Nonlinear stability of a boundary layer solution to the Euler-Poisson equations in plasma physics

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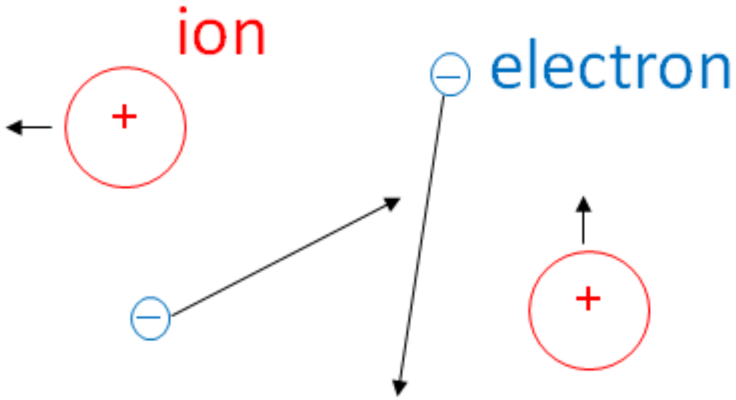


$u_e \gg u_i$

$(\because m_e \ll m_i)$

Nearly neutral : $\rho_e \fallingdotseq \rho_i$

$\phi \fallingdotseq 0$

m : mass

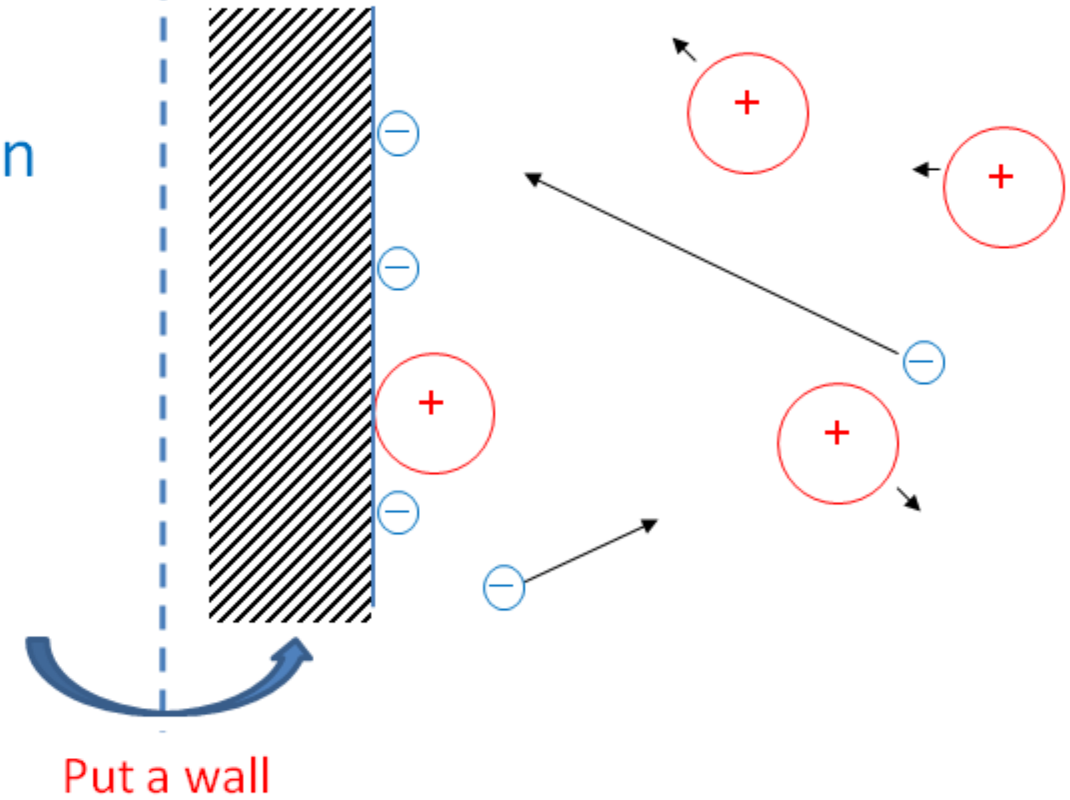
u : velocity

ρ : density

ϕ : electric potential

subscripts
