Practical CFL conditions for MUSCL schemes solving Euler equations

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Joint work with
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Outline

Issue

- **System of conservation laws** modelling a physical phenomenon like in fluid mechanics, ...

$$\begin{cases} \partial_t \mathbf{W} + \nabla \cdot \mathcal{F}(\mathbf{W}) = 0, \\ \mathbf{W}(0, \mathbf{x}) = \mathbf{W}_0(\mathbf{x}). \end{cases}$$

- **Physical constraints**: $\mathbf{W} \in \mathcal{W}$
  - Maximum principle (Euler for incompressible fluids: density)
  - Positivity (Euler for compressible fluids: density and pressure)
- These constraints must be satisfied:
  - at the **continuous** level (relevance of the mathematical model)
  - at the **discrete** level (robustness of the numerical scheme)

Example

Euler equations for a 2D perfect fluid:

$$\mathbf{W} = {}^t(\rho, \rho\mathbf{u}, \rho E), \quad \mathcal{F} = {}^t(\rho\mathbf{u}, \rho\mathbf{u}\otimes\mathbf{u} + p\mathcal{I}_2, (\rho E + p)\mathbf{u})$$

$$\mathcal{W} = \left\{ \mathbf{W} \in \mathbb{R}^4 \ : \ \rho = W_1 > 0 \ \text{ et } \ p = (\gamma - 1)\left[W_4 - \frac{W_2^2 + W_3^2}{2W_1}\right] > 0 \right\}$$

Riemann problem: $\mathbf{W}_0(\mathbf{x}) = \begin{cases} \mathbf{W}_l, & \text{if } x_1 < 0, \\ \mathbf{W}_r, & \text{if } x_1 > 0. \end{cases}$

Rarefaction waves and vacuum (Einfeldt, Munz, Roe & Sjgreen, 1991):

- $\rho_0 > 0,\ u_0 > 0,\ E_0 > u_0^2/2$
- $\mathbf{W}_l = (\rho_0, -\rho_0 u_0, 0, \rho_0 E_0)$ and $\mathbf{W}_r = (\rho_0, \rho_0 u_0, 0, \rho_0 E_0)$
- If $\dfrac{4\gamma}{3\gamma - 1} E_0 > u_0^2$, then density and pressure remain **positive**.

Positivity-preserving schemes

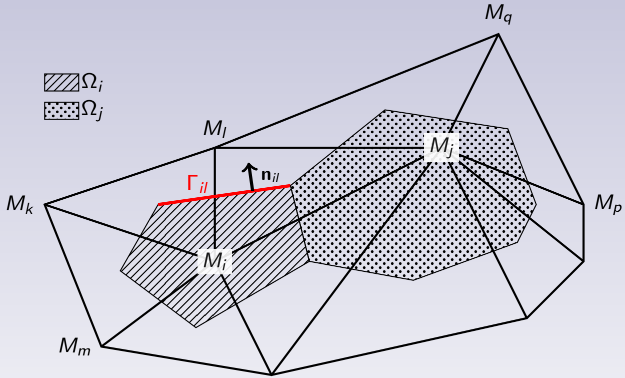
1st order

- Einfeldt *et al.* (1991), Bouchut (2004)
- **Godunov, Rusanov, HLLs** / **Roe**

2nd order: FV schemes + MUSCL strategy

- **Scalar equations**: Clain & Clauzon (2010), Calgaro *et al.* (2010)
  - Modification of limiters
  - Adaptation of the CFL condition
- **Systems of conservation laws**:
  - ... *monoslope* reconstruction: Perthame & Shu (1996)
  - ... *multislope* reconstruction: Berthon (2006)

MUSCL strategy



$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \Delta t^n \sum_{j \in \mathcal{V}(i)} \frac{|\Gamma_{ij}|}{|\Omega_i|} \mathcal{F}\left(\mathbf{W}_{ij}^n, \mathbf{W}_{ji}^n, \mathbf{n}_{ij}\right)$$

Modification of Berthon's strategy (2006)

