On variational kinetic formulations for scalar conservation laws and the Euler equations of gas dynamics.

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Content:

1. Scalar conservation laws:

$$\text{(S.C.L.)} \qquad \partial_t u + \operatorname{div} f(u) = 0, \quad (x,t) \in \mathbb{R}_+^{n+1},$$

   $f \in C^1(\mathbb{R})^n$.

   - Kinetic formulation (Lions-Perthame-Tadmor).
   - Variational kinetic formulation (Panov, Brenier).

2. Kinetic formulations for the Euler equations:

$$\text{(E.eqs.)} \qquad \begin{aligned} &\rho_t + \operatorname{div}(\rho u) = 0, \\ &(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p = 0, \\ &(\rho E)_t + \operatorname{div}(\rho E u + p u) = 0, \end{aligned}$$

   ρ – density, $u = (u^1, ..., u^n)$ – velocity, $E = \frac{|u|^2}{2} + e$, -total energy, e – internal energy,

$$p = (\gamma - 1)\rho e = R\rho T.$$

(S.C.L.) $$\partial_t u + \operatorname{div} f(u) = 0, \quad u(t=0) = u_0.$$

Entropy-entropy flux pair $(\eta, q) : q'(u) = f'(u)\eta'(u)$.
$u(x,t)$ is an entropy solution if for any convex entropy-entropy flux pair (η, q),

$$\partial_t \eta(u) + \operatorname{div} q(u) \leq 0, \quad \mathcal{D}'(\mathbb{R}^{n+1}_+).$$

(Kruzhkov) *For any* $u_0 \in L^\infty(\mathbb{R}^n)$, *there a unique entropy solution of* (S.C.L.) *and*

$$u \in C([0,+\infty); L^1_{loc}(\mathbb{R}^n)).$$

For any two entropy solutions u, v *with the data* u_0, $v_0 \in L^\infty \cap L^1(\mathbb{R}^n)$,

1. *for all* $t > 0$,

$$\int |u(x,t) - v(x,t)| \, dx \leq \int |u_0(x) - v_0(x)| \, dx;$$

2. *if* $u_0 \leq v_0$ *a.e.* $\mathbb{R}^n$,

$$u(x,t) \leq v(x,t), \quad \text{a.e. } \mathbb{R}^{n+1}_+.$$

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We will assume that $u(x,t)$ is L–periodic in x and for some $M > 0$,

$$0 < \operatorname{essinf} u \leq \operatorname{esssup} u < M.$$

Smooth solutions.

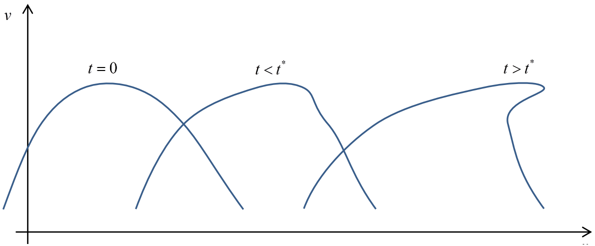
For the initial data u_0, choose a level set function $Y_0(x,v)$: $Y_0(x, u_0(x)) = \lambda$. Consider

$$\begin{cases} \partial_t Y + f'(v) \cdot \nabla_x Y = 0, \\ Y(t=0) = Y_0(x,v). \end{cases}$$

For all times $t \in (0, t^*)$ while there is $u(x,t)$ such that

$$\{(x,v) \,:\, Y(x,t,v) = \lambda\} = \{(x, u(x,t))\},$$

$u(x,t)$ is a classical solution of (S.C.L.).



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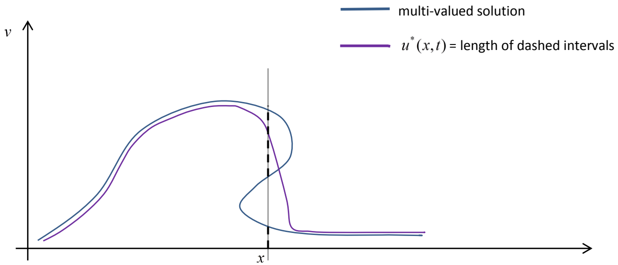

Averaging of multi-valued solutions.

