Relativistic Burgers equations on a curved spacetime

Baver Okutmustur

Middle East Technical University (METU)

HYP2012, June 25–29, 2012

Joint work with Philippe LeFloch and Hasan Makhlof, Université Paris 6

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Euler equations of compressible fluids

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = 0$$

ρ : density, u : velocity of the fluid, $p(\rho)$: pressure

Rewrite the second equation combining with the first one

$$\begin{aligned} 0 &= u\,\partial_t(\rho) + \rho\,\partial_t(u) + u^2\partial_x(\rho) + 2u\rho\,\partial_x(u) \\ &= \rho(\partial_t u + 2u\partial_x u) + u(\partial_t \rho + u\partial_x \rho) \\ &= \rho(\partial_t u + 2u\partial_x u) - u\rho\partial_x u = \rho(\partial_t u + u\partial_x u) \end{aligned}$$

The (inviscid) Burgers equation

$$\partial_t u + \partial_x(u^2/2) = 0, \qquad u = u(t,x),\ t > 0,\ x \in \mathbb{R}$$

Hyperbolic balance laws

Balance laws

$$\operatorname{div}_\omega(T(v)) = S(v)$$

$M = (M, \omega)$: $(n+1)$-dimensional curved spacetime (with boundary)
$\operatorname{div}_\omega$: the divergence operator associated with the volume form ω
$v : M \to \mathbb{R}$ unknown function (scalar field)
$T = T(v)$ flux vector field on M, $S = S(v)$ a scalar field on M.

The manifold M is assumed to be foliated by hypersurfaces :

$$M = \bigcup_{t \geq 0} H_t,$$

such that each slice H_t is an n-dimensional manifold.
H_t : spacelike, H_0 : initial slice

Lorentz invariant conservation law

Supposing that $n = 1$, $S(v) \equiv 0$

$M = [0, +\infty) \times \mathbb{R}$ covered by a single coordinate chart $(x^0, x^1) = (t, x)$ with $\omega = dx^0 dx^1$, it follows that

Hyperbolic conservation law

$$\partial_0 T^0(v) + \partial_1 T^1(v) = 0,$$

where $\partial_0 = \partial/\partial x^0$, $\partial_1 = \partial/\partial x^1$, $x^0 \in [0, \infty)$ and $x^1 \in \mathbb{R}$.

We search for the flux vector fields $T = T(v)$ for which solutions to the above equation satisfy Lorentz invariant property.

Derivation of a Lorentz invariant model

Lorentz transformations $(x^0, x^1) \mapsto (\bar{x}^0, \bar{x}^1)$

$$\bar{x}^0 := \gamma_\epsilon(V)\,(x^0 - \epsilon^2 V x^1),$$
$$\bar{x}^1 := \gamma_\epsilon(V)\,(-V\,x^0 + x^1), \qquad \gamma_\epsilon(V) = \left(1 - \epsilon^2\,V^2\right)^{-1/2},$$

$\epsilon \in (-1, 1)$ denotes the inverse of the (normalized) speed of light, $\gamma_\epsilon(V)$ is the so-called Lorentz factor associated with a given speed $V \in (-1/\epsilon, 1/\epsilon)$

v : fluid velocity component in the coordinate system (x^0, x^1) related to the component $\bar{v}$ in the coordinates $(\bar{x}^0, \bar{x}^1)$

$$\bar{v} = \frac{v - V}{1 - \epsilon^2 V\,v}.$$

Relativistic Burgers equations on Minkowski spacetime

Theorem

The conservation law

$$\partial_0 T^0(v) + \partial_1 T^1(v) = 0, \tag{1}$$

is invariant under Lorentz transformations if and only if after suitable normalization one has

$$T^0(v) = \frac{v}{\sqrt{1-\epsilon^2 v^2}}, \qquad T^1(v) = \frac{1}{\epsilon^2}\left(\frac{1}{\sqrt{1-\epsilon^2 v^2}} - 1\right),$$

where the scalar field v takes its value in $(-1/\epsilon, 1/\epsilon)$.

