Optimal Boundary Control for Nonlinear Hyperbolic Balance Laws

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Nonlinear Optimization

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SPP 1253: Optimization with partial differential equations

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Outline

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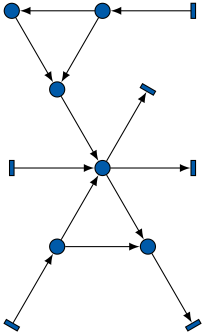
Optimal Control of Switched Networks for Nonlinear Hyperbolic Conservation Laws



Optimal Control of Switched Networks for Nonlinear Hyperbolic Conservation Laws

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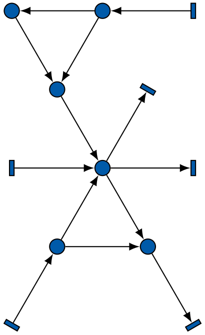
Setting

- directed graph $G = (V, E)$
- edges correspond to real intervals
- state $y = (y^i)_{e_i \in E}$



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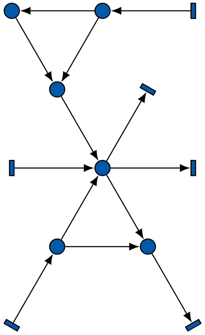
- conservation law on I_i
- initial conditions
- node conditions
- boundary conditions



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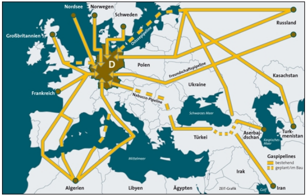

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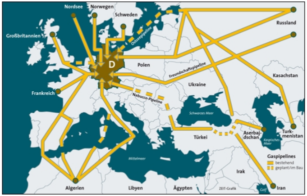

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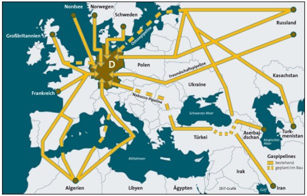
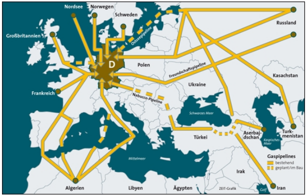


Optimal Control of Switched Networks for Nonlinear Hyperbolic Conservation Laws

Objective Functional

$$J(y(\bar{t},\cdot)) = \sum_{e_i \in E} \int_{a_i}^{b_i} \gamma_i(x)\Psi_i(y_i(\bar{t},x),x)\,dx$$

Covers usual tracking-type functionals



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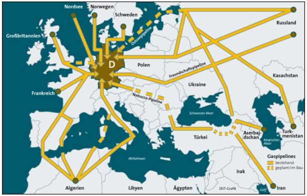
- initial value
- control of the source term
- boundary data
- node conditions
- switching times



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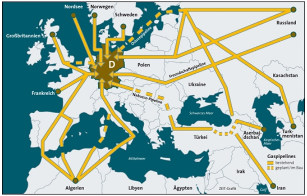

Considered Optimal Control Problem

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Optimal Control Problem

$$\min J(y(\bar{t}, \cdot))$$

$$\text{s.t. } u = (u_0, u_B, u_1) \in U_{ad}, \quad y = y(u) \text{ solves}$$

$$\begin{aligned} y_t + (f(y))_x &= g(\cdot, y, u_1), && \text{on } (0,T) \times \mathbb{R}^+ =: \Omega_T, \\ y(0,\cdot) &= u_0, && \text{on } \mathbb{R}^+ \\ y(\cdot, 0) &= u_B, && \text{on } (0,T). \end{aligned}$$

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Assumptions:

- Source Term $g \in C(0,T;C^1_{\text{loc}}(\mathbb{R}^{m+2}))$, $g \geq 0$, ...
- Flux $f \in C^2_{\text{loc}}(\mathbb{R})$, $\quad f'' \geq m_f > 0$

General Optimal Control Problem

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Original Problem

$$\begin{aligned} \min_{y \in Y, u \in U} \quad & J(y, u) \\ \text{s.t.} \quad & (y, u) \in W_{ad} \\ & C(y, u) = 0 \end{aligned}$$

General Optimal Control Problem

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$$\min_{y \in Y, u \in U} \quad J(y,u)$$
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Reduced Problem

$$\min_{u \in U} \quad \hat{J}(u) := J(y(u),u)$$
$$\text{s.t.} \quad (y(u),u) \in W_{ad}$$

General Optimal Control Problem

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We need the derivative of the reduced objective Function $\hat{J}$

- to derive optimality conditions
- to apply fast optimization methods

Derivative of the Reduced Objective Function (if $u \mapsto y(u)$ is differentiable!)

