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On the decay property for periodic entropy solutions to scalar conservation laws

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In the half-space $\Pi = (0,+\infty)\times\mathbb{R}^n$ we consider the conservation law

$$u_t + \operatorname{div}_x\varphi(u) = 0, \quad u = u(t,x),\ (t,x)\in\Pi, \tag{1}$$

where $\varphi(u) = (\varphi_1(u),\dots,\varphi_n(u)) \in C(\mathbb{R},\mathbb{R}^n)$.

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where $\varphi(u) = (\varphi_1(u), \dots, \varphi_n(u)) \in C(\mathbb{R}, \mathbb{R}^n)$. Recall the notion of entropy solution (e.s.) of (1) in the Kruzhkov sense

1. S. N. Kruzhkov, First order quasilinear equations in several independent variables, Engl. transl. in Math. USSR-Sb. 10 (1970) 217–243.

Definition 1

A function $u = u(t,x) \in L^\infty(\Pi)$ is called an entropy solution of (1) if $\forall k \in \mathbb{R}$

$$|u-k|_t + \operatorname{div}_x[\operatorname{sign}(u-k)(\varphi(u) - \varphi(k))] \le 0 \ \text{ in } \mathcal{D}'(\Pi).$$

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This relation means that for any test function $f = f(t,x) \in C_0^1(\Pi), f \ge 0$,

$$\int_\Pi [|u-k|f_t + \operatorname{sign}(u-k)(\varphi(u) - \varphi(k)) \cdot \nabla_x f] dtdx \ge 0$$

As was shown in

2. E. Yu. Panov, Existence of strong traces for generalized solutions of multidimensional scalar conservation laws, J. Hyperbolic Differ. Equ. 2 (4) (2005) 885–908,

an e.s. $u(t,x)$ always admits a strong trace $u_0 = u_0(x) \in L^\infty(\mathbb{R}^n)$ on the initial hyperspace $t = 0$ in the sense of relation

$$\operatorname*{ess\,lim}_{t\to 0} u(t,\cdot) = u_0 \ \text{ in } L^1_{loc}(\mathbb{R}^n),$$

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In the case $n > 1$ the uniqueness of e.s. to this Cauchy problem may be violated, see

3. S. N. Kruzhkov, E. Yu. Panov, First-order conservative quasilinear laws with an infinite domain of dependence on the initial data, English transl. in Soviet Math. Dokl. 42 (1991) 316–321.
4. S. N. Kruzhkov, E. Yu. Panov, Osgood's type conditions for uniqueness of entropy solutions to Cauchy problem for quasilinear conservation laws of the first order, Ann. Univ. Ferrara Sez. VII (N.S.) 40 (1994) 31–54.

But, if initial function is periodic (at least in $n-1$ independent directions), the uniqueness holds: an e.s. of (1), (2) is unique and space-periodic, see the proof in

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We assume that the requirement of space-periodicity holds: $u(t,x+e_i)=u(t,x)$ for almost all $(t,x)\in\Pi$ and all $i=1,\dots,n$, where $\{e_i\}_{i=1}^n$ is a basis of periods in $\mathbb{R}^n$. We denote by $P=[0,1)^n$ the corresponding fundamental parallelepiped (cube).

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under the conditions $\varphi(u)\in C^2(\mathbb{R},\mathbb{R}^n)$ and

$$\forall(\tau,\xi)\in\mathbb{R}^{n+1}, (\tau,\xi)\neq 0, \quad \operatorname{meas}\{\, u\in\mathbb{R} \mid \tau+\varphi'(u)\cdot\xi=0 \,\}=0, \qquad (3)$$

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the following decay result holds

$$\operatorname*{ess\,lim}_{t\to\infty} u(t,\cdot) = \text{const} = \int_P u_0(x)dx \ \text{ in } L^1(P). \tag{4}$$

We will say that equation (1) satisfies *the decay property* if (4) holds for every periodic e.s.