Sketch of the proof

Use Lorentz transformation with related change of coordinates
Apply the chain rule to write $\partial_0 T^0, \partial_1 T^1$
Substitute them in the conservation law equation (1)
Checking the Lorentz invariance property
Determine the general expression of T^0 and T^1

$$T^0(v) = T^0(\phi_\epsilon)(u) = \frac{e^{\epsilon u} - e^{-\epsilon u}}{2\epsilon} = \frac{1}{\epsilon}\sinh(\epsilon u) = u + O(\epsilon^2 u^3),$$
$$T^1(v) = T^1(\phi_\epsilon(u)) = \frac{e^{\epsilon u} + e^{-\epsilon u} - 2}{2\epsilon^2} = \frac{1}{\epsilon^2}\Big(\cosh(\epsilon u) - 1\Big) = \frac{u^2}{2} + O(\epsilon^2 u^4),$$
where $v = \frac{1}{\epsilon}\frac{e^{2\epsilon u} - 1}{e^{2\epsilon u} + 1} = \phi_\epsilon(u)$.

T^0 and T^1 are linear and quadratic, respectively.
Substitute $u = \frac{1}{2\epsilon}\ln\frac{1+\epsilon v}{1-\epsilon v}$, we get the desired result. (Note that in the limiting case $\epsilon \to 0$, we recover the (inviscid) Burgers equation).

Properties of the relativistic Burgers equation

$$\partial_0\Big(\frac{v}{\sqrt{1-\epsilon^2 v^2}}\Big) + \partial_1\Big(\frac{1}{\epsilon^2}\Big(\frac{1}{\sqrt{1-\epsilon^2 v^2}} - 1\Big)\Big) = 0 \tag{1}$$

1. The map $w = T^0(v) = \frac{v}{\sqrt{1-\epsilon^2 v^2}} \in \mathbb{R}$ is increasing and one-to-one from $(-1/\epsilon, 1/\epsilon)$ onto $\mathbb{R}$.
2. In terms of the new unknown $w \in \mathbb{R}$, (1) is equivalent to

$$\begin{aligned} &\partial_0 w + \partial_1 f_\epsilon(w) = 0, \\ &f_\epsilon(w) = \frac{1}{\epsilon^2}\big(-1 \pm \sqrt{1+\epsilon^2 w^2}\big), \end{aligned} \tag{2}$$

3. In the non-relativistic limit $\epsilon \to 0$, one recovers the Burgers equation

$$\partial_0 u + \partial_1(u^2/2) = 0, \qquad \text{where } u \in \mathbb{R}.$$

The proposed equation retains several key features of the relativistic Euler equations :

- Like the conservation of mass-energy in the Euler system, it has a conservative form.
- Like the velocity component in the Euler system, our unknown v is constrained to lie in the interval $(-1/\epsilon, 1/\epsilon)$ limited by the light speed parameter.
- Like the Euler system, by sending the light speed to infinity one recover the classical (non-relativistic) model.

The non-relativistic limit

We recover the Galilean transformations by relativistic case with $\epsilon \to 0$ given by

$$\bar{x}^0 = x^0, \quad \bar{x}^1 = x^1 - Vx^0, \quad \bar{v} = v - V.$$

We have the following :

The conservation law

$$\partial_0 T^0(v) + \partial_1 T^1(v) = 0,$$

is invariant under Galilean transformations iff the flux functions T^0 and T^1 are linear and quadratic, respectively. If $T^0(v) = v$, then after a suitable normalization one gets $T^1(v) = v^2/2$.

Relativistic Euler Equations

$$\partial_0\left(\frac{p+\rho c^2}{c^2}\frac{v^2}{c^2-v^2}+\rho\right)+\partial_1\left((p+\rho c^2)\frac{v}{c^2-v^2}\right)=0,$$
$$\partial_0\left((p+\rho c^2)\frac{v}{c^2-v^2}\right)+\partial_1\left((p+\rho c^2)\frac{v^2}{c^2-v^2}+p\right)=0,$$

p, ρ, v and c denote the pressure, density, velocity and speed of light.
Set ρ as a constant (and thus the pressure p) in the second equation :

$$\partial_0\left(\frac{v}{c^2-v^2}\right)+\partial_1\left(\frac{v^2}{c^2-v^2}\right)=0.$$

Use change of variable $z=\frac{v}{1-\epsilon^2v^2}$, with $c=1/\epsilon$, we get

$$\partial_0 z+\frac{1}{2\epsilon^2}\partial_1\left(-1\pm\sqrt{1+4\epsilon^2z^2}\right)=0. \qquad (3)$$

Vanishing pressure on flat spacetime

Assume that the pressure vanishes identically in the relativistic Euler equations, i.e.