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$$\langle \hat{J}'(u), s \rangle = \langle J_y(y(u), u), y'(u)s \rangle + \langle J_u(y(u), u), s \rangle$$

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$$\langle \hat{J}'(u), s \rangle = \langle J_y(y(u), u), \mathbf{y}'(\mathbf{u}) s \rangle + \langle J_u(y(u), u), s \rangle$$

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Implicit Function Theorem $\quad \Rightarrow \quad y'(u) = -C_y(y(u), u)^{-1} C_u(y(u), u)$

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Adjoint Gradient Representation

$$\hat{J}'(u) = C_u(y(u), u)^* p + J_u(y(u), u)$$

where the **adjoint state** p is a solution of the **adjoint equation**

$$C_y(y(u), u)^* p = -J_y(y(u), u)$$

Reduced Problem

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Initial-Boundary Value Problem

$$
\begin{aligned}
y_t + (f(y))_x &= g(\cdot, y, u_1) && \text{on } \Omega_T \\
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Reduced Problem

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Initial-Boundary Value Problem

$$\eta(y)_t + q(y)_x \eta'(y) - g(t,x,y,u_1) \leq 0, \qquad \text{in } \mathcal{D}'(\Omega_T)$$

$$\lim_{t \to 0+} \|y(t,\cdot) - u_0\|_{1,(0,R)} = 0, \qquad \forall R > 0$$

$$\min_{k \in I(y(t,0+),u_B)(t)} \operatorname{sgn}(u_B(t) - y(t,0+))(f(y(t,0+)) - f(k)) = 0, \qquad \text{a.e. on } [0,T]$$

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$\Rightarrow$ Existence, Uniqueness, Stability of solutions $y \in L^\infty(\Omega_T) \cap C([0,T]; L^1_{\text{loc}}(\mathbb{R}^+))$

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Control-to-State Mapping

$$u \in U_{\text{ad}} \longmapsto y(\bar{t}; u) := y(u)(\bar{t},\cdot) \in L^1_{\text{loc}}(\mathbb{R}^+)$$

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Reduced Optimal Control Problem

$$\min \hat{J}(u) := J(y(\bar{t}; u), u) \qquad \text{s.t.} \qquad u \in U_{ad}$$

Related Work

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- **Differentiability w.r.t. initial data:** Bressan, Guerra 97, Bouchut and James 99, Ulbrich 02, Ulbrich. 03, Giles 03, Bardos and Pironneau 05
- **Variational calculus for piecewise Lipschitz solutions of systems:** Bressan and Marson 95
- **Numerical approximation of sensitivity and adjoint equation:** Gosse and James 00, Ulbrich 02, Herty, Steffensen 11
- **Convergence of discrete sensitivities and adjoints:** Ulbrich 02, Giles 03, Pironneau 05, Giles, Ulbrich 11
- **Alternating descent for opt. ctrl. of Burgers' eq.:** Castro, Zuazua 09, 10
- **Optimization of IBVPs for a conservation law:** Colombo, Groli 02
- **Networks in case of strong solutions:** Dick, Gugat, Herty, Leugering et al.
- **Modal switchings in Networks:** Hante Leugering, Seidman 09
- **Controllability of systems:** Ancona, Bressan, Coclite; Dick, Gugat, Leugering

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Example: Burgers' Equation

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Consider the inviscid Burgers' equation

$$y_t^\varepsilon + \left(\frac{(y^\varepsilon)^2}{2} \right)_x = 0, \quad y(0, x) = \varepsilon - \operatorname{sgn}(x).$$

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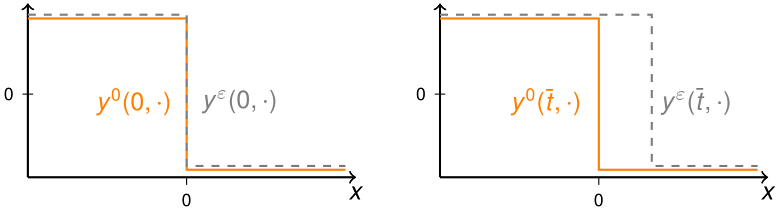

Shift-Variations

Let $v \in BV([a,b])$. For given $a < x_1 < ... < x_N < b$ we associate with $(\delta v, \delta x) \in L^1(a,b) \times \mathbb{R}^N$ the **shift-variation**

$$S_v^{(x_i)}(\delta v, \delta x) = \delta v + \Sigma_v^{(x_i)}(\delta x) := \delta v + \sum_{i=1}^{N} [v(x_i)] \operatorname{sgn}(\delta x_i) \mathbb{1}_{I(x_i, x_i+\delta x_i)},$$

where $[v(x_i)] := v(x_i-) - v(x_i+)$, $\quad I(a,b) = [\min(a,b), \max(a,b)]$.

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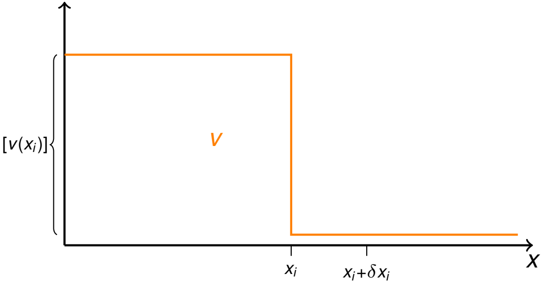

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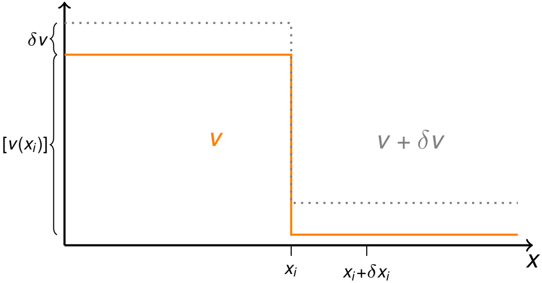