We propose the following necessary and sufficient condition for the decay property (by $\mathbb{Z}$ we denote the set of integers)

$$\forall \xi \in \mathbb{Z}^n, \xi \neq 0, \text{ the function } u \mapsto \varphi(u)\cdot\xi \text{ is not affine on non-empty intervals.} \quad (5)$$

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Remark that if the basis of periods is not fixed and may depend on a solution, our main result remain valid after replacement of condition (5) by the following stronger one:

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To prove our result, we use, as in [7], the strong pre-compactness property for the self-similar scaling sequence $u(kt, kx)$, $k \in \mathbb{N}$. This pre-compactness property will be obtained under condition (5) with the help of localization principles for H-measures with "continuous indexes", introduced in

8. E. Yu. Panov, On sequences of measure-valued solutions of first-order quasilinear equations, English transl. in Russian Acad. Sci. Sb. Math. 81 (1) (1995) 211–227.

Let u_0 be a strong trace of $u(t,x)$. In correspondence with [2] we can assume that $u(t,\cdot) \in C([0,+\infty), L^1(P))$.

Lemma 1

For all $t > 0$

$$\|u(t,\cdot)\|_\infty \le \|u_0\|_\infty,$$

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Lemma 1

For all $t > 0$

$$\|u(t,\cdot)\|_\infty \le \|u_0\|_\infty, \quad \int_P |u(t,x+h) - u(t,x)|dx \le \int_P |u_0(x+h) - u_0(x)|dx \quad (7)$$

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It follows from (7) that for any Lipschitz function $s(u)$ the set of values $v(t,\cdot) = s(u(t,\cdot))$, $t \ge 0$, is precompact in $L^2(P)$. This implies that the Fourier series of $v(t,\cdot)$

$$v(t,x) = \sum_{\kappa \in \mathbb{Z}^n} a_\kappa(t) e^{2\pi i \kappa \cdot x}, \ a_\kappa(t) = \int_P e^{-2\pi i \kappa \cdot x} v(t,x) dx \quad (8)$$

converges in $L^2(P)$ uniformly w.r.t. $t \ge 0$: $\sup\limits_{t>0} \sum\limits_{|\kappa|>N} |a_\kappa(t)|^2 \underset{N\to\infty}{\to} 0$.

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9. L. Tartar, *H*-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations, Proc. Roy. Soc. Edinburgh. Sect. A. 115 (3-4) (1990) 193–230,

there exists a subsequence $v_r = v_{k_r}$, $r \in \mathbb{N}$ weakly convergent to a function v^* and the Tartar's H-measure $\hat{\mu} = \hat{\mu}(t, x, \hat{\xi})$ corresponding to the scalar sequence $U_r = v_r - v^*$, that is, $\hat{\mu}$ is a locally finite Borel measure on $\Pi \times S$, $S = S^n$ being a unite sphere in $\mathbb{R}^{n+1}$, such that

$$\langle \hat{\mu}, \Phi_1(t,x)\overline{\Phi_2(t,x)}\psi(\hat{\xi})\rangle = \lim_{r\to\infty} \int_{\mathbb{R}^{n+1}} F(\Phi_1 U_r)(\hat{\xi})\overline{F(\Phi_2 U_r)(\hat{\xi})}\psi\left(\frac{\hat{\xi}}{|\hat{\xi}|}\right) d\hat{\xi}$$

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for all $\Phi_1(t,x), \Phi_2(t,x) \in C_0(\Pi)$ and $\psi(\hat{\xi}) \in C(S)$.
Here $\hat{\xi} = (\tau, \xi) \in \mathbb{R}^{n+1}$ are the dual variables, and

$$F(u)(\hat{\xi}) = \int_{\mathbb{R}^{n+1}} e^{-2\pi i(\tau t + \xi\cdot x)} u(t,x)\,dtdx$$

be the Fourier transform extended on $L^2(\mathbb{R}^{n+1})$.

It easily follows from relation (9) with $k = k_r$ that

Lemma 2

(i) The function $v^*(t,x) = v^*(t)$ does not depend on x (it is a weak-$*$ limit of the coefficient $a_0(k_r t)$;

(ii) $\operatorname{supp} \hat{\mu} \subset \Pi \times S_0$, where $S_0 = \left\{ \hat{\xi}/|\hat{\xi}| \in S \mid \hat{\xi} = (\tau, \xi) \neq 0,\ \tau \in \mathbb{R}, \xi \in \mathbb{Z}^n \right\}$.