Let $Y_0(x,v) = \begin{cases} 0 & v < u_0(x) \\ 1 & v \geq u_0(x) \end{cases}$,

$$u^*(h,x) = \int_0^{+\infty} (1 - Y_0(x - f'(v)h, v))\, dv.$$

Let $\omega(x)$ be a test function and compute

$$\begin{aligned}
&\int (u^*(h,x) - u_0(x))\omega(x)\, dx \\
&\qquad = \int \int_0^{+\infty} \left[(1 - Y_0(x - f'(v)h, v)) - (1 - Y_0(x,v))\right] \omega\, dxdv \\
&\qquad = \int_0^{+\infty} \int (1 - Y_0(x,v))(\omega(x + f'(v)h) - \omega(x))\, dxdv \\
&\qquad = h \int f(u_0(x))\omega_x\, dx + O(h^2).
\end{aligned}$$

u^* is approximately a weak solution of (S.C.L.) on $t \in [0,h]$.

Time discrete BGK-type approximation (Brenier, Giga-Miyakawa):

Define the kinetic function as (for $u > 0$):

$$Y(v,u) = \begin{cases} 0 & v < u \\ 1 & v \geq u. \end{cases}$$

$Y(v, u(x))$ – kenetic density of $u(x)$.
Let $h > 0$ – time step, $n \in \mathbb{N}$,

- Given $u^{n-1}(x)$, set $Y^{n-1}(x,v) = Y(v, u^{n-1}(x))$, solve

$$\begin{cases} \partial_t Y + f'(v) \cdot \nabla_x Y = 0, \\ Y(x,0,v) = Y^{n-1}(x,v), \end{cases}$$

and define

$$u^n(x,v) = \int_0^M (1 - Y(x,h,v))\, dv, \quad Y^n(x,v) = Y(v, u^n(x)).$$

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$$u^n(x,v) = \int_0^M (1 - Y(x,h,v))\, dv, \quad Y^n(x,v) = Y(v, u^n(x)).$$

- Setting $u^n = S_h(u^{n-1})$:
  - $\|S_h(u) - S_h(v)\|_{L^1} \leq \|u - v\|_{L^1}$;
  - $\|S_{h_1}(u) - S_{h_2}(u)\|_{L^1} \leq C|h_1 - h_2| TV(u)$.

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- Set $u^h(x, hn) = u^n(x)$ and linearly interpolate for $t \in (h(n-1), hn)$. Then

$$u^h \to u \text{ in } C([0,T); L^1(\mathbb{R}^n)), \ \forall T > 0,$$

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- (Vasseur) Convergence without BV bounds.

(Perthame-Tadmor) Continuous time BGK-type approximation.

$$\partial_t Y + f'(v) \cdot \nabla_x Y = \varepsilon^{-1}(Y(v, u(x,t)) - Y),$$

$$u(x,t) = \int_0^M (1 - Y(x,t,v))\, dv.$$

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Kinetic formulation of Lions-Perthame-Tadmor.

$u(x,t)$ is an entropy solution of (S.C.L.) *iff there is a nonnegative measure*

$$m \in \mathcal{M}_+(\mathbb{R}_+^{n+2}),$$

and $Y(x,t,v) = Y(v, u(x,t))$ solves:

$$\text{(K.eq.)} \qquad \partial_t Y + f'(v) \cdot \nabla_x Y = -\partial_v m,$$

$$Y(x,0,v) = Y(v, u_0(x)).$$

Applications:
(Lions-Perthame-Tadmor) $W^{s,1}_{t,x}$, $s \in (0, 1/3)$, -regularity of L^∞ solutions.
(De Lellis-Otto-Westdickenberg) Structure of L^∞ solutions.

Measure-valued solutions.