i : ion
e : electron

Plasma in Half Space



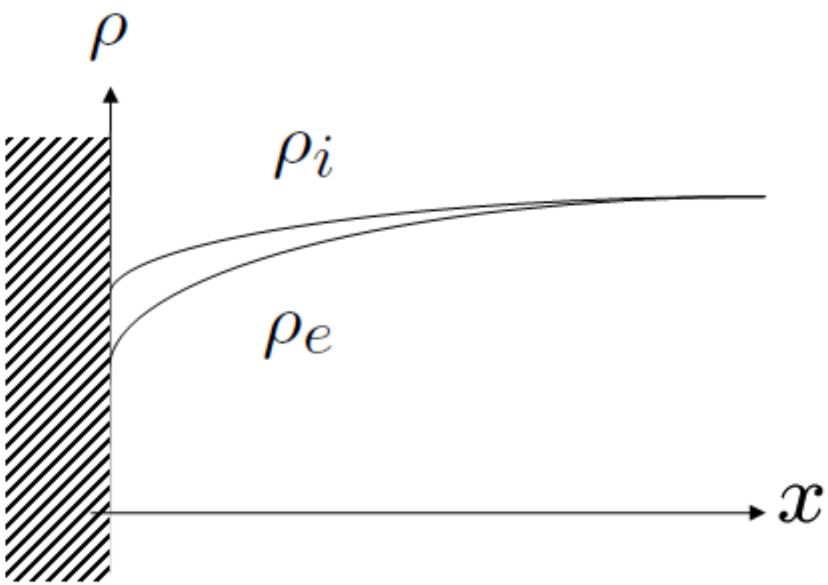
On the wall,
Electrons accumulate

$(\because u_e \gg u_i)$

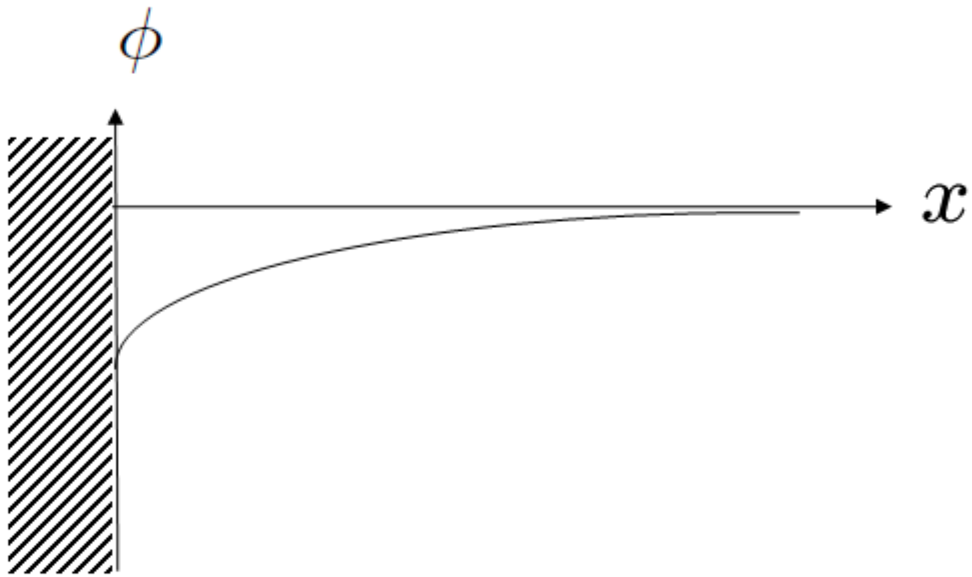
Elsewhere,
Ions dominate

Process of Sheath Formation(ii)

Density distribution



Potential distribution



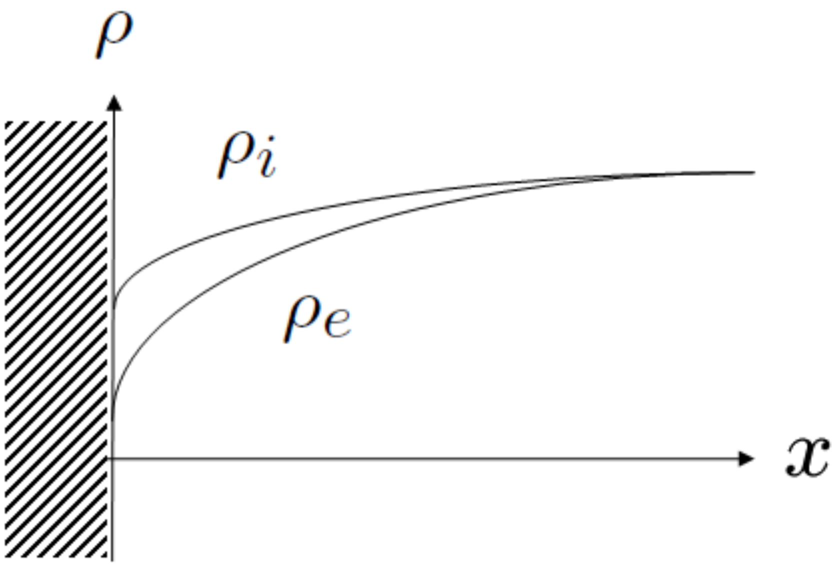
On the wall, electrons gather.
Elsewhere, ions dominate.

Lower potential on the wall.

