Control problems and conservation laws.

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Outline

Study from the viewpoint of control theory of:

$$
\begin{aligned}
\partial_t u + \partial_x(f(u)) = g(t), \qquad (t,x) \in (0,+\infty) \times (0,1), \\
u(0,x) = u_0 \quad \text{``}u(.,0) = u_l\text{''} \quad \text{``}u(.,1) = u_r\text{''},
\end{aligned}
\tag{1}
$$

with f convex.

- Simple model,
- situation rather general (non-linear transport, boundary controls+ source term),
- nonlinear phenomenon already present (shock waves appear even for regular initial data).

Functional framework:

Following Leroux (1D scalar, 1976), Bardos & Leroux & Nédélec (scalar multi D, 1977), $u \in L^\infty(0, +\infty, \mathrm{BV}(0,1))$ solution if for any number $k \in \mathbb{R}$ and any function $\phi \geq 0$ in $\mathcal{C}_c^1(\mathbb{R}^2)$ we have:

$$\begin{aligned}\int_0^{+\infty}\int_0^1 & |u-k|\phi_t + \mathrm{sgn}(u-k)(f(u)-f(k))\phi_x \\ & + \mathrm{sgn}(u-k)g(t)\phi dt dx + \int_0^1 |u_0(x)-k|\phi(0,x)dx \\ & + \int_0^{+\infty} \mathrm{sgn}(u_r(t)-k)(f(k)-f(u(t,1^-)))\phi(t,1) \\ & \qquad - \mathrm{sgn}(u_l(t)-k)(f(k)-f(u(t,0^+))\phi(t,0)dt \geq 0. \quad (2)\end{aligned}$$

Boundary conditions

- At $x = 1$: for almost all t and for any number k between $u_r(t)$ and $u(t, 1^-)$:

$$\mathrm{sgn}(u(t, 1^-) - u_r(t))(f(u(t, 1^-) - f(k)) \geq 0. \tag{3}$$

- En $x = 0$: for almost all t and for any number k between $u_l(t)$ and $u(t, 0^+)$

$$\mathrm{sgn}(u(t, 0^+) - u_l(t))(f(u(t, 0^+)) - f(k)) \leq 0. \tag{4}$$

- Geometric viewpoint: if the boundary trace of $u \neq$ boundary data, all the waves generated by the Riemann problem must leave the domain (system case: Dubois & Lefloch (1988)), boundary layer.

Control Theory

General control system:

$$\begin{cases} \dot{X}(t) = F(X(t), U(t)), \\ X(0) = X_0, \end{cases} \tag{5}$$

state: $X \in \mathcal{X}$, control: $U \in \mathcal{U}$. Two classical problems (among many):

- Exact controllability: given $T > 0, \quad X_0, X_1 \in \mathcal{X}$, find $u : [0, T] \mapsto \mathcal{U}$ such that:

$$X \text{ solution of (5)} \Rightarrow X(T) = X_1.$$

- Asymptotic stabilization through feedback law: given $(X_e, U_e) \in \mathcal{X} \times \mathcal{U}$ stationary state, find $\mathbb{U} : \mathcal{X} \mapsto \mathcal{U}$, such that X_e is asymptotically stable for the autonomous system:

$$\dot{X}(t) = F(X(t), \mathbb{U}(X(t))).$$

General method?

- If (5)=ODE: problems rather well understood.
- Si (5)=linear PDE: general method with duality, HUM Dolecki & Russell (1977), Lions (1988),

$$\text{Exact controllability} \iff \text{Observability.}$$

- If (5)=non-linear PDE, linearized system controllable "$\Rightarrow$" local controllability of non-linear system.
- Condition only sufficient, not necessary, lot of examples where nonlinearity is essential: return method, Coron (1992).

Previous results

Huge literature in the framework of classical solutions. Not so many results in the framework of entropy solution: main difficulty = linearization is tricky... Exact controllability:

- Ancona & Marson (1998): complete description of attainable states for a scalar conservation law on $(0, +\infty)$ with a convex flux, one boundary control and an initial data equal to 0.
- Horsin (1998): sufficient condition of exact controllability for Burgers' equation on an interval.
- Adimurthi & Ghoshal & Gowda (2011?)
- Case of systems: Bressan & Coclite (2002), Ancona & Coclite (2005), Ancona & Marson (2007), Glass (2007), Ancona & Nguyen & Priuli (2012?).

Feedback control???

Additional control.

