# Well-Balanced and Positivity Preserving DG Schemes for Shallow Water Flows with Shock Capturing by Adaptive Filtering Procedures

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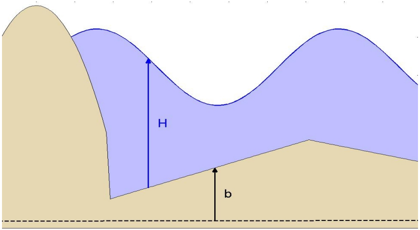
HYP 2012

# The 2D Shallow Water Equations




| | |
|---|---|
| $H$ | water height |
| $b$ | bottom elevation |

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathcal{F}(\mathbf{u}(\mathbf{x}, t)) = \mathbf{s}(\mathbf{u}, \mathbf{x}, t)$$

Where $\mathbf{u} = (H, Hv_1, Hv_2)^T, \quad \nabla \cdot \mathcal{F} = \frac{\partial}{\partial x}\mathbf{f}_1 + \frac{\partial}{\partial y}\mathbf{f}_2,$

$$\mathbf{f}_1 = \begin{pmatrix} Hv_1 \\ Hv_1^2 + \frac{1}{2}gH^2 \\ Hv_1 v_2 \end{pmatrix}, \quad \mathbf{f}_2 = \begin{pmatrix} Hv_2 \\ Hv_1 v_2 \\ Hv_2^2 + \frac{1}{2}gH^2 \end{pmatrix},$$

$$\mathbf{s} = (0, -gHb_x, -gHb_y)^T$$

# SWE – Challenges for a numerical scheme

Shock capturing, Well-balancedness, Positivity preservation

Xing, Zhang, Shu '10:

3rd order DG scheme on cartesian grids, WB + PP, Limiters

Bryson, Epshteyn, Kurganov, Petrova '10:

2nd order central-upwind scheme on triangular grids, WB + PP, Limiters

**In this talk:**

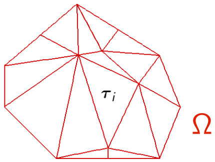
- 3rd order WB and PP DG scheme on triangular grids
- Shock capturing by (low cost) filtering procedures [*,+]
- Positivity preservation for implicit time integration [*]

Joint work with:

[*] Andreas Meister (Univ. Kassel)

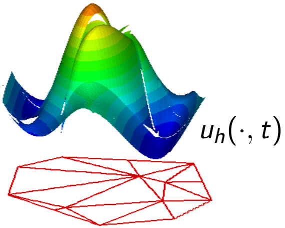
[+] Thomas Sonar (TU Braunschweig)

# The DG Scheme for SWE




$\mathcal{T}^h = \{\tau_1, \tau_2, \ldots, \tau_{\#\mathcal{T}^h}\}$

Triangulation




- Triangulation of computational domain
- Piecewise polynomial approximation $\mathbf{u}_h(\mathbf{x}, t)$ of degree $N$
- Variational formulation in space
- DG advantages: conservative method, highly local,
  unstructured grids easy to implement

# The DG Scheme for SWE

Variational formulation

$$\frac{d}{dt}\int_{\tau_i} \underset{\underset{\mathbf{u}_h}{\uparrow}}{\mathbf{u}}\,\Phi\,d\mathbf{x} + \int_{\partial\tau_i} \underset{\underset{\mathbf{F}^{num}(\mathbf{u}_h^-,\mathbf{u}_h^+,\mathbf{n})}{\uparrow}}{\mathcal{F}(\mathbf{u})\cdot\mathbf{n}}\;\Phi\,d\sigma - \int_{\tau_i} \underset{\underset{\mathcal{F}(\mathbf{u}_h)}{\uparrow}}{\mathcal{F}(\mathbf{u})}\cdot\nabla\Phi\,d\mathbf{x} = \int_{\tau_i} \underset{\underset{\mathbf{s}_h}{\uparrow}}{\mathbf{s}}\,\Phi\,d\mathbf{x}$$

# The DG Scheme for SWE

Variational formulation

$$\frac{d}{dt}\int_{\tau_i} \mathbf{u}_h\,\Phi\,d\mathbf{x} + \int_{\partial\tau_i} \mathbf{F}^{num}(\mathbf{u}_h^-,\mathbf{u}_h^+,\mathbf{n})\,\Phi\,d\sigma - \int_{\tau_i} \mathcal{F}(\mathbf{u}_h)\cdot\nabla\Phi\,d\mathbf{x} = \int_{\tau_i} \mathbf{s}_h\,\Phi\,d\mathbf{x}$$

Use orthogonal polynomials

$$\mathbf{u}_h|_{\tau_i}(\mathbf{x},t) = \sum_{k=1}^{q(N)} \hat{\mathbf{u}}_k^i(t)\Phi_k(\mathbf{x}), \qquad q(N) = (N+1)(N+2)/2$$