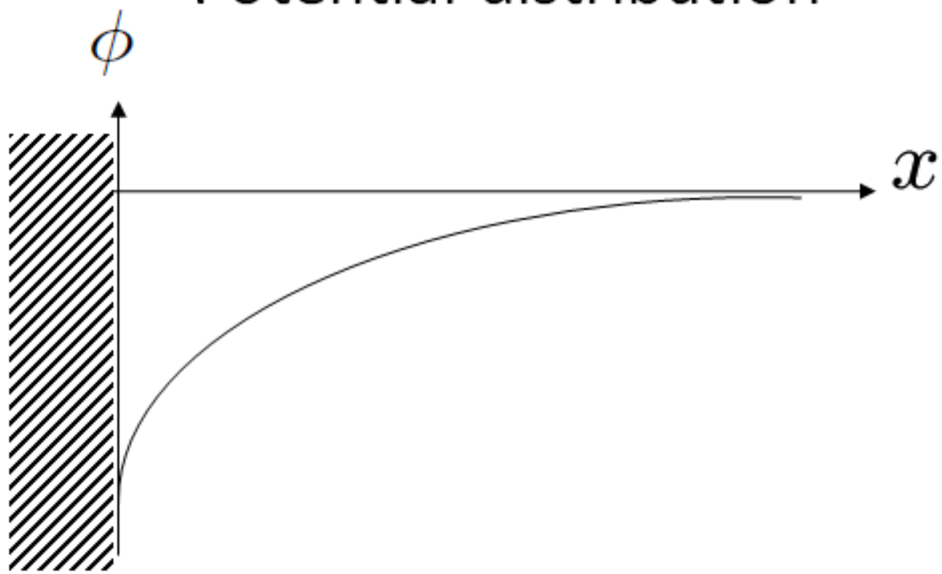
Toward the wall,
ions are accelerated
electrons are decelerated.

Process of Sheath Formation(iii)

Density distribution



Potential distribution



In the end, both flux to the wall coincide
and a steady state is attained.

This <u>stationary boundary layer</u> is called a SHEATH.

Remark : Physically, density & potential are monotone.
We focus our attention on sheath with monotonicity.

Bohm's Sheath Criterion

For the sheath formation, physical observation requires the **Bohm sheath criterion** (BSC):

$$\boldsymbol{u_+^2 \geq K + 1, \quad u_+ < 0,} \tag{BSC}$$

u_+ : Ion's velocity component normal to wall around sheath edge
K : Const. proportional to abs. temperature (= (Acoustic Velocity)^2)

$$(p(\rho) = K\rho, \quad K > 0, \quad \text{Isothermal})$$

Validate BSC from the mathematical point of view.

Remark : (BSC) $\Rightarrow$ Supersonic condition : $u_+^2 > K$.

1. Mathematical formulation of the problem

Euler-Poisson equations (dimension $N = 1$)

$$\rho_t + (\rho u)_x = 0, \tag{E.a}$$

$$(\rho u)_t + (\rho u u)_x + p(\rho)_x + \rho \phi_x = 0, \tag{E.b}$$

$$-\phi_{xx} = \rho - \boldsymbol{\rho_e}. \tag{E.c}$$

$t > 0$	:	Time variable
$x \in \mathbb{R}_+ := (0, \infty)$	:	Space variable
$\rho = \rho(t, x) > 0$	:	Ion density
$u = u(t, x) \in \mathbb{R}$	:	Ion velocity
$\phi = \phi(t, x) \in \mathbb{R}$	:	Electrostatic potential
$p(\rho) = K\rho \quad (K > 0)$	:	Pressure (Isothermal)
$\boldsymbol{\rho_e = e^{\phi} > 0}$ **(Boltzmann relation)**	:	**Electron density**

[Chen, Introduction to plasma physics, '77]

• Initial data

$$(\rho, u)(0, x) = (\rho_0, u_0)(x), \qquad \inf_{x \in \mathbb{R}_+} \rho_0(x) > 0, \tag{I.a}$$

$$\lim_{x \to \infty} (\rho_0, u_0)(x) = (\rho_+, u_+), \tag{I.b}$$

where $\rho_+ (> 0)$, $u_+ (< 0)$ are constants.

• Dirichlet Boundary data

$$\phi(t, 0) = \phi_b \tag{B}$$

where ϕ_b is constant.

• Reference point of potential

$$\lim_{x \to \infty} \phi(t, x) = 0, \tag{R}$$

◇ To construct classical solution to (E.c): $-\phi_{xx} = \rho - e^{\phi}$, it must be that

$$\rho_+ = 1. \tag{A}$$

Stationary problem

We define sheath by a stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\phi})(x)$ to (E):

$$(\tilde{\rho}\tilde{u})_x = 0, \tag{S.a}$$

$$\Big(\tilde{\rho}\tilde{u}^2 + p(\tilde{\rho})\Big)_x + \tilde{\rho}\tilde{\phi}_x = 0, \tag{S.b}$$

$$-\tilde{\phi}_{xx} = \tilde{\rho} - e^{\tilde{\phi}}, \tag{S.c}$$

with conditions (I.b), (B), (R), (A)

$$\inf_{x\in\mathbb{R}_+} \tilde{\rho}(x) > 0, \quad \lim_{x\to\infty} (\tilde{\rho}, \tilde{u}, \tilde{\phi})(x) = (\rho_+, u_+, 0), \quad \tilde{\phi}(0) = \phi_b.$$

Question

1. When does a stationary solution exist ?
2. Is the stationary solution asymptotically stable ?
3. What is the convergence rate ?