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We can assume that the sequence $u_r(t,x)$ weakly converges to a Young measure $\nu_{t,x}$:

$$\forall g(\lambda) \in C(\mathbb{R}) \quad g(u_r) \underset{r\to\infty}{\rightharpoonup} \langle \nu_{t,x}, g(\lambda) \rangle \quad \text{weakly-}* \text{ in } L^\infty(\Pi).$$

We introduce the measures $\gamma^r_{t,x}(\lambda) = \delta(\lambda - u_r(t,x)) - \nu_{t,x}(\lambda)$ and the corresponding distribution functions $U_r(t,x,p) = \gamma^r_{t,x}((p,+\infty))$ on $\Pi \times \mathbb{R}$. As was shown in [8], there exist a set $E \subset \mathbb{R}$ with at most countable complement such that $U_r(t,x,p) \underset{r\to\infty}{\rightharpoonup} 0$ weakly-$*$ in $L^\infty(\Pi)$ $\forall p \in E$ and

Proposition 1

1) There exists a family of locally finite complex Borel measures $\{\mu^{pq}\}_{p,q\in E}$ in $\Pi\times S$ and a subsequence $U_r(t,x,p)$ such that for all $\Phi_1(t,x),\Phi_2(t,x)\in C_0(\Pi)$ and $\psi(\hat{\xi})\in C(S)$

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3) For any $p_1,\dots,p_l\in E$ the matrix $\{\mu^{p_ip_j}\}_{i,j=1}^l$ is Hermitian and positive semidefinite, that is, for all $\zeta_1,\dots,\zeta_l\in\mathbb{C}$ the measure

$$\sum_{i,j=1}^{l}\mu^{p_ip_j}\zeta_i\overline{\zeta_j}\geq 0.$$

Let $s(u) \in C^1(\mathbb{R})$, $s'(u) \in C_0(\mathbb{R})$, and $v_r(t,x) = s(u_r(t,x))$, $r \in \mathbb{N}$. Then

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Theorem 1

For every $p, q \in E$ $\operatorname{supp} \mu^{pq} \subset \Pi \times S_0$.

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Let $L = L(p) \subset \mathbb{R}^{n+1}$ be the minimal linear subspace such that $\operatorname{supp} \mu^{pp} \subset \Pi \times L$. Since $u_r(t,x)$ is a bounded sequence of e.s. of (1) from the results of

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it follows the localization principle:

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it follows the localization principle:

Theorem 2

There exists $\delta > 0$ such that the function $u \mapsto \tau u + \xi \cdot \varphi(u)$ is constant on the interval $(p-\delta, p+\delta)$ for all $\hat{\xi} = (\tau, \xi) \in L$.

If the space $L = L(p)$ is not trivial then by Theorems 1,2 there exists a nonzero vector $\hat{\xi} = (\tau, \xi) \in (\mathbb{R} \times \mathbb{Z}^n) \cap L$ such that the function $u \mapsto \tau u + \xi \cdot \varphi(u)$ is constant on some interval $(p - \delta, p + \delta)$. This contradicts to our assumption (5).

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by the $L^1(P)$-contraction property. In view of (11) it follows from (12) that $\operatorname*{ess\,lim}_{t \to \infty} u(t, x) = u^*$ in $L^1(P)$. Finally, by the conservation of "mass" (see [5]), for all $t > 0$ $\int_P u(t, x) dx = \int_P u_0(x) dx$, where $u_0(x)$ is a strong trace of $u(t, x)$ on the initial hyper-space $t = 0$. Passing in this relation to the limit as $t \to \infty$, we obtain that $u^* = \int_P u_0(x) dx = \int_P u_0(x) dx.$

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Conversely, assume that equation (1) satisfies the decay property. Let us demonstrate that it satisfies condition (5). Assuming the contrary, we can find the segment $[a,b]$, $a<b$, and a nonzero point $(\tau,\xi)\in\mathbb{R}\times\mathbb{Z}^n$ such that the function $u\mapsto\tau u+\xi\cdot\varphi(u)$ is constant on the segment $[a,b]$.

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$$u(t,x) = \frac{a+b}{2} + \frac{b-a}{2}\sin(2\pi(\tau t + \xi\cdot x))$$

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Remark that the main result remain valid for unbounded renormalized solutions $u(t,\cdot)\in L^\infty(\mathbb{R}_+,L^1(P))$ in the sense of paper

11. P.V. Lysuho, E.Yu. Panov. Renormalized entropy solutions to the Cauchy problem for first order quasilinear conservation laws in the class of periodic functions, J. of Math. Sci. 177 (1) (2011) 27–49.

Thank you!