Semidiscrete equation for coefficients

$$\begin{aligned}\frac{d}{dt}\hat{\mathbf{u}}_k^i &= \left(-\int_{\partial\tau_i} \mathbf{F}^{num}(\mathbf{u}_h^-,\mathbf{u}_h^+,\mathbf{n})\ \Phi_k d\sigma + \int_{\tau_i} \mathcal{F}(\mathbf{u}_h)\cdot\nabla\Phi_k d\mathbf{x}\right)/\|\Phi_k\|_{L^2}^2 \\ &+ \int_{\tau_i} \mathbf{s}_h\Phi_k\,d\mathbf{x}/\|\Phi_k\|_{L^2}^2\end{aligned}$$

Quadrature rules

# The DG Scheme for SWE

Variational formulation

$$\frac{d}{dt}\int_{\tau_i} \mathbf{u}_h\,\Phi\,d\mathbf{x}+\int_{\partial\tau_i} \mathbf{F}^{num}(\mathbf{u}_h^-,\mathbf{u}_h^+,\mathbf{n})\,\Phi\,d\sigma-\int_{\tau_i} \mathcal{F}(\mathbf{u}_h)\cdot\nabla\Phi\,d\mathbf{x}=\int_{\tau_i} \mathbf{s}_h\,\Phi\,d\mathbf{x}$$

Use orthogonal polynomials

$$\mathbf{u}_h|_{\tau_i}(\mathbf{x},t)=\sum_{k=1}^{q(N)} \hat{\mathbf{u}}_k^i(t)\Phi_k(\mathbf{x}), \qquad q(N)=(N+1)(N+2)/2$$

Semidiscrete equation $\rightarrow$ Time integration: RK method

$$\begin{aligned}\frac{d}{dt}\hat{\mathbf{u}}_k^i &= \left(-\int_{\partial\tau_i} \mathbf{F}^{num}(\mathbf{u}_h^-,\mathbf{u}_h^+,\mathbf{n})\;\Phi_k d\sigma+\int_{\tau_i} \mathcal{F}(\mathbf{u}_h)\cdot\nabla\Phi_k d\mathbf{x}\right)/\|\Phi_k\|_{L^2}^2 \\ &+ \int_{\tau_i} \mathbf{s}_h\Phi_k\,d\mathbf{x}/\|\Phi_k\|_{L^2}^2\end{aligned}$$

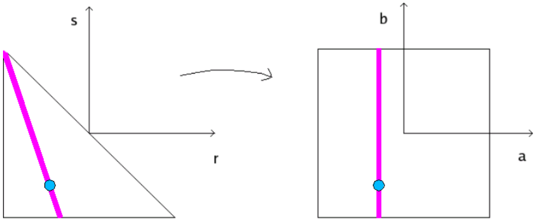
Quadrature rules

# Orthogonal Proriol-Koornwinder-Dubiner Basis

$$\Phi_{lm}(r,s) = P_l^{0,0}\left(2\frac{1+r}{1-s}-1\right)\left(\frac{1-s}{2}\right)^l P_m^{2l+1,0}(s), \quad l+m \le N$$

$P_n^{\alpha,\beta}$ Jacobi polynomials

Defined on reference element $\mathbb{T}$




$$\mathbb{T} = \{(r,s) \in \mathbb{R}^2 \,|\, -1 \le r,s;\ r+s \le 0\}$$

Proriol (1957), Koornwinder (1975), Dubiner (1991)

# New Modal Filtering Approach

- Damping step: Carried out after each time step

$$\hat{\mathbf{u}}_{lm}^{i,mod} = \exp\left(-\frac{const \cdot N\Delta t}{h_i}\left(\frac{l+m}{N}\right)^{2p}\right)\hat{\mathbf{u}}_{lm}^{i}$$

$\rightsquigarrow$ Multiplication by filter function $\sigma(\eta) = \exp\left(-\alpha_i \cdot \eta^{2p}\right)$

- Resulting from SV modified equation on $\mathbb{T}$

Meister, Ortleb, Sonar:

- On Spectral Filtering for Discontinuous Galerkin Methods on Unstructured Triangular Grids, NMPDE, '11

- New Adaptive Modal and DTV Filtering Routines for the DG Method on Triangular Grids applied to the Euler Equations, GEM, '12

# Shock Indication and Adaptivity

$$\hat{\mathbf{u}}_{lm}^{i,mod} = \sigma(\eta) \cdot \hat{\mathbf{u}}_{lm}^{i}$$

$$\sigma(\eta) = \exp\left(-s_i\, \alpha_i\, \eta^{2p}\right)$$ Modified filter function