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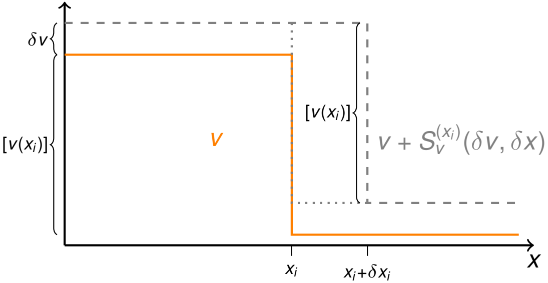

Definition: Shift-Differentiability

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Fréchet-Differentiability

$$u \in U \longmapsto v(u) \in Y$$

is **Fréchet-differentiable** at u if

$$\exists Dv(u) \in \mathcal{L}(U, Y) :$$

with $\delta v = Dv(u) \cdot \delta u$ holds

$$\|v(u + \delta u) - v(u) - \delta v\|_Y = o(\|\delta u\|_U).$$

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$$\exists a < x_1 < ... < x_N < b,$$

$$D_s v(u) \in \mathcal{L}(U, L^r(a, b) \times \mathbb{R}^N) \quad r > 1 :$$

with $(\delta v, \delta x) = D_s v(u) \cdot \delta u$ holds

$$\left\| v(u + \delta u) - v(u) - (\delta v + \Sigma_{v(u)}^{(x_i)}(\delta x)) \right\|_1$$

$$= o(\left\| \delta u \right\|_U).$$

Shift-Differentiability

Results on Shift-Differentiability (Ulbrich, SICON 2002):

- Shift-Differentiability of $u \mapsto v(u)$ implies Fréchet-Differentiability of tracking-type functionals

$$u \in U \longmapsto J(v(u)) := \int_a^b \gamma(x)\Psi(v(x), x)\, dx$$

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We will see that the latter result can be extended to IBVPs and thus the first is applicable.

Shift-Differentiability for IBVPs

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\begin{aligned}
y_t + (f(y))_x &= g(\cdot, y, u_1 + \delta u_1), && \text{on } \Omega_T, \\
y(0, \cdot) &= u_0 + \delta u_0 + \Sigma_{u_0}^{(x_i)}(\delta x), && \text{on } \mathbb{R}^+, \qquad (*) \\
y(\cdot, 0+) &= u_B + \delta u_B + \Sigma_{u_B}^{(t_j)}(\delta t), && \text{on } (0, T).
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Consider the space

$$W := PC^1(\mathbb{R}^+; x_1, \ldots, x_N) \times C(0, T; C^1(\mathbb{R}^m)) \times PC^1(0, T; t_1, \ldots, t_K) \times \mathbb{R}^N \times \mathbb{R}^K$$

and the mapping

$$\delta u = (\delta u_0, \delta u_1, \delta u_B, \delta x, \delta t) \in W \longmapsto y(\bar{t}, \cdot) \in L^1(a, b)$$

where $y = y(u)$ solves $(*)$.

Shift-Differentiability for IBVPs

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$$\begin{aligned} y_t + (f(y))_x &= g(\cdot, y, u_1 + \delta u_1), && \text{on } \Omega_T, \\ y(0,\cdot) &= u_0 + \delta u_0 + \Sigma_{u_0}^{(x_i)}(\delta x), && \text{on } \mathbb{R}^+, \qquad (*) \\ y(\cdot, 0+) &= u_B + \delta u_B + \Sigma_{u_B}^{(t_j)}(\delta t), && \text{on } (0,T). \end{aligned}$$

Consider the space

$$W := PC^1(\mathbb{R}^+; x_1, \dots, x_N) \times C(0,T; C^1(\mathbb{R}^m)) \times PC^1(0,T; t_1, \dots, t_K) \times \mathbb{R}^N \times \mathbb{R}^K$$

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where $y = y(u)$ solves $(*)$.

Then the mapping $\delta u \longmapsto y(\bar{t}, \cdot)$ is **shift-differentiable** in 0.

Generalized Characteristics and Characteristic Equation

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Generalized Characteristics (Dafermos)

A Lipschitz curve $t \mapsto (t, \xi(t))$ is a **generalized characteristic** on $[a, b]$ if

$$\dot{\xi}(t) \in \left[f'(y(t, \xi(t)+)), f'(y(t, \xi(t)-))\right] \quad \text{a.e. on } [a, b].$$

A generalized characteristic is called **genuine** if $y(t, \xi(t)+) = y(t, \xi(t)-)$ and a backward characteristic is called **maximal/minimal** if

$$\dot{\xi}(t) = f'(y(t, \xi(t)\pm)) \quad \text{for a.a. } t \in [a, b].$$

Generalized Characteristics and Characteristic Equation

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Characteristic Equation

$$\begin{aligned} \dot{\zeta}(t) &= f'(v(t)) \\ \dot{v}(t) &= g(t, \zeta(t), v(t), u_1(t, \zeta(t))) \end{aligned}$$