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$$\overline{\mathbf{W}} = \mathbf{W}^n - \frac{\Delta t}{\ell}\left[\boldsymbol{\mathcal{F}}(\mathbf{W}^n, \mathbf{V}^n, \mathbf{n}) - \boldsymbol{\mathcal{F}}(\mathbf{W}^n, \mathbf{W}^n, \mathbf{n})\right]$$

for a suitable 1D flux $\mathcal{F}$ and a small enough time step Δt

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$$\mathbf{W}_i^n = \eta_i^* \mathbf{W}_i^* + (1-\eta_i^*) \sum_{j\in\mathcal{V}(i)} \eta_{ij} \mathbf{W}_{ij}^n$$

$$\mathbf{W}_i^{n+1} = \eta_i^* \overline{\mathbf{W}}_i^* + (1-\eta_i^*) \sum_{j\in\mathcal{V}(i)} \eta_{ij} \overline{\mathbf{W}}_{ij}$$

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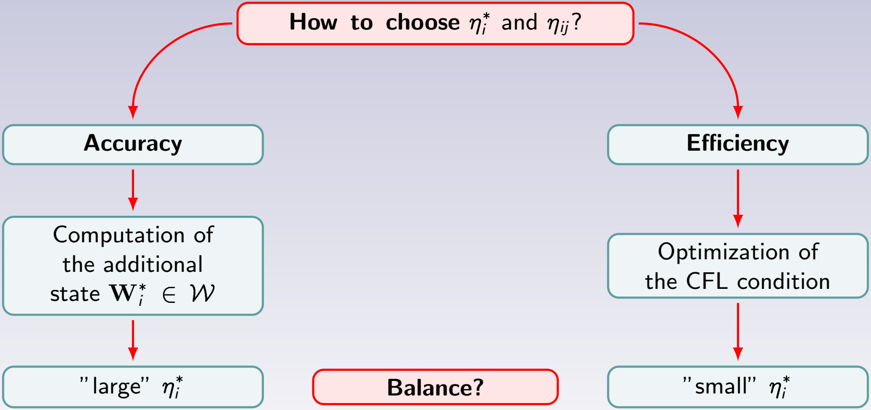

Outline

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1. Introduction
2. **Computation of the time step**
3. Admissibility
4. Simulations

Convex combinations

$$\mathbf{W}_i^{n+1} \quad = \quad \mathbf{W}_i^n \; - \; \Delta t^n \sum_{j \in \mathcal{V}(i)} \frac{|\Gamma_{ij}|}{|\Omega_i|} \boldsymbol{\mathcal{F}}\left(\mathbf{W}_{ij}^n, \mathbf{W}_{ji}^n, \mathbf{n}_{ij}\right)$$

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$$\overline{\mathbf{W}}_{ij} = \mathbf{W}_{ij}^n - \Delta t^n \sum_{k=1}^{4} \zeta_{ij,k} \boldsymbol{\mathcal{F}}\left(\mathbf{W}_{ij}^n, \mathbf{W}_{ij,k}^n, \mathbf{n}_{ij,k}\right), \quad j \in \mathcal{V}(i)$$

$$\begin{cases} \overline{\mathbf{W}}_{ij} = \displaystyle\sum_{k=1}^{4} \frac{\zeta_{ij,k}}{\mu_{ij,k}} \overline{\mathbf{W}}_{ij,k} \\ \overline{\mathbf{W}}_{ij,k} = \mathbf{W}_{ij}^n - \Delta t^n \mu_{ij,k} \left[\boldsymbol{\mathcal{F}}(\mathbf{W}_{ij}^n, \mathbf{W}_{ij,k}^n, \mathbf{n}_{ij,k}) - \boldsymbol{\mathcal{F}}(\mathbf{W}_{ij}^n, \mathbf{W}_{ij}^n, \mathbf{n}_{ij,k})\right] \end{cases}$$

Assumptions

Flux In addition to classical properties, we assume:
$\forall\,(V,W)\in\mathcal{W}^2,\ W-\frac{\Delta t}{\ell}[\mathcal{F}(W,V)-\mathcal{F}(W,W)]\in\mathcal{W}$
under the CFL condition $\Delta t\max_k|\lambda_k(V,W)|\leqslant\alpha_0\ell$.