2. Known facts about stationary sol. $\cdots$ [M.Suzuki '11]

Theorem 1 (Existence of monotone stationary solution)

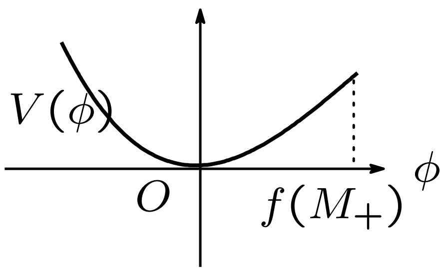
$i)$ *Let* $u_+^2 \le K$ *or* $K+1 = u_+^2$ *or* $\boldsymbol{K+1 < u_+^2}$.

$\boldsymbol{\phi_b \le f(|u_+|/\sqrt{K}),\ V(\phi_b) \ge 0} \Longleftrightarrow$ *Monotone stationary sol exists.*

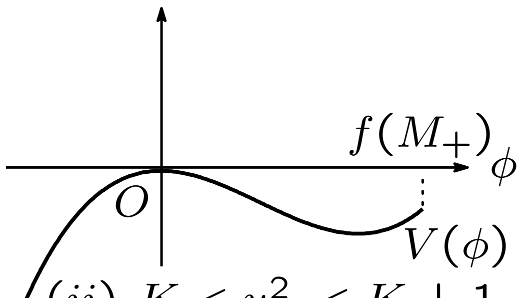
Moreover, assume monotonicity $\Rightarrow$ *unique.*

$ii)$ *Let* $K < u_+^2 < K+1$. *NO non-trivial stationary solution.*

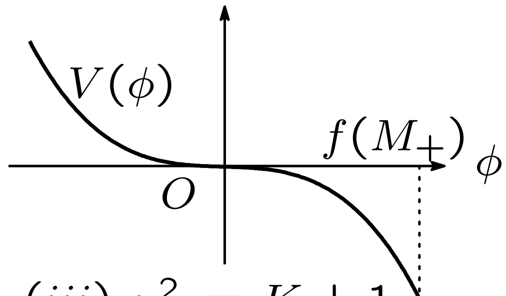
$$f(\tilde\rho) := -K \log\tilde\rho - \frac{u_+^2}{2\tilde\rho^2} + \frac{u_+^2}{2}, \qquad V(\tilde\phi) := \int_0^{\tilde\phi} [e^\eta - f^{-1}(\eta)]\, d\eta. \left(\begin{array}{c}\text{Sagdeev}\\ \text{potential}\end{array}\right)$$



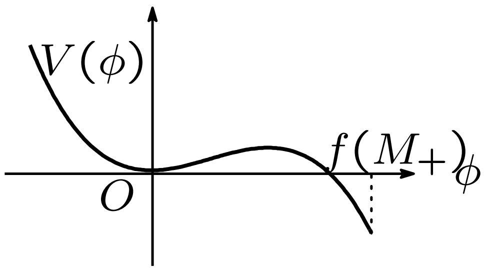
(i) $u_+^2 \le K$



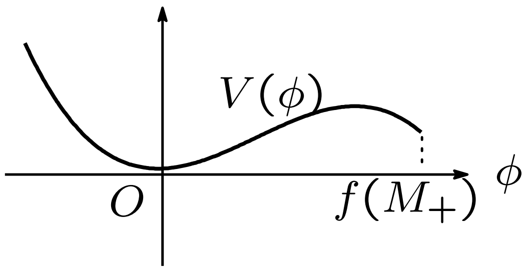
(ii) $K < u_+^2 < K+1$



(iii) $u_+^2 = K+1$



$\boldsymbol{(iv)\ K+1 < u_+^2 \le \omega}$



$\boldsymbol{(v)\ \omega < u_+^2}$

Related Works

(a) Results over bounded domain $(0,1)$

- **[A. Ambroso, F. Méhats, P.-A. Raviart, AA ’01]**
  shows existence of stationary solution under (BSC).

- **[A. Ambroso M3AS ’06]**
  numerically shows stability of stationary solution.

(b) Results over $\mathbb{R}_+$

- **[M. Suzuki ’11]**
  derives conditions for the existence of monotone stationary solution
  and shows its stability but with conditions stronger than (BSC).