$$\partial_0\left(\frac{\rho}{c^2 - v^2}\right) + \partial_1\left(\frac{\rho v}{c^2 - v^2}\right) = 0,$$
$$\partial_0\left(\frac{\rho v}{c^2 - v^2}\right) + \partial_1\left(\frac{\rho v^2}{c^2 - v^2}\right) = 0.$$

Rewriting these two equations :

$$(c^2 - v^2)(\partial_0 \rho + v\partial_1 \rho) + \rho(2v\partial_0 v + (v^2 + c^2)\partial_1 v) = 0,$$
$$v(c^2 - v^2)(\partial_0 \rho + v\partial_1 \rho) + \rho((v^2 + c^2)\partial_0 v + 2vc^2\partial_1 v) = 0.$$

Combining these equations we recover the classical Burgers equation (in flat spacetime)

$$\partial_0 v + \partial_1(v^2/2) = 0$$

Vanishing pressure on Schwarzshild spacetime

Suppose that $p \equiv 0$. We get the Euler system in $1+1$ dimensions on a Schwarzshild background takes the simplified form

$$\partial_t \left(\frac{r^2}{c^2} \, \widetilde{T}^{00} \right) + \partial_r \left(\frac{r(r-2m)}{c} \, \widetilde{T}^{01} \right) = 0,$$

$$\partial_t \left(\frac{r(r-2m)}{c} \, \widetilde{T}^{01} \right) + \partial_r \left((r-2m)^2 \, \widetilde{T}^{11} \right) - 3m \, \frac{(r-2m)}{r} \, \widetilde{T}^{11} + m \, \frac{(r-2m)}{r} \, \widetilde{T}^{00} = 0,$$

where $\widetilde{T}^{00} := \frac{c^2 \rho + p(\rho) v^2/c^2}{c^2 - v^2} c^2$, $\widetilde{T}^{01} := \frac{c^2 \rho + p(\rho)}{c^2 - v^2} c v$, $\widetilde{T}^{11} := \frac{v^2 \rho + p(\rho)}{c^2 - v^2} c^2$.
Combining these two equations, we arrive at

$$\partial_t v + \left(1 - \frac{2m}{r} \right) v \partial_r v = \frac{m}{r^2} (v^2 - c^2), \qquad \text{or equivalently ,}$$

Burgers equation on Schwarzshild spacetime

$$\partial_t (r^2 v) + \partial_r \left(r(r-2m) \frac{v^2}{2} \right) = r v^2 - m c^2 \tag{4}$$

Static solutions of Burgers equation on Schwarzschild spacetime

Consider

$$\partial_r \left(r(r - 2m) \frac{v^2}{2} \right) = rv^2 - mc^2$$

We find that all static solutions are described by

Static solutions of Burgers equation

$$v_s(r) = \pm\sqrt{c^2 - K^2 \left(1 - \frac{2m}{r}\right)}, \quad \text{or} \quad \frac{c^2 - v_s^2}{1 - \frac{2m}{r}} = K^2, \; K \in (0, c) \tag{5}$$

Relativistic Burgers equation on Schwarzshild spacetime

We set $M = \mathbb{R}_+ \times \mathbb{R}$.
In coordinates (x^0, x^1) with $\partial_\alpha := \partial/\partial x^\alpha$ (with $\alpha = 0, 1$), the hyperbolic balance laws under consideration reads

$$\partial_0(\overline{\omega}\, T^0(v, x^0, x^1)) + \partial_1(\overline{\omega}\, T^1(v, x^0, x^1)) = \overline{\omega}\, S(v, x^0, x^1),$$

where $v : M \to \mathbb{R}$ is the unknown function and $T^\alpha = T^\alpha(v, x^0, x^1)$ and $S = S(v, x^0, x^1)$ are prescribed (flux and source) fields on M, while $\overline{\omega} = \overline{\omega}(x^0, x^1)$ is a positive weight-function.
Set $x^0 = ct, x^1 = r$, with $c = 1/\epsilon$, we propose the following model

Relativistic Burgers equation on Schwarzshild spacetime

$$\partial_t(r^2 w) + \partial_r\Big(r(r-2m)f_\epsilon(\omega)\Big) = 0, \quad f_\epsilon(\omega) = \frac{1}{\epsilon}\Big(-1 + \sqrt{1+\epsilon^2 w^2}\Big) \quad (6)$$

Static solution of Relativistic Burgers equation on Schwarzshild spacetime

Consider

$$\partial_r\Big(r(r-2m)f_\epsilon(\omega)\Big)=0,$$

where f_ϵ is strictly convex. We take positive branch of f_ϵ^{-1}.
It follows that