**Resolution indicator $\omega^{res}$**

$$\omega_i^{res} = \frac{\sum\limits_{l+m=N} (\hat{u}_{lm}^i)^2}{\sum\limits_{l+m<N} (\hat{u}_{lm}^i)^2 + \epsilon}$$

$$\tilde{s}_i = \min\left\{1000(5N^4+1)\omega_i^{res}, 1\right\}$$

Persson, Peraire '06; Barter, Darmofal '07

$$s_i = \begin{cases} \tilde{s}_i & \text{if } \tilde{s}_i > tol \\ 0 & \text{otherwise} \end{cases}$$

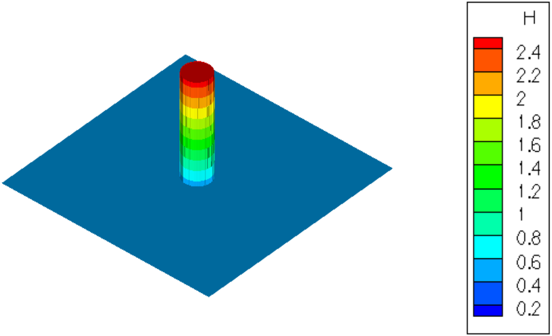
$$tol = 0.01$$

# Circular Dam-Break Problem

- Computational domain: $[0, 40] \times [0, 40]$
- Initial conditions:

$$H(x, y, 0) = \begin{cases} 2.5 & \text{if } (x-20)^2 + (y-20)^2 \leq 2.5^2 \\ 0.5 & \text{else} \end{cases}$$

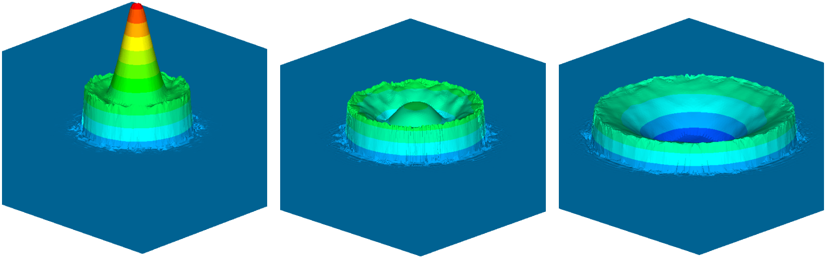
- Boundaries: outflow conditions



# Circular Dam-Break Problem

Modally filtered DG solution: Water height

$N = 6, \ K = 2200.$
$\alpha_i = 2\frac{N\Delta t}{h_i}, \ p = 1.$



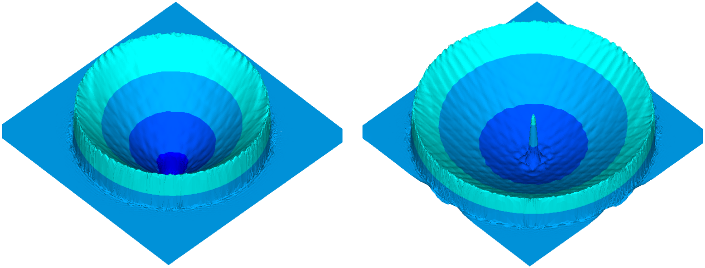
$t = 0.4$ $t = 0.7$ $t = 1.4$

# Circular Dam-Break Problem

Modally filtered DG solution: Water height

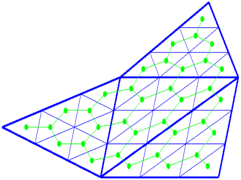
$N = 6,\ K = 2200.$

$\alpha_i = 2\frac{N\Delta t}{h_i},\ p = 1.$



$t = 3.5$ $\qquad$ $t = 4.7$

# DTV Filtering on Non-Cartesian Graphs



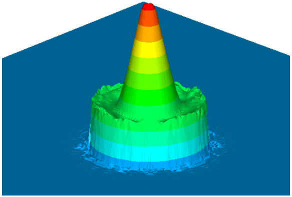
$$\mathbf{U}^{(0)} = [u_h(\mathbf{x}_j)]_{j \in DTV\ nodes}$$

for k = 0,1,2,...

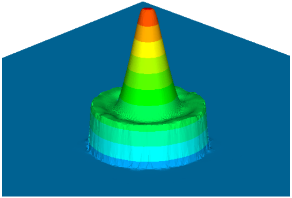
$$\mathbf{U}^{(k+1)} = DTV(\mathbf{U}^{(k)})$$