3. Asymptotic stability of sheath under (BSC)

Perturbation (ψ, η, σ) satisfies equations

$$\begin{pmatrix} \sqrt{K}\psi \\ \eta \end{pmatrix}_t + \begin{pmatrix} u & \sqrt{K} \\ \sqrt{K} & u \end{pmatrix} \begin{pmatrix} \sqrt{K}\psi \\ \eta \end{pmatrix}_x + \begin{pmatrix} 0 \\ \sigma \end{pmatrix}_x + \eta \begin{pmatrix} \sqrt{K}\tilde{v} \\ \tilde{u} \end{pmatrix}_x = 0, \tag{P.a}$$

$$-\Delta\sigma = e^{\psi+\tilde{v}} - e^{\tilde{v}} - e^{\sigma+\tilde{\phi}} + e^{\tilde{\phi}}. \tag{P.b}$$

with initial and boundary data to (P)

$$(\psi, \eta)(0, x) = (\psi_0, \eta_0)(x) := (\log \rho_0 - \log \tilde{\rho}, u_0 - \tilde{u}),$$

$$\lim_{x\to\infty} (\psi_0, \eta_0)(x) = (0, 0), \tag{PI}$$

$$\sigma(t, 0) = 0, \quad \lim_{x\to\infty} \sigma(t, x) = 0. \tag{PB}$$

When perturbation is small ((BSC) $\Rightarrow$ supersonic)

In x direction, characteristics of hyperbolic equations (P.a) are

$$\begin{aligned} \lambda_1 &:= u - \sqrt{K} < 0, \\ \lambda_2 &:= u + \sqrt{K} < 0. \end{aligned}$$

$$\Big(\because u = u_+ + (\tilde{u} - u_+) + \eta, \quad \text{(BSC)} : u_+ \leq -\sqrt{K+1}\Big)$$

- For hyperbolic equation (P.a), no boundary condition is necessary.
- For elliptic equation (P.b), one boundary condition is necessary.

$\Rightarrow$ Well-posed with 1 boundary condition (PB),

$$\sigma(t, 0) = 0.$$

Difficulty to show asymptotic stability

System of linearized equations of (P)
around asymptotic state $(\rho, u, \phi) = (\rho_+, u_+, 0)$ is

$$\begin{pmatrix} \sqrt{K}\psi \\ \eta \end{pmatrix}_t + \begin{pmatrix} u_+ & \sqrt{K} \\ \sqrt{K} & u_+ \end{pmatrix} \begin{pmatrix} \sqrt{K}\psi \\ \eta \end{pmatrix}_x + \begin{pmatrix} 0 \\ \sigma \end{pmatrix}_x = 0, \quad -\sigma_{xx} = \psi - \sigma. \tag{L}$$

Spectrums of (L) are given by

$$\mu(i\xi) = i\left(-\xi u_+ \pm |\xi|\sqrt{K + \frac{1}{1+|\xi|^2}}\right), \quad \xi \in \mathbb{R}.$$

Real parts of all spectrums are ZERO.

To resolve this difficulty, we employ weighted energy method.

◇ All characteristics go into boundary.

◇ Decay of (ψ_0, η_0) as $x \to \infty \Rightarrow$
convergence of solution towards stationary solution as $t \to \infty$.

Introduce new variables $(\Psi, H, \Sigma) := (e^{\beta x/2}\psi, e^{\beta x/2}\eta, e^{\beta x/2}\sigma)$.

Rewrite systems of equation (P) w.r.t. $(\Psi, H, \Sigma) \Rightarrow (P')$.

Linearize (P') around asymptotic state $(\rho, u, \phi) = (\rho_+, u_+, 0) \Rightarrow (L')$.

Spectrums of (L') are given by

$$\mu(i\xi) = \frac{\beta u_+}{2} + i\left(-\xi u_+ \pm \sqrt{K\zeta - \frac{1}{\zeta} + 1 - K}\right),$$

$$\text{where} \quad \zeta = 1 + |\xi|^2 - \frac{\beta^2}{4} + i\beta\xi \quad \text{for } \xi \in \mathbb{R}.$$

$$\text{Linearly Stable} \;\Leftrightarrow\; \sup_{\xi\in\mathbb{R}} Re\,(\mu(i\xi)) < 0 \;\Leftrightarrow\; u_+^2 > K + \frac{1}{1-\beta^2/4}. \qquad (\natural)$$

$$\left(\because \quad \sup_{\xi\in\mathbb{R}} Re\,(\mu(i\xi)) = Re\,(\mu(0)) \qquad \right)$$

$\therefore$ If $u_+^2 > K + 1$, setting $\beta \ll 1$ ensures $(\natural)$.