Static solution of Relativistic Burgers equation

$$w_s(r)=f_\epsilon^{-1}\Big(\frac{K}{r(r-2m)}\Big)=\pm\frac{K}{r(r-2m)}\sqrt{\epsilon^2+2\frac{r(r-2m)}{K}}. \quad (7)$$

Geometric formulation of finite volume schemes

(M, ω) : $(1+1)$-dimensional curved spacetime, globally hyperbolic, foliated by spacelike, compact, oriented hypersurfaces $H_t, (t \in \mathbb{R})$:

$$M = \bigcup_{t \in \mathbb{R}} H_t.$$

$\mathcal{T}^h = \bigcup_{K \in \mathcal{T}^h} K$: a triangulation of M, which is made of (compact) spacetime elements K.

- The boundary ∂K of an element K is piecewise smooth $\partial K = \bigcup_{e \subset \partial K} e$ and contains exactly two spacelike faces, denoted by e_K^+ and e_K^-, and "timelike" elements

$$e^0 \in \partial^0 K := \partial K \setminus \{e_K^+, e_K^-\}.$$

- $|K|$ and $|e_K^+|, |e_K^-|, |e^0|$ represent the measures of K and e_K^+, e_K^-, e^0, respectively.

Consider a hyperbolic balance law posed on M :

$$\mathrm{div}_\omega(T(v)) = \frac{1}{\omega}(\partial_0(\omega T^0(v,x)) + \partial_1(\omega T^1(v,x)) = S(v,x)$$

Integrating this equation in space and time

$$\int_K (\omega S) dV_M = \int_K \mathrm{div}_\omega(T(v))\, dV_M,$$

which is equal to

$$\int_{e_K^+} T^0 \omega(n_{e_K^+}, \cdot) = \int_{e_K^-} (T^0)\omega(n_{e_K^-}, \cdot) - \sum_{e^0 \in \partial^0 K} \int_{e^0} T^1 \omega(n_{e^0}, \cdot) + \int_{\partial^0 K} S(v)\omega.$$

We introduce the approximations

$$\overline{T}_e(v) \simeq \frac{1}{|e_K^-| \overline{\omega}_{e_K^-}} \int_{e_K^-} T^0(v)\omega(n_{e_K^+}, \cdot), \quad \overline{S}_K \simeq \frac{1}{|K|\omega_K} \int_K S(v)\omega,$$

and

$$\int_{e^0} T^1(v)\omega(n_{e^0}, \cdot) \simeq |e^0|\, \overline{\omega}_{e^0}\, Q_{K,e^0}(v_K^-, v_{K_{e^0}}^-),$$

where $Q_{K,e^0} : \mathbb{R}^2 \to \mathbb{R}$ is a numerical flux function.

We obtain the finite volume approximations

$$|e_K^+|\,\overline{\omega}_{e_K^+}\,\overline{T}_{e_K^+}(v_K^+) = |e_K^-|\overline{\omega}_{e_K^-}\,\overline{T}_{e_K^-}(v_K^-) - \sum_{e^0\in\partial^0 K} |e^0|\overline{\omega}_{e^0}\,Q_{K,e^0}(v_K^-, v_{K_{e^0}}^-) + \omega_K|K|\,\overline{S}_K.$$

and in local coordinates it is of the form

$$\overline{\omega}_j\,\overline{T}_j^{n+1} = \overline{\omega}_j\,\overline{T}_j^n - \lambda\Big(\overline{\omega}_{j+1/2}\,Q_{j+1/2}^n - \overline{\omega}_{j-1/2}\,Q_{j-1/2}^n\Big) + \overline{\omega}_j|\Delta t|\,\overline{S}_j^n$$

where $\lambda := \Delta t/\Delta r$, $\overline{T}_j^n := \overline{T}(v_j^n)$

Well-balanced scheme on Schwarzshild spacetime

We focus attention on defining a well-balanced scheme specifically in the case

$$\omega(r) = r(r-2m), \ \ m \geq 0$$

The discrete version of Burgers equation on Schwarzschild spacetime reads

$$\overline{T}_j^{n+1} = \overline{T}_j^n - \frac{\Delta t}{\Delta r}\left(\omega_{j+1/2}\, Q_{j+1/2} - \omega_{j-1/2}\, Q_{j-1/2}\right) + \Delta t\, \overline{S}_j$$

where the mess size $\Delta r = r_{j+1/2} - r_{j-1/2}$, and

$$r_{j-1/2} = 2m + j\Delta, \ r_j = 2m + (j+1/2)\Delta r, \ r_{j+1/2} = 2m + (j+1/2)\Delta r$$

and the averaged weights are $\omega_{j\pm1/2} = r_{j\pm1/2}(r_{j\pm1/2} - 2m)$.