Meister, Ortleb, Sonar '11

$t = 0.4$:



before DTV filtering



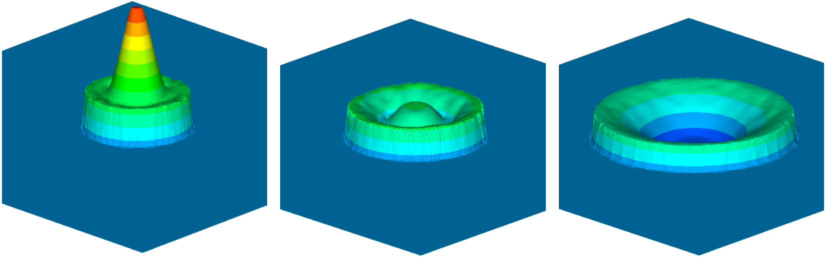
after DTV filtering

# Circular Dam-Break Problem

DTV filtered solutions: Water height

$N = 6, \ K = 2200.$

49 subtris, $\lambda = 1.$



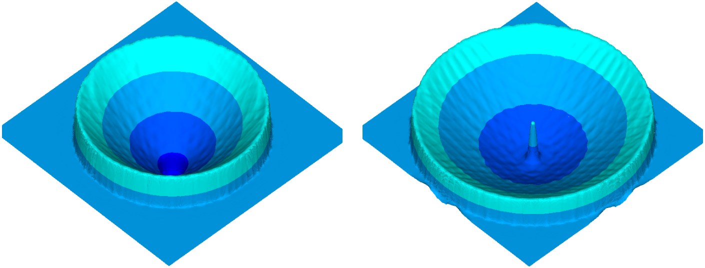
$t = 0.4$ $\qquad$ $t = 0.7$ $\qquad$ $t = 1.4$

# Circular Dam-Break Problem

DTV filtered solutions: Water height

$N = 6,\ K = 2200.$

49 subtris, $\lambda = 1.$



$t = 3.5$ $\qquad$ $t = 4.7$

# Well-balanced Scheme

$$\frac{d}{dt}\int_{\tau_i} \mathbf{u}_h\,\Phi\,d\mathbf{x} + \int_{\partial\tau_i} \mathbf{F}^{WB}(\mathbf{u}_*^-, \mathbf{u}_*^+, H^-, \mathbf{n})\,\Phi\,d\sigma - \int_{\tau_i} \mathcal{F}(\mathbf{u}_h)\cdot\nabla\Phi\,d\mathbf{x} = \int_{\tau_i} \mathbf{s}_h\,\Phi\,d\mathbf{x}$$

- $\mathbf{F}^{WB}(\mathbf{u}_*^-, \mathbf{u}_*^+, H^-, \mathbf{n}) = \mathbf{F}^{num}(\mathbf{u}_*^-, \mathbf{u}_*^+, \mathbf{n}) + \begin{pmatrix} 0 \\ \frac{n_1\,g}{2}\left((H^-)^2 - (H_*^-)^2\right) \\ \frac{n_2\,g}{2}\left((H^-)^2 - (H_*^-)^2\right) \end{pmatrix}$

  Hydrostatic reconstruction  Audusse et al. '04

$$\begin{aligned} \mathbf{u}_*^\pm &= \left(H_*^\pm, H_*^\pm\cdot(v_1)^\pm, H_*^\pm\cdot(v_2)^\pm\right)^T, \\ H_*^\pm &= \max\left\{0, H^\pm + b^\pm - \max\left\{b^-, b^+\right\}\right\} \end{aligned}$$

# Well-balanced Scheme

$$\frac{d}{dt}\int_{\tau_i} \mathbf{u}_h\,\Phi\,d\mathbf{x} + \int_{\partial\tau_i} \mathbf{F}^{WB}(\mathbf{u}_*^-,\mathbf{u}_*^+,H^-,\mathbf{n})\,\Phi\,d\sigma - \int_{\tau_i} \mathcal{F}(\mathbf{u}_h)\cdot\nabla\Phi\,d\mathbf{x} = \int_{\tau_i} \mathbf{s}_h\,\Phi\,d\mathbf{x}$$

- $\mathbf{F}^{WB}(\mathbf{u}_*^-,\mathbf{u}_*^+,H^-,\mathbf{n}) = \mathbf{F}^{num}(\mathbf{u}_*^-,\mathbf{u}_*^+,\mathbf{n}) + \begin{pmatrix} 0 \\ \frac{n_1\,g}{2}\left((H^-)^2 - (H_*^-)^2\right) \\ \frac{n_2\,g}{2}\left((H^-)^2 - (H_*^-)^2\right) \end{pmatrix}$

  Hydrostatic reconstruction     Audusse et al. '04

- Higher order quadrature necessary

  $\int_\tau \left(\frac{1}{2}gH^2\right)_x \Phi d\mathbf{x} = \int_\tau gHH_x\,\Phi d\mathbf{x} = -\int_\tau gHb_x\,\Phi d\mathbf{x}$