Main result (Asymptotic stability of sheath)

$\cdots$ [S.Nishibata, M.O., M.Suzuki, *SIAM. J. Math. Anal. '12*]

Theorem 2 (with exponential weight : 1D case)

Suppose

$$\text{(BSC)}: \quad u_+ < 0, \quad u_+^2 > K+1, \quad K > 0$$

$$(e^{\lambda x/2}\psi_0, e^{\lambda x/2}\eta_0) \in H^2(\mathbb{R}_+)$$

$$\lambda + \left(|\phi_b| + \|(e^{\lambda x/2}\psi_0, e^{\lambda x/2}\eta_0\|_{H^2} \right) / \lambda \ll 1.$$

Then $^{\exists 1}$ *Time global solution* (ψ, η, σ)

$$e^{\lambda x/2}\psi, e^{\lambda x/2}\eta \ \in \ \bigcap_{i=0}^{2} C^i\left([0,\infty); H^{2-i}(\mathbb{R}_+)\right),$$

$$e^{\lambda x/2}\sigma \ \in \ \bigcap_{i=0}^{2} C^i\left([0,\infty); H^{4-i}(\mathbb{R}_+)\right).$$

Moreover, $^{\exists}C, \gamma > 0$ *s.t.*

$$\|(e^{\lambda x/2}\psi, e^{\lambda x/2}\eta)(t)\|_{H^2} + \|e^{\lambda x/2}\sigma(t)\|_{H^4} \leq C\|(e^{\lambda x/2}\psi_0, e^{\lambda x/2}\eta_0)\|_{H^2}\, e^{-\gamma t}.$$

Outline of proof

(Local existence) + (A-priori estimate) $\Rightarrow$ (Global existence)

Lemma 3 (Local existence)
$(e^{\lambda x/2}\psi_0, e^{\lambda x/2}\eta_0) \in H^2(\mathbb{R}_+)$ *with*

$$\eta_0(0) + \tilde{u}(0) + \sqrt{K} < 0, \quad \sup_{x \in \mathbb{R}_+} |\psi_0(x)| + |\phi_b| \ll 1.$$

$\Rightarrow {}^{\exists}\alpha, T > 0$, *s.t.*, ${}^{\exists}1$ *sol.* (ψ, η, σ) *s.t.*
$(e^{\alpha x/2}\psi, e^{\alpha x/2}\eta) \in C([0,T]; H^2(\mathbb{R}_+))$, $e^{\alpha x/2}\sigma \in C([0,T]; H^4(\mathbb{R}_+))$.

Lemma 4 (A-priori estimate)
$N_{e^{\alpha x}}(T) := \sup_{0 \le t \le T} \|(e^{\alpha x/2}\psi, e^{\alpha x/2}\eta)(t)\|_{H^2}$.
$\beta \in (0, \alpha]$, $\beta + (N_{e^{\alpha x}}(T) + |\phi_b|)/\beta \ll 1 \Rightarrow {}^{\exists}C, \gamma > 0$ *s.t.*

$$e^{\gamma t}\left(\|e^{\beta x/2}(\psi,\eta)(t)\|_{H^2}^2 + \|e^{\beta x/2}\sigma(t)\|_{H^4}^2\right) + \int_0^t e^{\gamma\tau}\left(\|e^{\beta x/2}(\psi,\eta)(\tau)\|_{H^2}^2 + \|e^{\beta x/2}\sigma(\tau)\|_{H^4}^2\right) d\tau \le C\|e^{\beta x/2}(\psi_0,\eta_0)\|_{H^2}^2$$

4. Asymptotic stability of sheath in multi-dimensions

$$\rho_t + \mathrm{div}\,(\rho u) = 0, \tag{E.a}$$

$$(\rho u)_t + \mathrm{div}\,(\rho u \otimes u) + \nabla p(\rho) + \rho \nabla \phi = 0, \tag{E.b}$$

$$-\Delta \phi = \rho - \rho_e. \tag{E.c}$$

$$x = (x_1, x') = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N_+ := (0, \infty) \times \mathbb{R}^{N-1}$$

- Initial data $\lim_{x_1 \to \infty} (\rho_0, u_0)(x_1, x') = (\rho_+, u_+, 0, \ldots, 0), \quad {}^\forall x' \in \mathbb{R}^{N-1}$
- Boundary data $\phi(t, 0, x') = \phi_b, \quad {}^\forall x' \in \mathbb{R}^{N-1}$
- Reference point of potential $\lim_{x_1 \to \infty} \phi(t, x_1, x') = 0, \quad {}^\forall x' \in \mathbb{R}^{N-1}$

Planar Stationary solution

Embed 1D stationary solution in multi-dimensional space as

$$\tilde{\tilde{v}}(x) = \tilde{v}(x_1), \quad \tilde{\tilde{u}}(x) = (\tilde{u}(x_1), 0, \ldots, 0), \quad \tilde{\tilde{\phi}}(x) = \tilde{\phi}(x_1). \quad (x \in \mathbb{R}^N_+)$$

Hereafter, we simply write $(\tilde{\tilde{v}}, \tilde{\tilde{u}}, \tilde{\tilde{\phi}})$ by $(\tilde{v}, \tilde{u}, \tilde{\phi})$