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$\updownarrow$

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Generalized Characteristics

Generalized Characteristics

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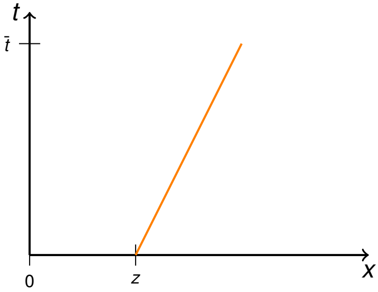
Properties of $\xi(0) = z$ (Dafermos 1977)

$$
\begin{array}{rcccl}
\xi(t) & = & \zeta(t) & & t \in [0, \bar{t}] \\
y(t, \xi(t)) & = & v(t) & & t \in (0, \bar{t}) \\
u_0(\xi(0)-) & \leq & v(0) & \leq & u_0(\xi(0)+) \\
y(\bar{t}, \xi(\bar{t})-) & \geq & v(\bar{t}) & \geq & y(\bar{t}, \xi(\bar{t})+) \\
 & & \vdots & &
\end{array}
$$



Generalized Characteristics

TECHNISCHE UNIVERSITÄT DARMSTADT

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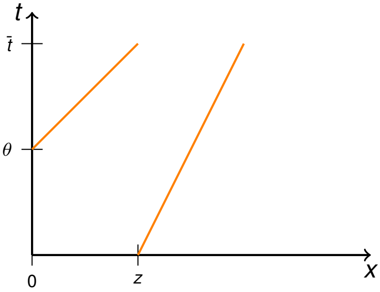
 & & \vdots & &
\end{array}
$$

Properties of $\xi(\theta) = 0$ (c.f. Perrollaz 2012)

$$
u_B(\xi(\theta)-) \quad \geq \quad v(\theta) \quad \geq \quad u_B(\xi(\theta)+)
$$



Construction of Local Solutions

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$(\zeta, v)(\cdot; \theta, w, \hat{u}_1)$ solves

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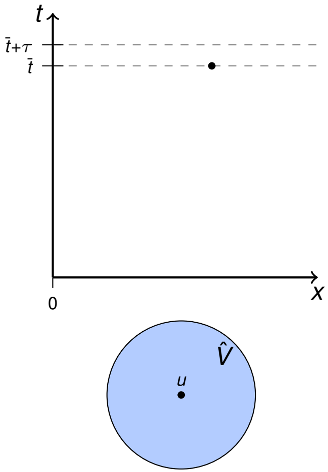

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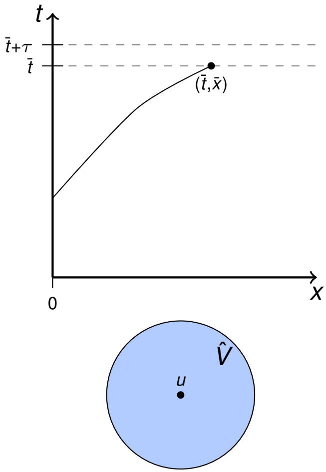

Construction of Local Solutions

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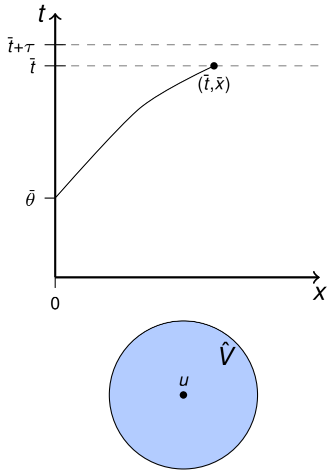

Construction of Local Solutions

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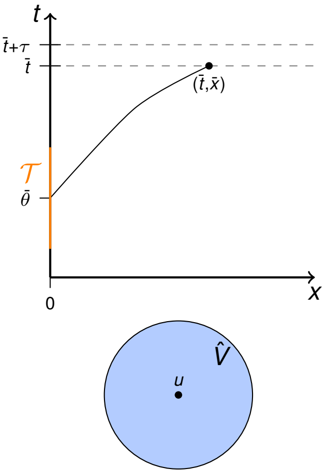

Construction of Local Solutions

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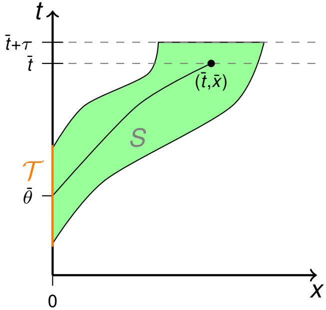

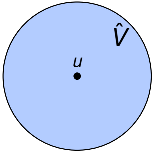

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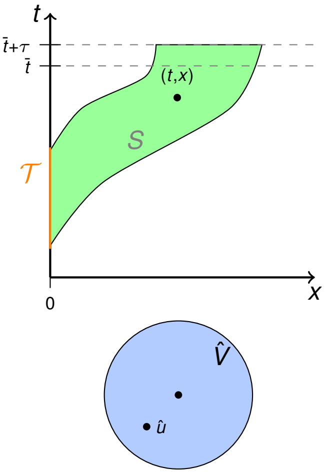
- Define $\Theta(t, x, \hat{u}) \in \mathcal{T}$ as the **unique solution** of

$$x = \zeta(t; \theta, \hat{u}_B(\theta), \hat{u}_1), \quad \theta \in \mathcal{T}$$



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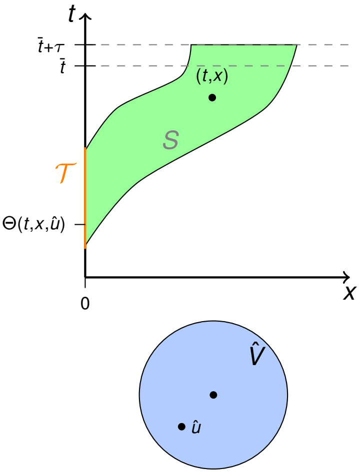