- In the end, boundary controls alone not completely satisfying: some reasonable states are not reachable, furthermore we have a minimum time for controllability.
- Additional source term $g(t)$ efficient, Chapouly (2007):

$$\partial_t u + \partial_x(\frac{u^2}{2}) = g(t), \tag{6}$$

  any regular state is attainable from any other regular state and in arbitrarily small time.
- Using extension method of Russel (1974), g only explicit control, boundary controls = under determined equation.

Exact controllability result

Theorem

Let f be a convex, regular and satisfies either:

$$\frac{f'(M)}{\sup\limits_{z\in[0,M]} f''(z)} \to +\infty \text{ if } M \to +\infty \quad \text{or} \quad \frac{f'(M)}{\sup\limits_{z\in[M,0]} f''(z)} \to -\infty \text{ if } M \to -\infty.$$

Let $T > 0$, $u_0 \in \mathrm{BV}(0,1)$ *and* $u_1 \in \mathrm{BV}(0,1)$ *satisfy for some* $K > 0$*:*

$$\forall x \in (0,1),\ 0 < h < 1 - x, \quad u_1(x+h) - u_1(x) \leq Kh.$$

Then there exist two functions g and u such that u is an entropy solution of (1) *on* $(0,T) \times (0,L)$ *and furthermore:*

$$u(0,.) = u_0 \quad \text{and} \quad u(T,.) = u_1 \quad \text{on } (0,1).$$

Remarks

- Condition on the final state related to the non reversibility in time of some entropy solutions.
- Compared to Ancona & Marson, decoupling between hypothesis on u_1 and on f.
- Strategy: g allows us to "push" u_0 and u_1 out of $(0, 1)$ and replace them by constant states.
- Control of approximation generated with a wave front tracking algorithm (Dafermos (1972), DiPerna (1976), Bressan (1992)).

Feedback loop

- Natural stationary states to stabilize: constant states.
- For a flux f strictly convex such that:

$$\lim_{z\to+\infty} f(z) = \lim_{z\to-\infty} f(z) = +\infty,$$

and a constant $\bar{u} \in \mathbb{R}$ satisfying $f'(\bar{u}) \neq 0$:

$$u(t,.) \in L^1(0,1) \mapsto \begin{cases} g(u(t,.)) = \frac{f'(\bar{u})}{2} ||u(t,.) - \bar{u}||_{L^1(0,1)}, \\ u_l(u(t,.)) = \bar{u}, \\ u_r(u(t,.)) = \bar{u}. \end{cases}$$

Asymptotic stabilization theorem

Theorem

The closed loop feedback has the following properties.

- *For any initial data* $u_0 \in \mathrm{BV}(0,1)$*, there is a unique entropy solution* u *and it belongs to:* $L^\infty((0,+\infty);\mathrm{BV}(0,1)) \cap \mathrm{Lip}([0,+\infty);L^1(0,1))$.
- *We have two constants* $C_1 > 0$ *and* $C_2 > 0$ *depending only on* $\bar{u}$ *such that:*

$$||u(t,.) - \bar{u}||_{L^1(0,1)} \leq C_1 e^{-\frac{f'(\bar{u})}{2}t} ||u_0 - \bar{u}||_{L^1(0,1)},$$

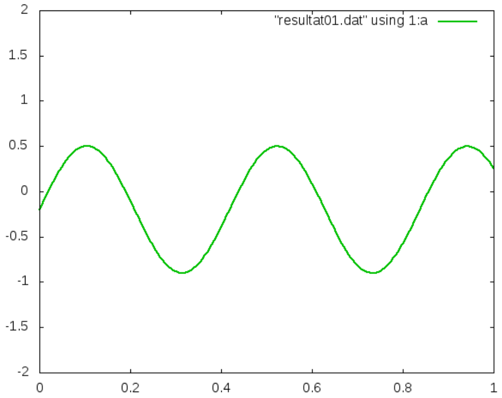
and also:

$$||u(t,.) - \bar{u}||_{L^\infty(0,1)} \leq C_2 e^{-\frac{f'(\bar{u})}{2}t} ||u_0 - \bar{u}||_{L^\infty(0,1)}.$$