Perturbation (ψ, η, σ) satisfies equations

$$\begin{pmatrix} \sqrt{K}\psi \\ \eta \end{pmatrix}_t + \sum_{j=1}^{N} M_j \begin{pmatrix} \sqrt{K}\psi \\ \eta \end{pmatrix}_{x_j} + \begin{pmatrix} 0 \\ \nabla\sigma \end{pmatrix} + \eta_1 \begin{pmatrix} \sqrt{K}\tilde{v} \\ \tilde{u} \end{pmatrix}_{x_1} = 0, \tag{P.a}$$

$$-\Delta\sigma = e^{\psi+\tilde{v}} - e^{\tilde{v}} - e^{\sigma+\tilde{\phi}} + e^{\tilde{\phi}}. \tag{P.b}$$

$$M_j(u_j) := u_j \boldsymbol{I}_{N+1} + \begin{pmatrix} 0 & & \sqrt{K} & \\ & 0 & \vdots & 0 \\ \sqrt{K} & \cdots & 0 & \cdots \\ & 0 & \vdots & 0 \end{pmatrix}_{< j+1} \quad \left(\begin{array}{c} \text{matrix of} \\ \text{size (N+1)} \end{array} \right)$$

$$\widehat{j+1}$$

with initial and boundary data to (P)

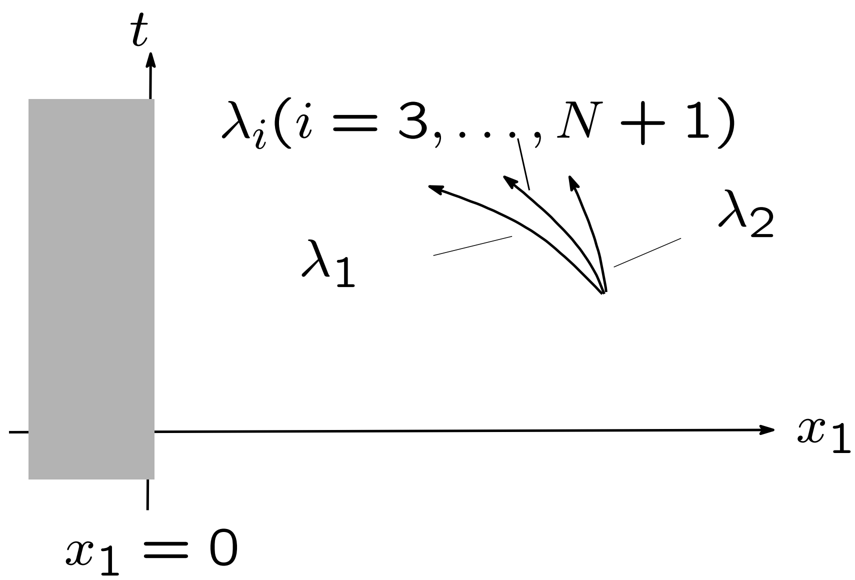
$$(\psi, \eta)(0, x) = (\psi_0, \eta_0)(x) := (\log \rho_0 - \log \tilde{\rho}, u_0 - \tilde{u}),$$

$$\lim_{x_1 \to \infty} (\psi_0, \eta_0)(x) = (0, 0), \tag{PI}$$

$$\sigma(t, 0, x') = 0, \qquad \lim_{x_1 \to \infty} \sigma(t, x_1, x') = 0, \qquad {}^{\forall} x' \in \mathbb{R}^{N-1}. \tag{PB}$$

In x_1 direction, characteristics of hyperbolic equations (P.a) are

$$\begin{aligned} \lambda_1 &:= u_1 - \sqrt{K} < 0, \\ \lambda_2 &:= u_1 + \sqrt{K} < 0, \\ \lambda_i &:= u_1 < 0 \quad (i = 3, \ldots, N+1). \end{aligned}$$



Spectrums of (L) are given by

$$\mu(i\xi) = i\left(-\xi_1 u_+ \pm |\xi|\sqrt{K + \frac{1}{1+|\xi|^2}}\right), \; -i\xi_1 u_+ \text{(N-1 multiple)} \;\; \xi \in \mathbb{R}^N.$$

Energy method with weight function of normal coordinate works!

- Derivation of the basic estimate (a) $\left(\omega = \begin{pmatrix} \psi \\ \eta \end{pmatrix} \in \mathbb{R}^{N+1} \quad D_t := \partial_t + (u \cdot \nabla)\right)$

$$\omega_t + \sum_{j=1}^{N} M_j \omega_{x_j} + \begin{pmatrix} 0 \\ \nabla\sigma \end{pmatrix} + \eta_1 \begin{pmatrix} \tilde{v} \\ \tilde{u} \end{pmatrix}_{x_1} = 0 \qquad \text{(P.a)}$$

$$-\Delta\sigma = e^{\psi+\tilde{v}} - e^{\tilde{v}} - e^{\sigma+\tilde{\phi}} + e^{\tilde{\phi}} \qquad \text{(P.b)}$$

$$\int dx \quad \left[e^{\beta x_1}\ {}^t\omega \cdot \text{(P.a)}\right] \Rightarrow \qquad \left(\delta := N_{e^{\beta x_1}}(T) + |\phi_b| + \beta^2\right)$$