Let for every (x,t), $Y(x,t,v)$ be non-decreasing in v and $Y(x,t,0)=0$, $Y(x,t,M)=1$ and

$$\partial_t Y + f'(v)\cdot\nabla_x Y = -\,\partial_v m, \quad m \in \mathcal{M}_+(\mathbb{R}_+^{n+2}).$$

- $Y(x,t,v)$ defines a probability measure $\nu_{x,t}$ on $\mathbb{R}$:

$$\nu_{x,t}((v_1,v_2]) = Y(x,t,v_2) - Y(x,t,v_1),$$

  and for any convex entropy-entropy flux pair (η,q) :

$$\partial_t\langle\eta,\nu_{x,t}\rangle + \partial_x\langle q,\nu_{x,t}\rangle \le 0, \quad \mathcal{D}'(\mathbb{R}^{n+1}).$$

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- (Tartar) Compensated compactness method.
- (Schochet) Entropy mv-solutions (with given $\nu_{0,x}$) are not unique.

- (DiPerna) MV-solutions with

  $$\nu_{0,x} = \delta_{u_0(x)},$$

  coincide with weak entropy solutions.

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- (Tartar) Compensated compactness method.
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  - Take $Y(x,0,v)=v/M$, independent of x.
  - Take $m_1(x,t,v)\equiv 0$ and $m_2(x,t,v)=m(v)\ge 0$, $m'(0)=m'(M)=0$.
  - Obtain two solutions v/M and $v/M - tm'(v)$.
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Variational property of the kinetic solutions $Y = Y(v, u(x,t))$.

- $$\frac{d}{dt}\int_0^L\int_0^M Y^2\,dxdv = -\frac{d}{dt}\int_0^L u(x,t)\,dx = 0.$$

  Consider

  $$\partial_t Y + f'(v)\cdot\partial_x Y = -\partial_v m.$$

- Let $\tilde{Y}(x,v)$ be non-decreasing in v test function, then

  (V.K.eq.) $$\int_0^L\int_0^M (\tilde{Y} - Y)(\partial_t Y + f'(v)\cdot\partial_x Y)\,dxdv \geq 0.$$

- (V.K.eq.) *is equivalent to* (K.eq.) *if* $Y = Y(x, u(x,t))$, *but more restrictive when* Y *comes from mv-solution.*
- For mv-solutions (V.K.eq.) imposes a non-linear constraint

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- For mv-solutions (V.K.eq.) imposes a non-linear constraint
  $$\frac{d}{dt}\int_0^L\int_0^M Y^2\,dxdv = 0.$$
- Stability: $\|Y_1(\cdot,t,\cdot) - Y_2(\cdot,t,\cdot)\|_{L^2_{x,v}} \leq \|Y_1(\cdot,0,\cdot) - Y_2(\cdot,0,\cdot)\|_{L^2_{x,v}}$.
- (Panov) Existence/uniqueness of mv-solutions verifying (V.K.eq.).

(Panov, Brenier) Y *is a solution of* (V.K.eq.) *iff the level curves of* $Y(x,t,\cdot)$,

$$u_\lambda(x,t) = \sup\{v \,:\, Y(x,t,v) \le \lambda\}, \quad \forall \lambda \in [0,1],$$

are weak entropy solutions of (S.C.L.)

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Consider

$$\int_0^L \int_0^M (\tilde{Y} - Y)(\partial_t Y + f'(v) \cdot \partial_x Y)\, dx dv \ge 0.$$

- Take

$$\tilde{Y} = Y + c_0 \phi'(Y) \eta'(v) \omega(x,t),$$

$c_0 > 0$, $\phi' \ge 0$, $\eta'' \ge 0$. Test function $\omega \ge 0$, smooth, L–periodic in x.