Comparison between schemes and models

Geometric Burgers equation I (Rel. Burg. eqn. on Sch. spacetime)

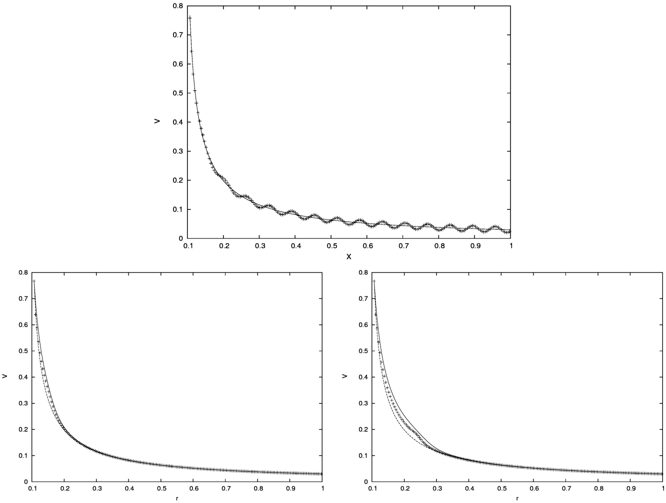
$$\partial_t(r^2 w) + \partial_r\Big(r(r-2m)(-1+\sqrt{1+w^2}\Big) = 0 \quad \text{(Conservative)}$$

Geometric Burgers equation II (Burg. eqn. on Sch. spacetime)

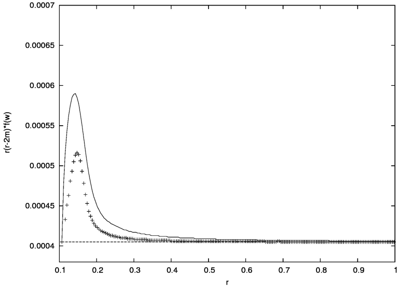
$$\partial_t(r^2 v) + \partial_r\Big(r(r-2m)v^2/2\Big) = rv^2 - mc^2 \qquad \text{(Non-conservative)}$$

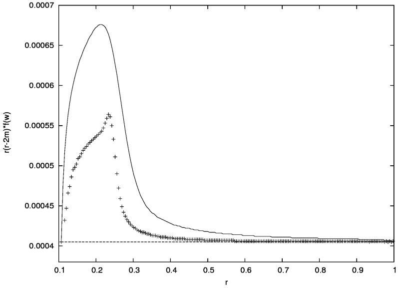
Normalize $c = 1/\epsilon = 1$, $r \in (2m, R)$ where R is an upper bound for the spatial variable. We use the Godunov flux at the boundary.
For the numerical calculations : We take $2m = 0.1$, $R = 1.0$, $CFL = 0.9$
We compare the numerical solutions based on the three schemes :
-A first order Lax-Friedrichs scheme (plotted with $+$)
-A second order Lax-Friedrichs scheme (plotted with dots)
-A well-balanced second-order Lax-Friedrichs scheme (plotted with $-$)

Numerical solutions (model I)

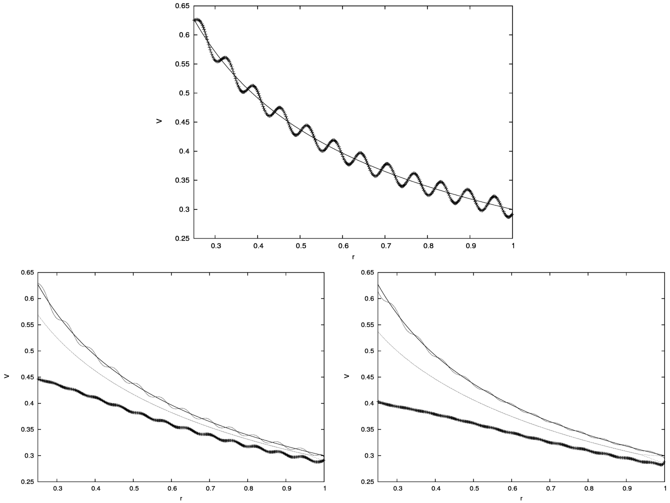


Comparison of the numerical flux by the three schemes (model I)