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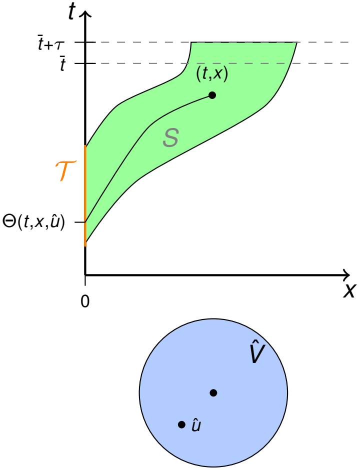

Construction of Local Solutions

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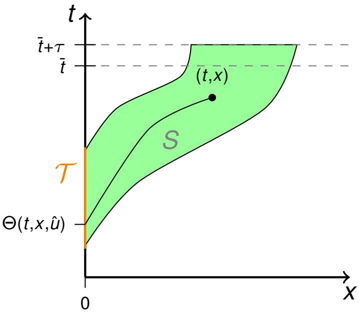
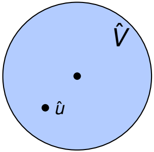
- Define $Y(t, x, \hat{u}) := v(t; \theta, \hat{u}_B(\theta), \hat{u}_1)$.





Construction of Local Solutions

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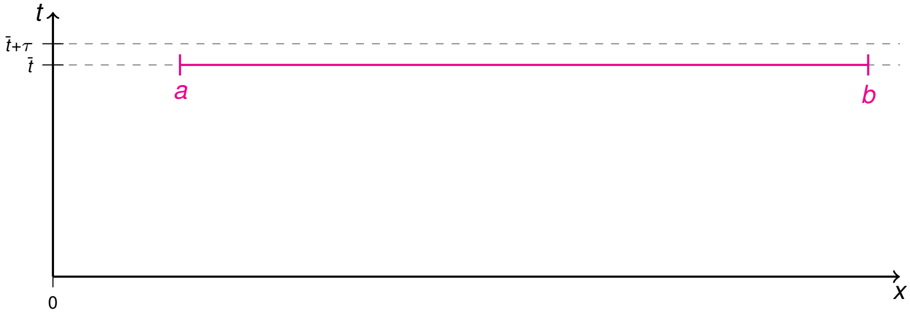
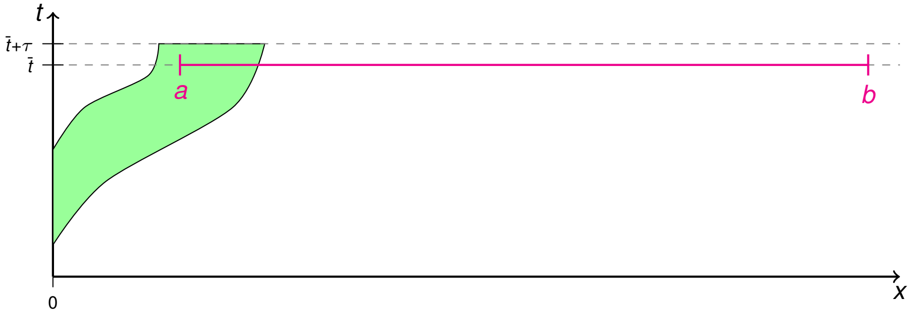
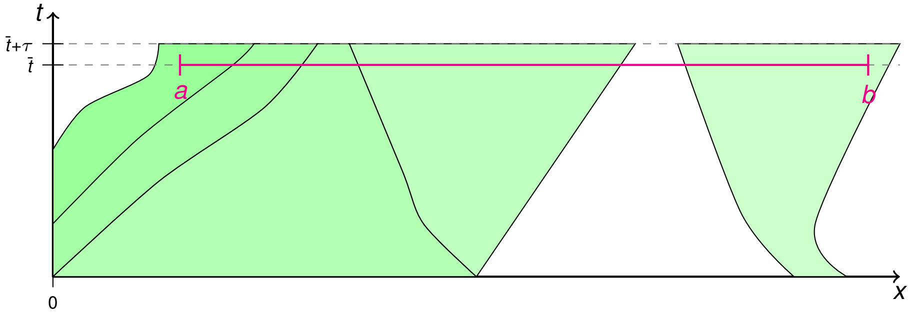


Construction of Local Solutions



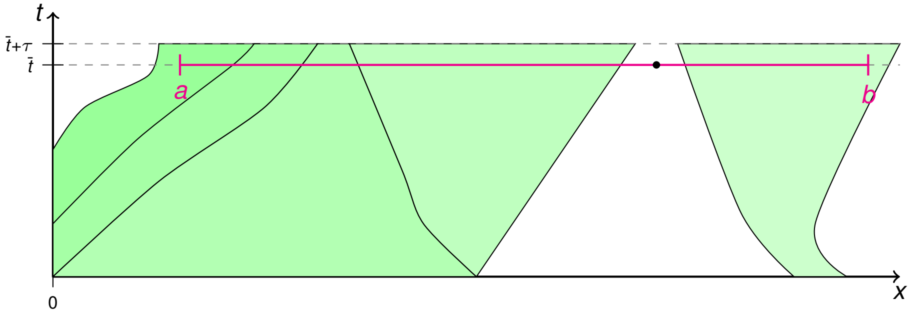
Construction of Local Solutions



For continuity points $x \in [a, b]$ of $y(\bar{t}, \cdot; u)$ we have:

- $y(\bar{t}, x; \hat{u}) = Y(\bar{t}, x, \hat{u}) + o(\hat{u} - u)$
- $(x, \hat{u}) \mapsto Y(\bar{t}, x, \hat{u})$ is continuously Fréchet-differentiable

Construction of Local Solutions



Construction of Local Solutions

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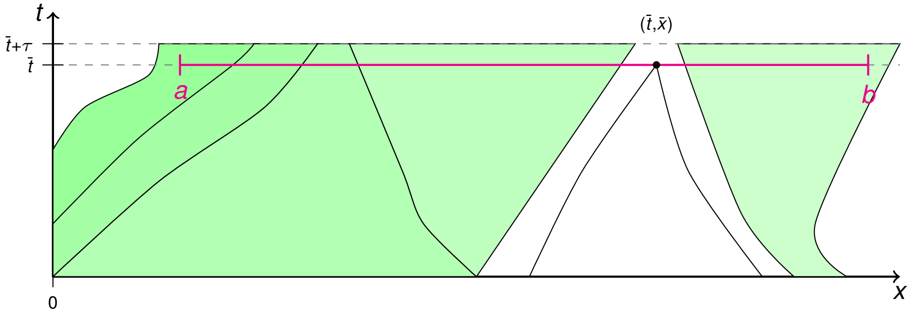
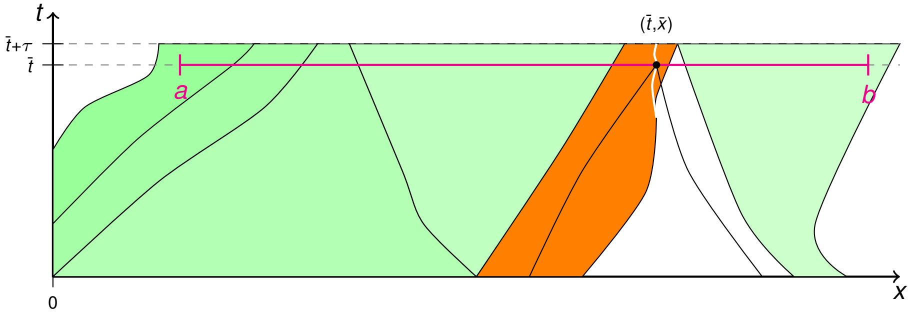


Construction of Local Solutions



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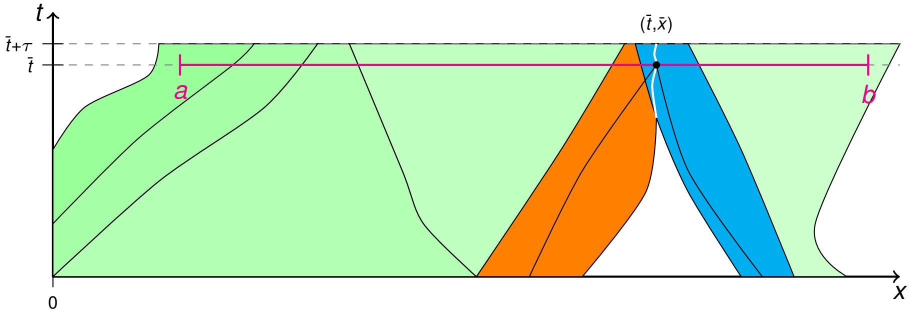

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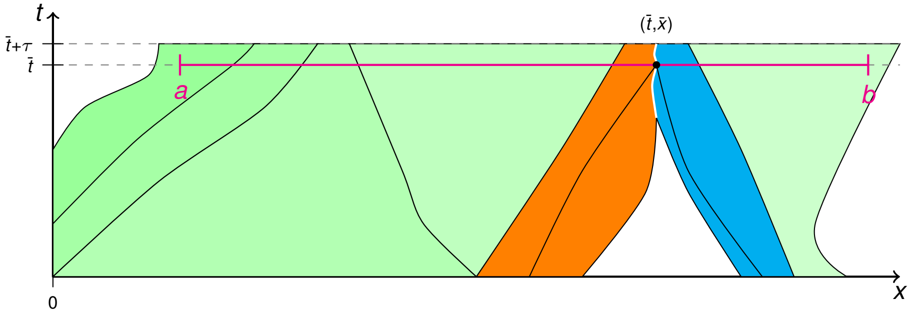

Construction of Local Solutions

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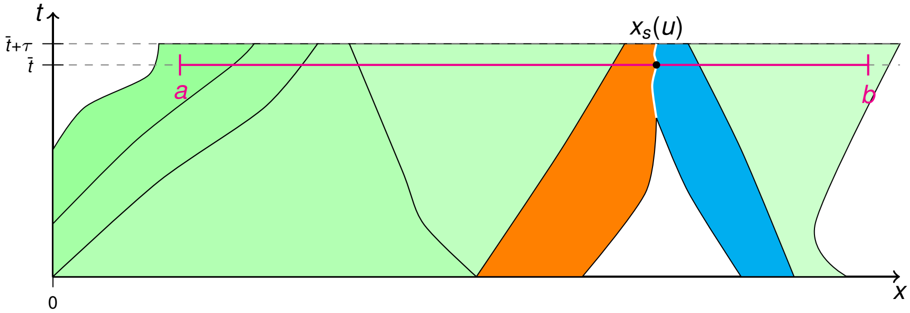