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- Compute
$$\partial_v \tilde{Y} = \partial_v Y(1 + c_0 \phi'' \eta' \omega) + \phi' \eta'' \omega \geq 0.$$

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$$\begin{aligned} 0 \quad &\le \quad \int_0^L \int_0^M \omega \eta'(v) \partial_t \phi(Y) + \omega q'(v) \partial_x \phi(Y)\, dv dx \\ &= \quad \int_0^L \omega_t \left(\int_0^M \eta(v) \phi'(Y) Y_v\, dv \right) dx + \int_0^L \omega_x \left(\int_0^M q(v) \phi'(Y) Y_v\, dv \right) dx \\ &\quad\quad ...\lambda = Y(\cdot,\cdot,v)... \\ &= \quad \int_0^L \omega_t \int_0^1 \eta(u_\lambda) \phi'(\lambda)\, d\lambda + \int_0^L \omega_x \int_0^1 q(u_\lambda) \phi'(\lambda)\, d\lambda, \end{aligned}$$

$\forall\ \phi'(\lambda) \ge 0.$

(Brenier) Define

$$\mathbb{H} = \{Y \in L^2((0,L)\times(0,M)),\ L-\text{periodic}\},$$
$$K = \{Y \in \mathbb{H},\ \text{ nondecreasing in } v.\}$$

- $K-$ closed convex cone.
  $\partial K(Y) = \{Z \in \mathbb{H}, \int_0^L \int_0^M (\tilde{Y} - Y)\cdot Z\, dvdx \le 0\}.$

$$\int_0^L \int_0^M (\tilde{Y} - Y)(\partial_t Y + f'(v)\cdot \partial_x Y)\, dxdv \ge 0,$$

(Diff.incl.) $\partial_t Y \in\ -(f'(v)\cdot\nabla_x Y + \partial K(Y)).$

$f'(v)\cdot\nabla_x Y + \partial K(Y)$ monotone (and maximal if $|f'(v)| \neq 0$).

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1. (Existence/Uniqueness) For any initial data $Y_0 \in K$, there a unique solution $Y \in C([0,+\infty);\mathbb{H})$.
2. (Regularity) If $\partial_x Y_0 \in \mathbb{H}$ then
   $$\partial_x Y,\ \partial_t Y \in L^\infty((0,+\infty);\mathbb{H}).$$
   If, in addition, $\partial_v Y_0 \in \mathbb{H}$, then
   $$\partial_v Y \in L^\infty((0,+\infty);\mathbb{H}).$$
3. (Stability) L^p stability: for any $p \in [1,+\infty]$, and two solutions Y_i
   $$\|Y_1(t) - Y_2(t)\|_{L^p} \le \|Y_1(0) - Y_2(0)\|_{L^p}, \quad \forall t > 0.$$

(Brenier) *Solutions* Y^h *of a time-discrete BGK-type approximation converge to a solution of* (Diff.incl.).

Projection-type approximation of (Diff.incl.):

1. $h > 0$ – time step. Given $Y^{n-1}(x, v) \in K$, define

$$Y^n = \mathrm{Proj}_K(Y^{n-1}(x - hf'(v), v)).$$

2. $$\|\nabla_{x,v} Y^n\| \leq \|\nabla_{x,v} Y^0\|, \quad \|\frac{Y^n - Y^{n-1}}{h}\| \leq C\|\partial_x Y^0\|.$$

3. Define $Y^h : Y^h(x, nh, v) = Y^n(x, v)$, and linear for $t \in [(n-1)h, nh]$.

$$\|Y^h(t) - Y^h(s)\| \leq C|t - s|,$$

$$Y^h \to Y, \quad \text{in } C([0, T]; \mathbb{H}), \ \forall T > 0,$$

Y – solution of (Diff.incl.).

Strong solutions Y of (Diff.incl.) are minimal solutions:

$$\|\partial_t Y\| = \min_{Z \in \partial K(Y)} \|f'(v)\partial_x Y + Z\|, \ \forall t > 0.$$

Dual formulation: define the tangent cone

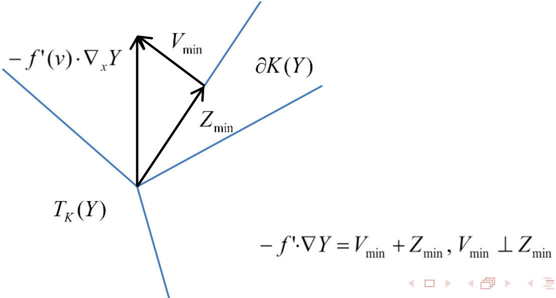
$$T_K(Y) = \mathbb{H} - \text{closure of } \{h(\tilde{Y} - Y),\ h \geq 0,\ \tilde{Y} \in K\}.$$

- $\partial_t Y \in T_K(Y)$.
- $T_K(Y)$ is a polar cone to $\partial K(Y)$.