CFL Condition $\Delta t^n\times\max_{j\in\mathcal{V}(i)}\left\{\mu^*_{ij},\max_{1\leq k\leq 4}\mu_{ij,k}\right\}\times\bar{\lambda}^n_i\leq\alpha_0$

$$\bar{\lambda}^n_i:=\max_{\substack{j\in\mathcal{V}(i)\\1\leq k\leq 4}}\left\{|\mathbf{u}^n_{ij}\cdot\mathbf{n}_{ij,k}|+c^n_{ij},|\mathbf{u}^n_{ij,k}\cdot\mathbf{n}_{ij,k}|+c^n_{ij,k}\right\}$$

Optimization The optimal coefficient reads

$$\mu_i^{\mathrm{opt}}(\eta_i^*,\eta_{ij})=\begin{cases}\dfrac{2}{(1-\eta_i^*)|\Omega_i|}\max_{j\in\mathcal{V}(i)}\dfrac{|\Gamma_{ij}|}{\eta_{ij}}, & \text{si }\eta_i^*\geqslant\overline{\eta_i^*}\\[2ex]\dfrac{|\partial\Omega_i|}{|\Omega_i|}\left[\min_{j\in\mathcal{V}(i)}\left\{\eta_i^*\left(1-\dfrac{2|\Gamma_{ij}|}{|\partial\mathrm{T}_{ij}|}-\dfrac{\eta_{ij}|\partial\Omega_i|}{|\partial\mathrm{T}_{ij}|}\right)+\dfrac{\eta_{ij}|\partial\Omega_i|}{|\partial\mathrm{T}_{ij}|}\right\}\right]^{-1}\end{cases}$$

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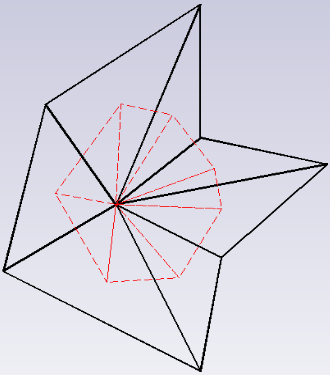
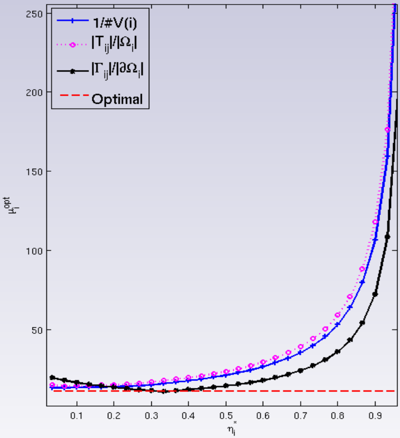
Optimization An optimal bound for the solution is given by:

$$\mu_i^{\text{opt}}\left(\eta^*_i=\frac{1}{3},\eta_{ij}=\frac{|\Gamma_{ij}|}{|\partial\Omega_i|}\right)=3\frac{|\partial\Omega_i|}{|\Omega_i|}.$$

Example




1/#V(i)
$|T_{ij}|/|\Omega_i|$
$|\Gamma_{ij}|/|\partial\Omega_i|$
Optimal
μ_i^{opt}
η_i^*
250
200
150
100
50
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

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Key-point

Reconstructed states

$\mathbf{W}_i^n + \Delta\mathbf{W}_{ij}^n \in \mathcal{W}$ provided a suitable τ-limiter is used.

Making the additional state admissible

$$\mathbf{W}_{ij}^n = \mathbf{W}_i^n + \Delta\mathbf{W}_{ij}^n,$$

$$\mathbf{W}_i^* = \frac{1}{\eta_i^*}\left[\mathbf{W}_i^n - (1-\eta_i^*)\sum_{j\in\mathcal{V}(i)} \eta_{ij}\mathbf{W}_{ij}^n\right] = \mathbf{W}_i^n - \frac{1-\eta_i^*}{\eta_i^*}\sum_{j\in\mathcal{V}(i)} \eta_{ij}\Delta\mathbf{W}_{ij}^n$$

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$$\beta_i^n \in [0,1] \implies \mathbf{W}_{ij}^n = (1-\beta_i^n)\mathbf{W}_i^n + \beta_i^n\left[\mathbf{W}_i^n + \Delta\mathbf{W}_{ij}^n\right] \in \mathcal{W}$$

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$$\rho_i^* > 0,\ p_i^* > 0 \quad \implies \quad \beta_i^n = \min\left\{1, \frac{\eta_i^*}{1-\eta_i^*}\xi_i^*\right\}.$$