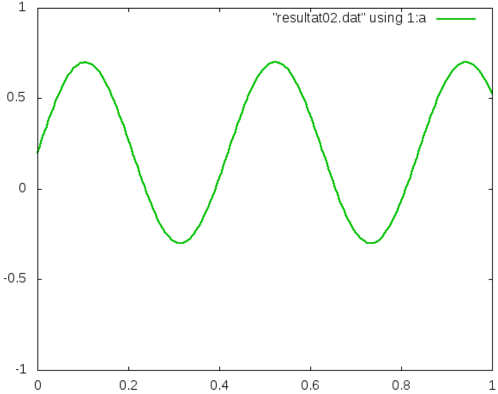
Remarks

- For the viscous equation without feedback, $\bar{u}$ is asymptotically stable but we only have:
$$||u(t,.) - \bar{u}||_{L^2(0,1)} \leq e^{-\epsilon t}||u_0 - \bar{u}||_{L^2(0,1)}.$$
- Regularizing property of the feedback law after a certain time depending only on $\bar{u}$.
- Result for $f'(\bar{u}) \neq 0$, but slower.
- A posteriori analysis via generalized characteristics of Dafermos (1977).
- Interaction of generalized characteristics with boundaries, boundary layers act "mainly" at the beginning.

Numerical simulations: Burgers' equation, Lax Friedrichs scheme, $\bar{u} = 1$



Numerical simulations: Burgers' equation, Lax Friedrichs scheme, $\bar{u} = 0$



Current work

- Stabilization of stationary shocks:

$$u_\alpha(x) = \begin{cases} \bar{u} & \text{if } x \leq \alpha, \\ -\bar{u} & \text{if } x > \alpha, \end{cases}$$

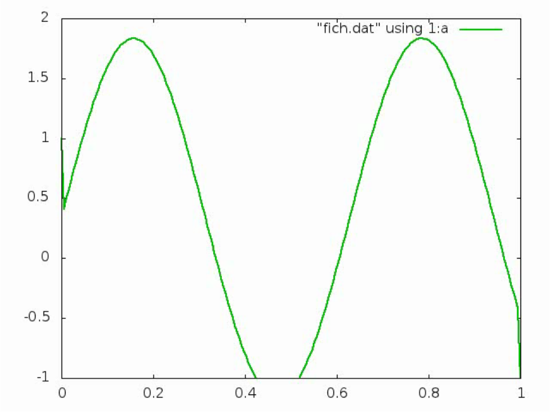
using the observation:

$$o(t) = \frac{1}{2\delta} \int_{\alpha-\delta}^{\alpha+\delta} u(t,x)dx,$$

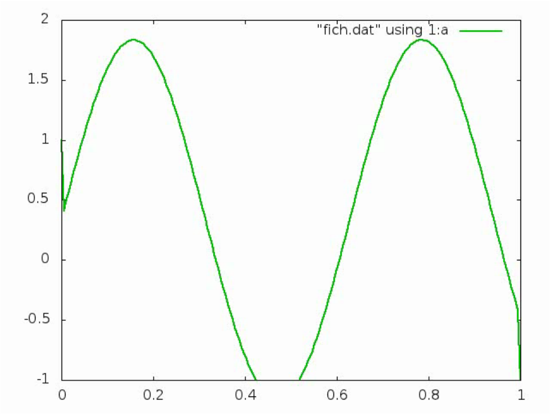
and without g.

- Boundary layer $\Rightarrow$ no Lyapunov function: robustness with respect to viscosity?
- Exact controllability for the isentropic Euler Poisson system.

Stationary shock: Burgers' equation, Lax Friedrichs scheme, good parameters



Stationary shock: Burgers' equation, Lax Friedrichs scheme, bad parameters



Isentropic Euler-Poisson system (electron in semi-conductor)

$$\begin{cases} \rho_t + j_x = 0, \\ j_t + (\frac{j^2}{\rho} + P(\rho))_x = -\sigma j + \frac{q}{\mu}\rho\phi_x, \\ \phi_{xx} = \frac{q}{\epsilon}(\rho - n), \end{cases} \tag{7}$$

ρ electron density, j current density, ϕ electric charge, q elementary charge, μ elementary mass, ϵ medium's permitivity, σ damping coefficient and n background charge (doping).

- Degond & Markowich (1990): existence of nontrivial regular stationary states,
- Bo Zhang (1992) existence via compensated compactness and Godunov scheme,
- Poupaud & Rascle & Vila (1995) global existence via Glimm's scheme.

Perspectives

- CNS on (u_0, u_1, T) to go from u_0 to u_1 in time T for the scalar equation, (use of the dissipative structure).
- Scalar case with a non convex flux with only one inflection point. Dafermos (1984)
- Asymptotic stabilization by feedback law for systems of conservation laws (Generalized characteristics for systems Trivisa).
- For Camassa-Holm and Hunter-Saxton: weak solutions Bressan & Constantin (2007).
- Fluid structure interaction Andreianov & Lagoutière & Seguin & Takahashi (2008), (2011).

Thank you for your attention