In a neighbourhood of shock points $\bar{x} \in [a, b]$ of $y(\bar{t}, \cdot; u)$ we have:

- $y(\bar{t}, x; u) = Y_\pm(\bar{t}, x, u)$ if $\operatorname{sgn}(x - x_s(u)) = \pm 1$
- $(x, \hat{u}) \mapsto Y_\pm(\bar{t}, x, \hat{u})$ is continuously Fréchet-differentiable

Construction of Local Solutions

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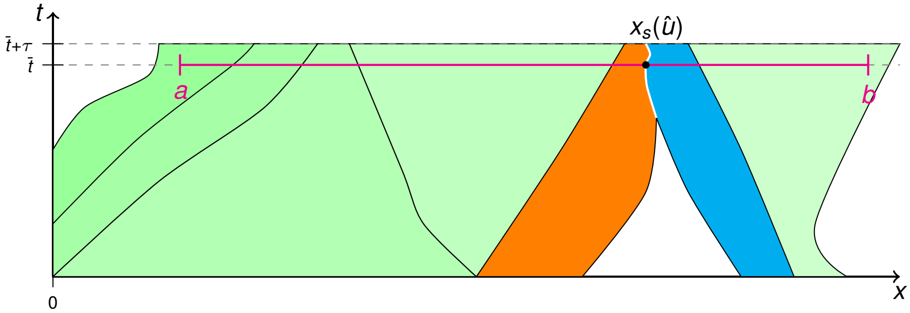

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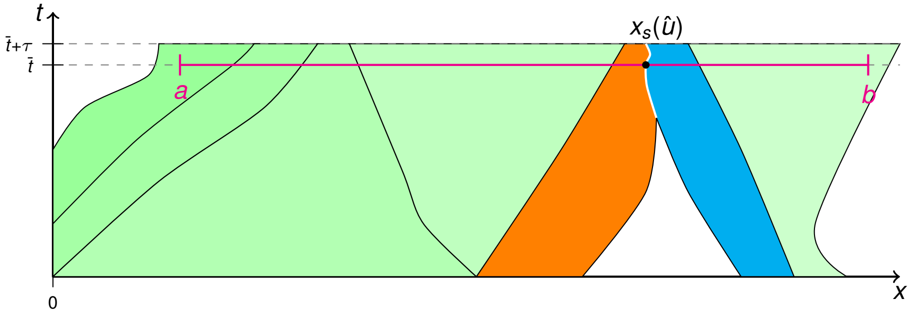
Construction of Local Solutions



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- $(x, \hat{u}) \mapsto Y_{\pm}(\bar{t}, x, \hat{u})$ **and** $\hat{u} \mapsto x_s(\hat{u})$ are continuously Fréchet-differentiable

Optimal Control of a IBVP

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$$\min J(y,u) = \int_a^b \gamma(x)\Psi(y(\bar{t},x),x)\,dx$$

$$\text{s.t. } u \in U_{ad},\ y = y(\delta u) \text{ solves}$$

$$\begin{aligned}
y_t + (f(y))_x &= g(t,x,y,u_1+\delta u_1), && (t,x) \in (0,T)\times\mathbb{R}^+ =: \Omega_T,\\
y(0,\cdot) &= u_0 + \delta u_0 + \Sigma_{u_0}^{(x_i)}(\delta x), && \text{on } \mathbb{R}^+,\\
y(\cdot,0+) &= u_B + \delta u_B + \Sigma_{u_B}^{(t_j)}(\delta t), && \text{on } (0,T).
\end{aligned}$$

Optimal Control of a IBVP

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\end{aligned}$$

- Shift-differentiability in 0 of the control-to-state mapping

$$\delta u = (\delta u_0, \delta u_1, \delta u_B, \delta x, \delta t) \in W \longmapsto y(\bar{t};\delta u) \in L^1(a,b)$$

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- Shift-differentiability in 0 of the control-to-state mapping

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- Fréchet-differentiability in 0 of the reduced cost functional

$$\hat{J} : W \to \mathbb{R} \quad \delta u \in W \longmapsto J(y(\delta u), u + \delta u).$$

Outline

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Adjoint Gradient Representation of $\hat{J}$

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Gradient Representation

$$\hat{J}'(0) \cdot \delta u = \big(p, g_{u_1}(\cdot, y, u_1)\delta u_1\big)_{2,(0,\bar{t})\times\mathbb{R}^+} + (p(0,\cdot), \delta u_0)_{2,\mathbb{R}^+} + \big(p(\cdot,0), f'(u_B)\delta u_B\big)_{2,(0,\bar{t})} + \sum_{i=1}^{N} p(0,x_i)[u_0(x_i)]_+ \delta x_i + \sum_{j=1}^{K} p(t_j,0)[f(u_B(t_j))]\delta t_j$$

Adjoint Gradient Representation of $\hat{J}$

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where p is an adequate solution **on** $[0,\bar{t}]\times\mathbb{R}^+$ of the

Adjoint Equation

$$p_t + f'(y)p_x = -g_y(t,x,y,u_1)p$$

$$p(\bar{t},x) = \gamma(x)\int_0^1 \psi_y(y(\bar{t},x+) + \tau[y(\bar{t},x)], x)\, d\tau.$$

Linear Transport Equations with Discontinuous Coefficients

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on a bounded domain

$$\begin{aligned} p_t + ap_x &= -bp \quad \text{on } (0,\bar{t}) \times \mathbb{R}^+ \\ p(\bar{t},\cdot) &= p^{\bar{t}} \quad \text{on } \mathbb{R}^+ \end{aligned}$$

Solution?