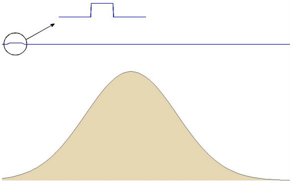
- WB filter: Indicator $\omega_i^{res} = \sum_{l+m=N} (\hat{u}_{lm}^i)^2 \cdot \left(\sum_{l+m<N} (\hat{u}_{lm}^i)^2 + \epsilon\right)^{-1}$

  now depending on $\hat{u}_{lm}^i = \hat{H}_{lm}^i + \hat{b}_{lm}^i$

# Small Perturbation Test

Smooth bottom elevation
on $\Omega = [0,2] \times [0,1]$:

$$b(\mathbf{x}) = 0.8e^{-5(x-0.9)^2 - 50(y-0.5)^2}$$



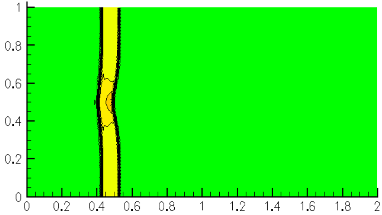
Initial water height

$$H(\mathbf{x},0) = \begin{cases} 1 - b(\mathbf{x}) + 0.01, & \text{if } 0.05 \le x \le 0.15, \\ 1 - b(\mathbf{x}), & \text{else.} \end{cases}$$

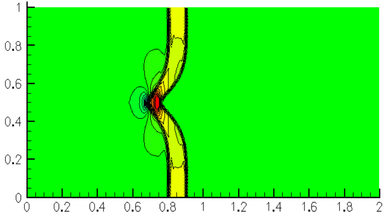
Initial velocity: $\mathbf{v} = \mathbf{0}$

# Small Perturbation Test

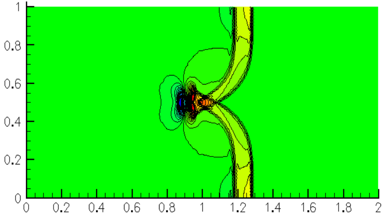
Water surface $w = H + b$ (for $N = 2,\ K = 46360,\ \alpha_i = 5\frac{N\Delta t}{h_i},\ p = 1$)



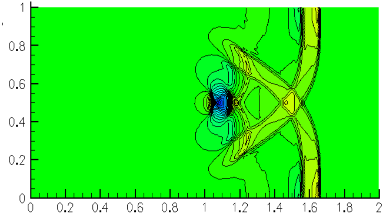
$t = 0.12$



$t = 0.24$



$t = 0.36$



$t = 0.48$

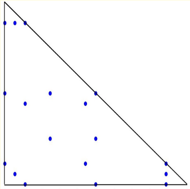
(30 contour levels from 0.99 to 1.01)

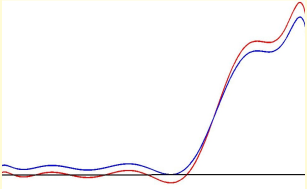
Problem: boundary conditions

# Positivity Treatment

## Xing/Zhang/Shu-approach assumes

- Non-negative cell means $\bar{H}_i^n$ at time $t^n$
- Positivity preserving flux (e.g. HLL, Lax-Friedrichs)
- Non-negative values of $H$ at certain quadrature nodes





Zhang, Xia, Shu '11: Maximum-principle-satisfying and PP DG schemes on triangular grids (scalar & Euler equations)

$\Rightarrow$ Non-negative cell means at next time level $t^{n+1}$

$$\bar{H}_i^{n+1} = \bar{H}_i^n - \frac{\Delta t}{|\tau_i|} \int_{\partial \tau_i} F_1^{WB}(\mathbf{u}_*^{n,-}, \mathbf{u}_*^{n,+}, \mathbf{n})\, d\sigma$$

Under suitable CFL-like time step restriction

# Positivity Treatment

**Xing/Zhang/Shu-approach assumes**

- Non-negative cell means $\bar{H}_i^n$ at time $t^n$
- Positivity preserving flux (e.g. HLL, Lax-Friedrichs)
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$\Rightarrow$ Non-negative cell means at next time level $t^{n+1}$

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Under suitable CFL-like time step restriction

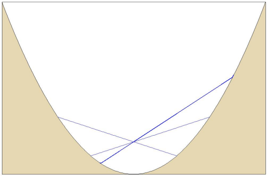
- Comp. velocities via $\mathbf{v} = \begin{cases} \mathbf{0} & \text{if } H < 10^{-6} \\ \frac{2H \cdot (H\mathbf{v})}{H^2 + \max\{H^2, \epsilon\}} & \textit{else} \end{cases}$ (Bryson et al.)
- Filter modification: $\hat{\mathbf{u}}_{lm}^{i,mod} = \exp\left(-s_i\, \alpha_i\, \eta^{2p}\right) \cdot \hat{\mathbf{u}}_{lm}^i$

$$s_i = (\bar{H}_i)^{-1} \cdot \min\left\{1000(5N^4 + 1)\omega_i^{res}, 0\right\}$$