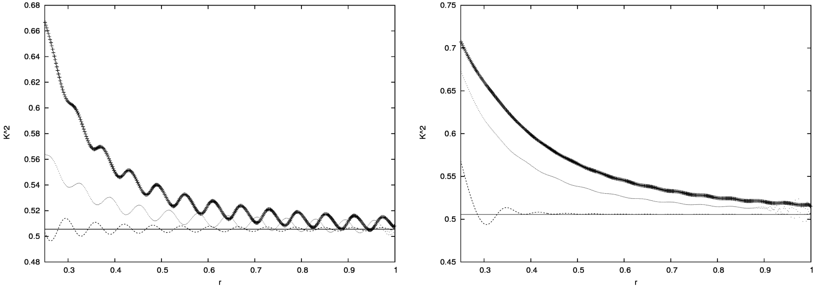




Numerical solutions (model II)

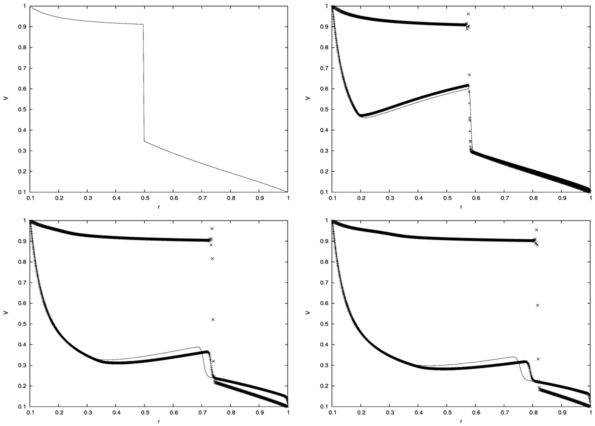


Comparison of the numerical values of K^2 based on the three schemes (model II)



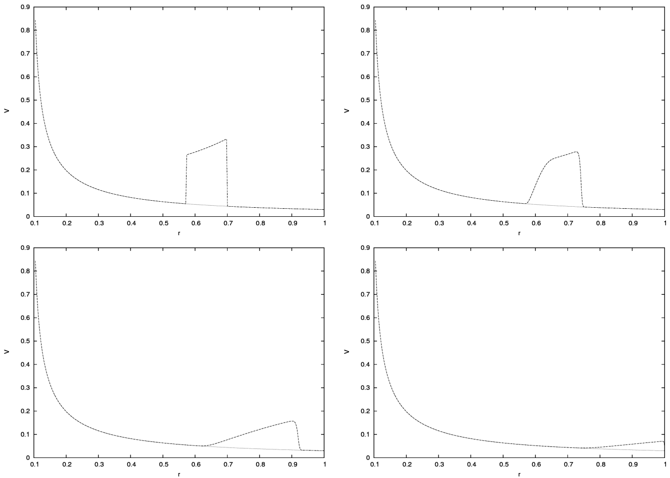
Comparison of schemes for a single shock at $r = 0.5$ (model II)

Perturbation of shock to the right away from the singularity

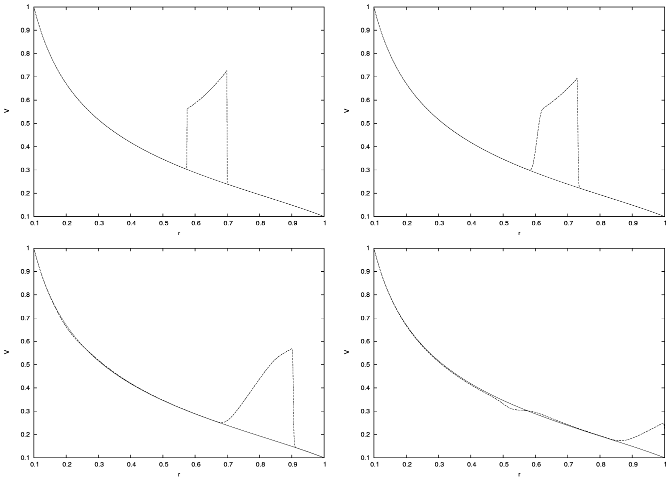


Late-time asymptotics–perturbed static solutions (model I)

Impose an initial perturbation



Late-time asymptotics-perturbed static solutions (model II)



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