Linear Transport Equations with Discontinuous Coefficients

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on $\mathbb{R}$ without source term

$$\begin{aligned} p_t + a p_x &= \mathbf{0} && \text{on } (0,\bar{t}) \times \mathbb{R} \\ p(\bar{t},\cdot) &= p^{\bar{t}} && \text{on } \mathbb{R} \end{aligned}$$

Bouchut & James 99: $p = p_1 - p_2$ where p_i are Lipschitz solutions with $\partial_x p_i \geq 0$.

on a bounded domain

$$\begin{aligned} p_t + a p_x &= -bp && \text{on } (0,\bar{t}) \times \mathbb{R}^+ \\ p(\bar{t},\cdot) &= p^{\bar{t}} && \text{on } \mathbb{R}^+ \end{aligned}$$

Solution?

Linear Transport Equations with Discontinuous Coefficients

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on $\mathbb{R}$ without source term

$$\begin{aligned} p_t + a p_x &= \mathbf{0} && \text{on } (0,\bar{t}) \times \mathbb{R} \\ p(\bar{t},\cdot) &= p^{\bar{t}} && \text{on } \mathbb{R} \end{aligned}$$

Bouchut & James 99: $p = p_1 - p_2$ where p_i are Lipschitz solutions with $\partial_x p_i \geq 0$.

on $\mathbb{R}$ with source term

$$\begin{aligned} p_t + a p_x &= -bp + c && \text{on } (0,\bar{t}) \times \mathbb{R} \\ p(\bar{t},\cdot) &= p^{\bar{t}} && \text{on } \mathbb{R} \end{aligned}$$

Ulbrich 02: p is a solution along generalized characteristics

on a bounded domain

$$\begin{aligned} p_t + a p_x &= -bp && \text{on } (0,\bar{t}) \times \mathbb{R}^+ \\ p(\bar{t},\cdot) &= p^{\bar{t}} && \text{on } \mathbb{R}^+ \end{aligned}$$

Solution?

Reversible Solution - Example: $y_t + \frac{1}{2}(y^2)_x = 5$

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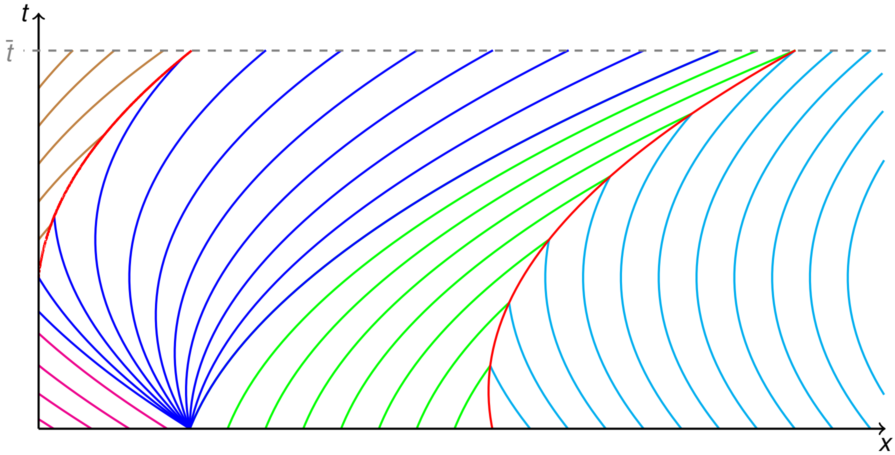

Outline

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Outlook

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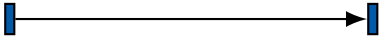
The next Steps

- Shift of Rarefaction Centers
  (Appropriate Definition of p)



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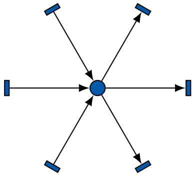

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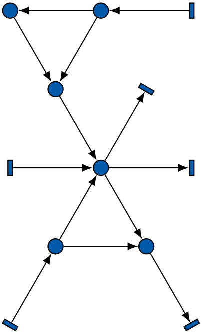

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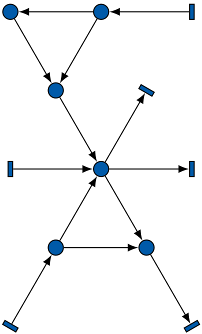
- (Shift)-Differentiability of the Trace $y(\cdot; 0+)$



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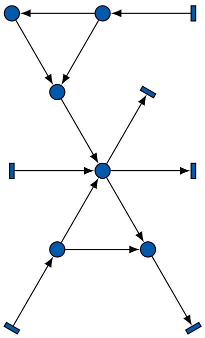

Further Questions

- Systems of Conservation Laws
- Consistency with Parabolic Regularization

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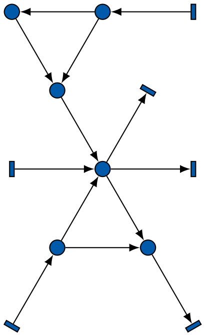

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Thank you for your attention!