# Oscillating Lake

Paraboloidal vessel
on $\Omega = [-2, 2]^2$:

$$b(\mathbf{x}) = 0.1(x^2 + y^2).$$



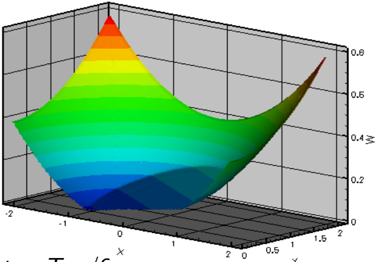
2D cut at $y = 0$

Periodic analytical solution known

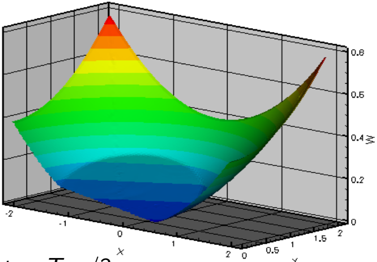
$$\begin{aligned}
H(\mathbf{x}, t) &= \max\{0, 0.05(2x\cos(\omega t) + 2y\sin(\omega t)) + 0.075 - b(\mathbf{x}))\}, \\
v_1(\mathbf{x}, t) &= -0.1\omega\sin(\omega t), \\
v_2(\mathbf{x}, t) &= 0.1\omega\cos(\omega t), \qquad \omega = \sqrt{0.2g}
\end{aligned}$$

# Oscillating Lake

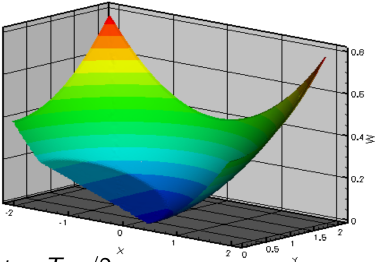
Water surface $w = H + b$ (for $N = 2,\ K = 23138,\ \alpha_i = 10\frac{N\Delta t}{h_i},\ p = 1$)



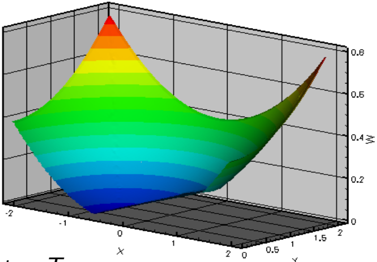

$t = T_{per}/6$




$t = T_{per}/3$




$t = T_{per}/2$




$t = T_{per}$

# Positivity Treatment – Implicit Euler Case

**Extension of Xing/Zhang/Shu-approach**

- Non-negative cell means $\bar{H}_i^n$ at time $t^n$
- Positivity preserving flux (e.g. HLL, Lax-Friedrichs)
- Non-negative values of $H$ at certain quadrature nodes

$\Rightarrow$ Non-negative cell means at next time level $t^{n+1}$

$$\bar{H}_i^{n+1} = \bar{H}_i^n - \frac{\Delta t}{|\tau_i|} \int_{\partial \tau_i} F_1^{WB}(\mathbf{u}_*^{n+1,-}, \mathbf{u}_*^{n+1,+}, \mathbf{n}) \, d\sigma$$

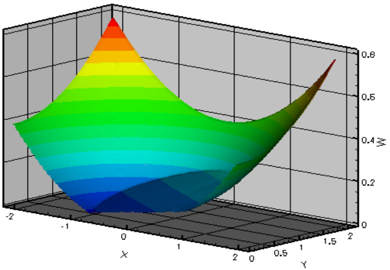
No time step restriction!
Implicit Euler is unconditionally SSP

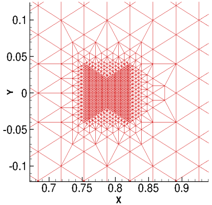
Higueras '05

# Why implicit?

$N = 0$ (first order): explicit vs. implicit Euler time integration

output time $t = T_{per}/6 \approx 0.7475$




stiff grid, $S = \frac{max\_area}{min\_area} = 413.7$

| Stiffness | Avg. $\Delta t$ (EX) | Avg. $\Delta t$ (IM) | $\frac{\text{CPU}_{EX}}{\text{CPU}_{IM}}$ | Avg. iter. per $\Delta t$ Newton | Avg. iter. per $\Delta t$ GMRES |
|---|---|---|---|---|---|
| 6.5 | 9.0e-4 | 1.1e-2 | 1.10 | 4.6 | 6.6 |
| 25.9 | 4.6e-4 | 5.5e-3 | 1.31 | 4.0 | 5.1 |
| 103.4 | 2.3e-4 | 2.7e-3 | 1.32 | 4.0 | 5.0 |
| 413.7 | 1.1e-4 | 1.4e-3 | 2.07 | 2.9 | 3.0 |

# Unconditionally PP schemes: implicit and higher order?

# Unconditionally PP schemes: implicit and higher order?

**Implicit SSP RK schemes**

No unconditionally SSP scheme of order $\geq 2$

Gottlieb, Shu, Tadmor '01

Hence no unconditionally PP implicit RK scheme of order $\geq 2$

# L-stable SDIRK2

Semidiscrete DG scheme

$$\frac{d}{dt}\mathbf{u}(t) = \mathcal{L}_h(\mathbf{u}(t))$$

(boundary terms neglected)

$$\begin{array}{c|cc} \alpha & \alpha & \\ 1 & 1-\alpha & \alpha \\ \hline & 1-\alpha & \alpha \end{array} \qquad \alpha = 1 - \frac{\sqrt{2}}{2}$$

$$\mathbf{u}^{(1)} = \mathbf{u}^n + \alpha \Delta t \mathcal{L}_h\left(\mathbf{u}^{(1)}\right) \qquad \text{OK}$$