Then,

$$\|\partial_t Y + f'(v) \cdot \nabla_x Y\| = \min_{V \in T_K(Y)} \|V + f'(v) \cdot \nabla_x Y\|.$$

$\partial_t Y$ minimizes an "interaction functional" $\min_{V \in T_K(Y)} \|V + f'(v) \cdot \nabla_x Y\|$.



Example. Let $x \in \mathbb{R}$, $f'' \geq 0$ and $u^- < u^+$.

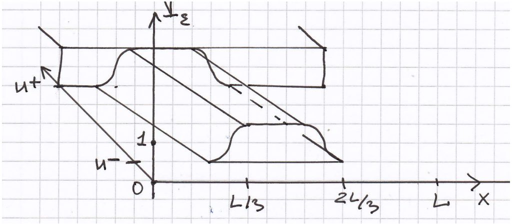
$$u_0(x) = \begin{cases} u^+ & x \in [0, L/3) \\ u^- & x \in [L/3, 2L/3) \\ u^+ & x \in [2L/3, L] \end{cases} .$$

Set $Y_0 = Y(x, u_0(x))$, and $Y_\varepsilon(x, v) = Y_0(x, v) * \omega_\varepsilon(x)$.
Let $\partial_t Y_\varepsilon$ be the solution of the minimization problem

$$\min_{V \in T_K(Y_\varepsilon)} \|V + f'(v) \cdot \nabla_x Y_\varepsilon\|,$$

then

$$\partial_t Y_\varepsilon = \begin{cases} -f'(v) \cdot \nabla_x Y_\varepsilon & x \text{ close to } 2L/3 \\ -\sigma \partial_x Y_\varepsilon & x \text{ close to } L/3 \\ 0 & \text{otherwise} \end{cases} , \; \sigma = \frac{f(u^+) - f(u^-)}{u^+ - u^-}.$$



Part II

Admissible set $K = \{$ maxwellians $\}$.
Motivation – gradient flows in the spaces of prob. measures.

$$\partial_t f + \operatorname{div}_v(\xi f) = 0, \quad \xi \in \partial\Phi(f),$$

$\Phi(f)$ – displacement convex functional.

- (Otto) The heat eq. and porous medium eq.

$$\Phi = -\int f \ln f \, dv, \; \Phi = \frac{1}{m-1}\int \rho^m \, dv.$$

- (Kinderleher-Jordan-Otto) Fokker-Planck equation.

$$\Phi = -\int f \ln\left(\frac{f}{\bar{f}}\right) dv, \quad \bar{f} \in K.$$

- (Carlen-Gangbo) Time-discrete scheme for a model Boltzmann equation:

$$\partial_t f + v \cdot \nabla_x f - T\operatorname{div}_v(f \nabla_v \ln \frac{f}{M_f}) = 0.$$

The Euler equations in $\mathbb{R}^3$ for monatomic gas ($\gamma = 5/3$).

- $(\rho, u, T)(x,t)$ – density, velocity and temperature of the gas.
- Kinetic density: $f = \frac{\rho}{(2\pi)^{3/2}} e^{-\frac{|v-u|^2}{2T}}$.
- The Euler equations:

$$\text{(E.eqs.)} \qquad \int \begin{bmatrix} 1 \\ v \\ |v|^2 \end{bmatrix} (\partial_t f + v \cdot \nabla_x f)\, dv = 0.$$

Maxwellian densities:

$$K = \left\{ \mu = \frac{e^{-\frac{|v-u|^2}{2T}}}{(2\pi T)^{3/2}} \; : \; u \in \mathbb{R}^3,\, T \in \mathbb{R}^+ \right\}$$

in the metric space

$\mathcal{P}_r^2 = \{$ abs. continuous prob. measures on $\mathbb{R}^3$ with finite second moments$\}$.

