2001; Sinai, J.Stat. Phys., 2005).

- Instead of discussing the difficult problem of convergence, one can consider the truncated series

$$u(t) = \sum_{j=0}^{J} u_j t^j \quad \text{or} \quad u(t) = \sum_{\ell=0}^{L} \frac{1}{\nu^\ell} U_\ell(\nu t)$$

and regard them as approximate solutions, to be analyzed with the method of control inequalities.

- For example, for the BNW datum , extensive symbolic calculations (by Python or Mathematica) allow to compute the previous truncated series for Euler with $J \simeq 50$, or NS with $L \simeq 10$ (Morosi, Pernici and P, arXiv:1203.6865v1 [math.AP], 2012; Morosi and P, in preparation).

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Here are some previous works related to this talk:

- Morosi and P, *Rev. Math. Phys* 2004, 2008 (on approximate solutions and control inequalities for Euler/NS equations or other PDEs)
- Morosi and P, *Nonlinear Analysis* 2011 (a fully quantitative setting for NS equations on $\mathbf{T}^3$ based on $\mathbb{H}^1_{\Sigma 0}$).

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