(cut-off timestep with implicit Euler)

$$\mathbf{u}^{(2)} = \underbrace{\mathbf{u}^n + (1-\alpha)\Delta t \mathcal{L}_h\left(\mathbf{u}^{(1)}\right)}_{\text{NOT positivity preserving}} + \alpha \Delta t \mathcal{L}_h\left(\mathbf{u}^{(2)}\right)$$

NOT positivity preserving

$\longrightarrow$ modification necessary

$$\mathbf{u}^{n+1} = \mathbf{u}^{(2)}$$

# The DG scheme ...

... for cell means of water height $\bar{H}_i(t) = \frac{1}{|\tau_i|}\int_{\tau_i} H_h(\mathbf{x},t)\,d\mathbf{x}$

$$\begin{aligned}
\frac{d}{dt}(|\tau_i|\bar{H}_i) &= -\sum_{j\in N(\tau_i)} \int_{\Gamma_{ij}} F_1^{WB}(\mathbf{u}_*^-,\mathbf{u}_*^+,\mathbf{n})\,d\sigma \\
&= \sum_{j\in N(\tau_i)} p_{ij} - \sum_{j\in N(\tau_i)} d_{ij}
\end{aligned}$$

$$\begin{aligned}
p_{ij} &= \max\left\{0, -\int_{\Gamma_{ij}} F_1^{WB}(\mathbf{u}_*^-,\mathbf{u}_*^+,\mathbf{n})\,d\sigma\right\} = d_{ji} \geq 0 \\
d_{ij} &= \max\left\{0, +\int_{\Gamma_{ij}} F_1^{WB}(\mathbf{u}_*^-,\mathbf{u}_*^+,\mathbf{n})\,d\sigma\right\} = p_{ji} \geq 0
\end{aligned}$$

$$\Rightarrow \begin{cases} p_{ij} - d_{ij} = -\int_{\Gamma_{ij}} F_1^{WB}(\mathbf{u}_*^-,\mathbf{u}_*^+,\mathbf{n})\,d\sigma \\ p_{ij} = d_{ji} \end{cases}$$

# What's the use of this reformulation?

# What's the use of this reformulation?

The Patankar Trick for production-destruction equations:

$$\frac{d}{dt}c_i = \sum_{i=1}^{I} p_{ij}(\mathbf{c}) - \sum_{i=1}^{I} d_{ij}(\mathbf{c}), \quad p_{ij} = d_{ji} \quad (i,j = 1,\dots,I)$$

positive quantities $c_i$, e.g. concentrations

explicit Euler

$$c_i^{n+1} = c_i^n + \Delta t \left( \sum_{i=1}^{I} p_{ij}(\mathbf{c}^n) - \sum_{i=1}^{I} d_{ij}(\mathbf{c}^n) \right)$$

NO positive scheme

Patankar-Euler (PE)

$$\begin{aligned} c_i^{n+1} &= c_i^n + \Delta t \left( \sum_{i=1}^{I} p_{ij}(\mathbf{c}^n) - \sum_{i=1}^{I} d_{ij}(\mathbf{c}^n) \frac{c_i^{n+1}}{c_i^n} \right) \\ \Rightarrow &\quad \underbrace{\left(1 + \Delta t \sum d_{ij}(\mathbf{c}^n)(c_i^n)^{-1}\right)}_{>0} c_i^{n+1} = \underbrace{c_i^n + \Delta t \sum p_{ij}(\mathbf{c}^n)}_{>0} \end{aligned}$$

positive but NOT conservative

# What's the use of this reformulation?

The Patankar Trick for production-destruction equations:

$$\frac{d}{dt} c_i = \sum_{i=1}^{I} p_{ij}(\mathbf{c}) - \sum_{i=1}^{I} d_{ij}(\mathbf{c}), \qquad p_{ij} = d_{ji}$$

positive quantities $c_i$, e.g. concentrations

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$$c_i^{n+1} = c_i^n + \Delta t \left( \sum_{i=1}^{I} p_{ij}(\mathbf{c}^n) - \sum_{i=1}^{I} d_{ij}(\mathbf{c}^n) \right)$$