(Refs.: Ambrosio-Gigli-Savaré, Villani.)
Metric $W_2(\mu,\nu)$ given by

$$W_2^2(\mu,\nu) = \int |t_\mu^\nu(v) - v|^2 \mu\, dv = \min_{t(v):\nu=t\#\mu} \int |t(v) - v|^2 \mu\, dv,$$

$t_\mu^\nu(v)$ – optimal map, carrying μ to ν.

Tangent vector: let $\mu(t)$ be a smooth curve in $\mathcal{P}_r^2$. The tangent vector to $\mu(t)$ is defined as

$$\mathrm{v}(t,v) = \lim_{h\to 0} \frac{t_{\mu(t)}^{\mu(t+h)}(v) - v}{h}, \quad \text{in } L^2_{\mu(t)}.$$

- $\mathrm{v}(t,v)$ is the transport velocity:

$$\partial_t \mu + \mathrm{div}_{\,v}(\mathrm{v}\mu) = 0.$$

For μ_1, μ_2 in K,

$$t_{\mu_1}^{\mu_2}(v) = \sqrt{\frac{T_2}{T_1}}v + u_2 - \sqrt{\frac{T_2}{T_1}}u_1.$$

For $\mu(t) \subset K$, tangent vectors $\mathrm{v} = \alpha + \beta v$, $\alpha \in \mathbb{R}^3$, $\beta \in \mathbb{R}$.
Tangent plane to K at μ :

$$T_K(\mu) = \{\alpha + \beta v \,:\, \alpha \in \mathbb{R}^3,\, \beta \in \mathbb{R}\}.$$

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Convexity (McCann):
Given $\mu_1, \mu_2 \in K$, and $\alpha \in [0, 1]$, let

$$t_\alpha(v) = \alpha v + (1 - \alpha) t_{\mu_1}^{\mu_2}(v),$$

- $t_\alpha \# \mu_1$ is a constant speed geodesic between μ_1 and μ_2.
- $t_\alpha \# \mu_1$ coincides the optimal map $t_{\mu_1}^{\mu_\alpha}$, where μ_α is the maxwellian with (u_α, T_α) given by

$$\begin{cases} u_\alpha = (1 - \alpha) u_2 + \alpha u_1, \\ \sqrt{T_\alpha} = (1 - \alpha)\sqrt{T_2} + \alpha \sqrt{T_1}. \end{cases}$$

For

$$\Phi(\mu) = \begin{cases} 0 & \mu \in K, \\ +\infty & \mu \notin K, \end{cases}$$

- Φ is displacement (geodesically) convex: for any μ_1, $\mu_2 \in K$,

$$\Phi(t_\alpha \# \mu_1) \leq \alpha \Phi(\mu_1) + (1 - \alpha)\Phi(\mu_2).$$

Subdifferential:
Let $\Phi : \mathcal{P}_r^2 \to \mathbb{R} \cup \{+\infty\}$ be displacement convex. An element $\xi \in L^2_\mu$ belongs to $\partial\Phi$ for any $\nu \in \mathcal{P}_r^2$,

$$\Phi(\nu) \geq \Phi(\mu) + \int \xi(v) \cdot (t_\mu^\nu(v) - v)\, d\mu.$$

$\xi \in \partial K(\mu)$, with $\mu \in K$, if $\xi \in L^2_\mu$ and

$$\int \xi \cdot (\alpha + \beta v)\mu\, dv = 0,\ \alpha \in \mathbb{R}^3, \beta \in \mathbb{R},$$

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Given $(\rho, u, T)(x,t)$ smooth solution on (E.eqs.) and f its kinetic density compute

(K.E.eq.) $$\partial_t f + v \cdot \nabla_x f + \mathrm{div}\,_v(\xi f) = 0,$$

with

$$\xi = -\left(3 - \frac{|v-u|^2}{T}\right)\frac{\nabla_x T}{2} + (v-u)^t\left(\mathbb{D} - \frac{1}{3}\mathrm{tr}(\mathbb{D})\mathbb{I}\right),$$

where $\mathbb{D} = (\nabla_x u + \nabla_x^t u)/2$.