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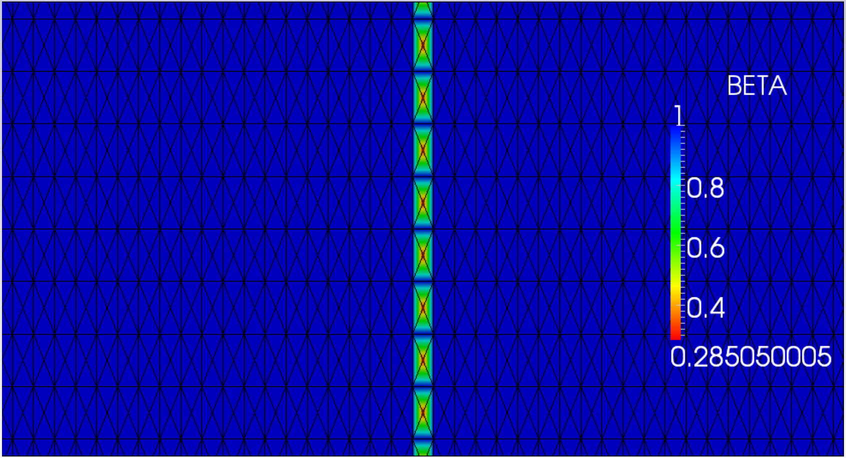
1-2-3 test case



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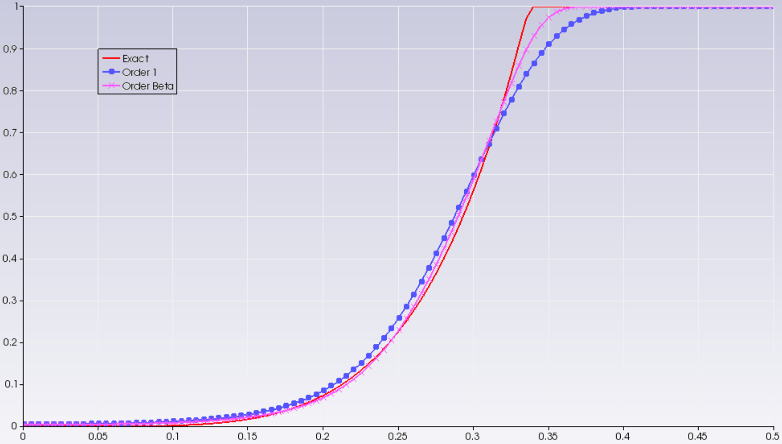

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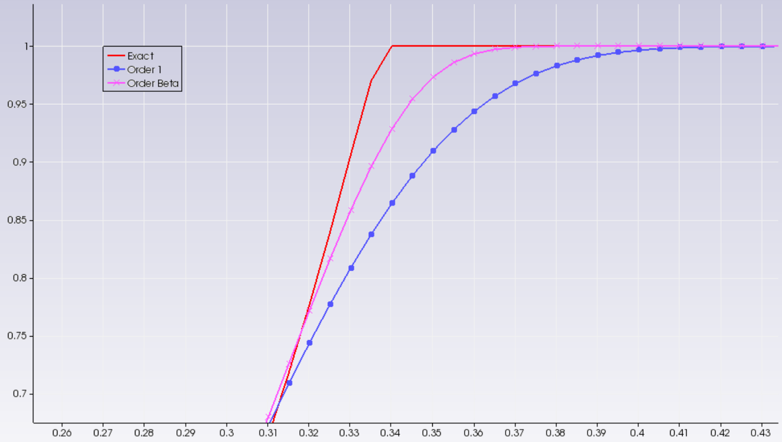

Perspectives

Done

- ✓ Analysis of robustness of general MUSCL strategies
- ✓ Derivation of (sufficient) conditions to preserve positivity
- ✓ Explicit CFL conditions
- ✓ Easy adaptation of industrial codes

C. Calgaro, E. Creusé, T. Goudon & Y. Penel, *Positivity-preserving schemes for Euler equations: sharp and practical CFL conditions* (under revision).

To do

- ❧ Influence of the numerical flux on β_i^n
- ❧ Determination of sets of parameters that activate β_i^n
- ❧ Application to other systems of conservation laws