NO positive scheme

Modified PE scheme    Burchard, Deleersnijder, Meister '03

$$c_i^{n+1} = c_i^n + \Delta t \left( \sum_{i=1}^{I} p_{ij}(\mathbf{c}^n) \frac{c_j^{n+1}}{c_j^n} - \sum_{i=1}^{I} d_{ij}(\mathbf{c}^n) \frac{c_i^{n+1}}{c_i^n} \right)$$

positive and conservative

# L-stable SDIRK2

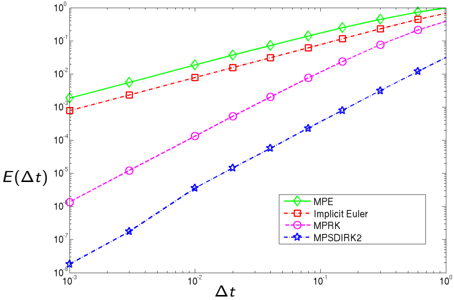
$$
\begin{aligned}
\frac{d}{dt}\mathbf{u}(t) &= \mathcal{L}_h\left(\mathbf{u}(t)\right) \\
\mathbf{u}^{(1)} &= \mathbf{u}^n + \alpha\Delta t\mathcal{L}_h\left(\mathbf{u}^{(1)}\right) \qquad \text{OK} \\
& \quad \text{(cut-off timestep with implicit Euler)} \\
\mathbf{u}^{(2)} &= \underbrace{\mathbf{u}^n + (1-\alpha)\Delta t\mathcal{L}_h\left(\mathbf{u}^{(1)}\right)}_{\text{replace by } \mathbf{u}_*^{(2)} \geq 0} + \alpha\Delta t\mathcal{L}_h\left(\mathbf{u}^{(2)}\right) \qquad \text{OK} \\
u_{*,i}^{(2)} &= u_i^n + (1-\alpha)\Delta t \sum_{j\in N(\tau_i)}\left(p_{ij}\frac{u_{*,j}^{(2)}}{\tilde{u}_j} - d_{ij}\frac{u_{*,i}^{(2)}}{\tilde{u}_i}\right) \\
\tilde{u}_i &= \max\left\{\epsilon, u_i^n + (1-\alpha)\Delta t\left[\mathcal{L}_h\left(\mathbf{u}^{(1)}\right)\right]_i\right\}, \qquad \epsilon = 10^{-10} \\
\mathbf{u}^{n+1} &= \mathbf{u}^{(2)}
\end{aligned}
$$

# Accuracy Study for Production-Destruction ODE

Simple nonlinear model
$\mathbf{c}(t)$ positive constituents

$$\begin{aligned} c_1' &= -\frac{c_2 c_1}{c_1 + 1} = -d_{12} \\ c_2' &= \frac{c_2 c_1}{c_1 + 1} - a c_2 = p_{21} - d_{23} \\ c_3' &= a c_2 = p_{32} \end{aligned}$$

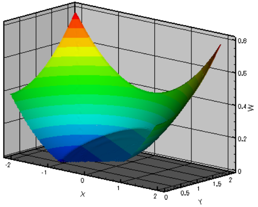
$a = 0.3$
$\mathbf{c}^0 = (9.98, 0.01, 0.01)^T$



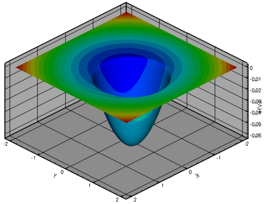

Truncation errors:

$$E(\Delta t) = \frac{\sqrt{\frac{1}{N_{\Delta t}} \sum_{n=1}^{N_{\Delta t}} \left(c_1(n\Delta t) - c_1^n\right)^2}}{\frac{1}{N_{\Delta t}} \sum_{n=1}^{N_{\Delta t}} c_1(n\Delta t)}$$

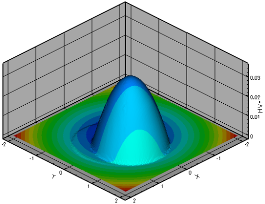
# MPSDIRK2 for Oscillating Lake Test



$w = H + b$



$Hv_1$



$Hv_2$

Output time $t = T_{per}/6$, $N = 2$, avg. $\Delta t = 0.0054$

Stiffness $S = 25.9$ ($K = 23284$), Filter parameters $\alpha_i = 10\frac{N\Delta t}{h_i}$, $p = 1$

| No. $\Delta t$ | Avg. $\Delta t$ | CPU in $s$ | Patankar steps | Avg. iter. per $\Delta t$ Newton | GMRES |
|---|---|---|---|---|---|
| 139 | 0.0054 | 599 | 0 | 9 | 16 |
| 28 | 0.0267 | 1552 | 1 | 26 | 229 |
| 17 | 0.044 | 16071 | 8 | 60 | 4027 |

# Conclusion and Outlook

## Conclusion

- Well-balanced and positivity preserving DG scheme for SWE
- Shock capturing by filtering procedures
- Unconditionally positiv implicit time integration

## Outlook

- Improvement of iterative solver
- Higher order implicit PP schemes
- Inclusion of bottom friction $\longrightarrow$ stiff source terms
- IMEX time integration