With $\mu = f(x,t,\cdot)/\rho(x,t) \in K$,

$$\xi(x,t,\cdot) \in L^2_{\mu(x,t,\cdot)}(\mathbb{R}^3), \quad \xi(x,t,\cdot) \in \partial K\left(\mu(x,t,\cdot)\right).$$

Minimization.

Consider a normalized transport curve $\eta_h \in \mathcal{P}_r^2$,

$$\eta_h = \frac{f(x - vh, t, v)}{\int f(x - hv, t, v)\, dv},$$

and $\mu_h = f(x, t + h, v)/\rho(x, t + h) = \mu(t + h) \in K$.
Let $\xi_2, \xi_1 \in L^2_{\mu_0}$ be the tangent vectors to η_h and μ_h at $h = 0$:

$$\begin{aligned} \partial_h \mu_h + \mathrm{div}\,_v(\xi_1 \mu_h) &= 0, \\ \partial_h \eta_h + \mathrm{div}\,_v(\xi_2 \eta_h) &= 0. \end{aligned}$$

Then, with ξ from the (K.E.eq.),

$$\xi_2 = \xi_1 + \xi, \quad \xi_1 \perp \xi \quad \text{in } L^2_{\mu(t)},$$

and

$$\|\xi_1 - \xi_2\|_{L^2_{\mu(t)}} = \min_{\tilde{\xi} \in T_K(\mu(t))} \|\tilde{\xi} - \xi_2\|_{L^2_{\mu(t)}}.$$

Discrete time projection scheme:

1. (Transport) Given $f^{n-1}(x,\cdot)$ and $h>0$, define

$$\rho^n(x) = \int f^{n-1}(x-hv,v)\,dv,\ \tilde{\mu}^n(x,v) = \frac{f^{n-1}(x-vh,v)}{\rho^n(x)}.$$

2. (Projection) Find a minimizer $\mu^n \in K$:

$$W_2(\mu^n,\tilde{\mu}^n) = \min_{\eta\in K} W_2(\eta,\tilde{\mu}^n).$$

3. Set

$$f^n(x,v) = \rho^n(x)\mu^n(x,v).$$

- There is unique minimizer μ^n in Step 2 and (compare with BGK)

$$\begin{aligned}
\int f^n\,dv &= \int f^{n-1}(x-hv,v)\,dv,\\
\int vf^n\,dv &= \int vf^{n-1}(x-hv,v)\,dv,\\
\int |v|^2 f^n\,dv &< \int |v|^2 f^{n-1}(x-hv,v)\,dv.
\end{aligned}$$

Local error estimate.

Let $(\rho, u, T) \in C^2_{t,x}(\mathbb{R}^3 \times [0, T_0])$ *be a solution of the Euler equations with*

$$\inf \rho = \inf_{\mathbb{R}^3 \times [0,T_0]} \rho(x,t) > 0,$$
$$\inf T = \inf_{\mathbb{R}^3 \times [0,T_0]} T(x,t) > 0.$$

Let $f(x,t,v)$ *be the kinetic density of* (ρ, u, T), $\mu = f/\rho$, *and* $h > 0$.
With μ^1 *– first iteration of the discrete scheme,*

$$W_2(\mu^1(x,\cdot), \mu(x,h,\cdot)) = O(h^2),$$

uniformly in $x \in \mathbb{R}^3$. *Additionally, uniformly in* $x \in \mathbb{R}^3$,

$$\int \begin{bmatrix} 1 \\ v_i \\ |v|^2 \end{bmatrix} (f^1(x,v) - f(x,h,v))\, dv = O(h^2),\ i = 1..3.$$

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Open questions.

- Convergence of the scheme to a smooth solution.
- Variational formulation for weak solution.

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