$$\frac{d}{dt}\left(\int_{\mathbb{R}^N_+} dx e^{\beta x_1} \frac{K\psi^2 + \eta^2}{2}\right) + \beta \int_{\mathbb{R}^N_+} dx e^{\beta x_1} \left[{}^t\omega \frac{-M_1}{2}\omega - \eta_1\sigma\right]$$

$$+ \int_{x=0} dx' \left[{}^t\omega \frac{-M_1}{2}\omega\right] - \int_{\mathbb{R}^N_+} dx e^{\beta x_1} [\sigma \mathrm{div}\eta] \leq C\delta \|e^{\beta x_1/2}\omega\|_1^2.$$

$$\text{(P.b)} \quad \Rightarrow \quad |-\Delta\sigma - (\psi - \sigma)| \leq C\delta|(\psi, \sigma)|$$

$$\mathrm{div}\left((\text{P.a})_2 \cdots (\text{P.a})_{N+1}\right) \quad \Rightarrow \quad |D_t(\mathrm{div}\eta) + K\Delta\psi + \Delta\sigma| \leq C\delta|(\eta, \nabla\eta)|$$

$$\therefore \quad |D_t(\mathrm{div}\eta) + K\Delta\psi - \psi + \sigma| \leq C\delta|(\omega, \nabla\omega, \sigma)|$$

$$\therefore \quad -\int_{\mathbb{R}_+^N} dx e^{\beta x_1} [\sigma \mathrm{div}\eta]$$

$$\geq \int_{\mathbb{R}_+^N} dx e^{\beta x_1} [(D_t (\mathrm{div}\eta) + K\Delta\psi - \psi) \,\mathrm{div}\eta] - C\delta \| e^{\frac{\beta x_1}{2}} \omega\|_1^2 \qquad (\because \text{elliptic estimate})$$

$$\geq \int_{\mathbb{R}_+^N} dx e^{\beta x_1} [D_t (\mathrm{div}\eta)\, \mathrm{div}\eta] + \int_{\mathbb{R}_+^N} dx e^{\beta x_1} [(K\Delta\psi - \psi)(-D_t\psi)] - C\delta \| e^{\frac{\beta x_1}{2}} \omega\|_1^2 \qquad (\because (\mathrm{P.a})_1)$$

$$\geq \frac{d}{dt}\left(\int_{\mathbb{R}_+^N} dx e^{\beta x_1} \frac{(\mathrm{div}\eta)^2 + K(\nabla\psi)^2 + \psi^2}{2}\right) + \beta \int_{\mathbb{R}_+^N} dx \left[e^{\beta x_1} f(t, x_1, x')\right] + \int_{x_1=0} dx' \left[f(t, 0, x')\right]$$

$$-C\delta \| e^{\beta x_1/2} \omega\|_1^2. \qquad f(t, x_1, x') = \frac{-u_1}{2} (\mathrm{div}\eta)^2 - K\psi_x \mathrm{div}\eta + \frac{-Ku_1}{2} (\nabla\psi)^2 + \frac{-u_1}{2}\psi^2$$

basic estimate (a) $\left(\begin{array}{c} \text{blue terms} > 0 \Leftrightarrow u_+^2 > K+1 \\ \text{green terms} > 0 \Leftarrow u_+^2 > K \end{array} \right) \quad \omega := \begin{pmatrix} \psi \\ \eta \end{pmatrix} \in \mathbb{R}^{N+1}$

$$\frac{d}{dt}\left(\int_{\mathbb{R}_+^N} dx \frac{e^{\beta x_1}}{2} \left[(K+1)\psi^2 + \eta^2 + K(\nabla\psi)^2 + (\mathrm{div}\eta)^2\right]\right)$$

$$+ \quad \beta \int_{\mathbb{R}_+^N} dx e^{\beta x_1} \left[{}^t\omega \frac{-M_1}{2} \omega - \eta_1 \sigma + \frac{-u_1}{2}\psi^2 + \frac{-Ku_1}{2} (\nabla\psi)^2 - K\psi_x \mathrm{div}\eta + \frac{-u_1}{2} (\mathrm{div}\eta)^2 \right]$$

$$+ \quad \int_{x=0} dy \left[{}^t\omega \frac{-M_1}{2} \omega + \frac{-u_1}{2}\psi^2 + \frac{-Ku_1}{2} (\nabla\psi)^2 - K\psi_x \mathrm{div}\eta + \frac{-u_1}{2} (\mathrm{div}\eta)^2 \right]$$

$$\leq \quad C\delta \| e^{\beta x_1/2} \omega\|_1^2.$$

- <u>Derivation of supplementary inequality (b)</u> $\omega := {}^t(\psi, \eta) \in \mathbb{R}^{N+1}$

$$\int dx \left[e^{\beta x_1} \, {}^t\omega \cdot (\mathrm{P.a}) \right] + \sum_{i=1}^{N} \int dx \left[e^{\beta x_1} \, {}^t\partial_i \omega \cdot \partial_i (\mathrm{P.a}) \right] \Rightarrow$$

$$\text{(b)} \quad \frac{d}{dt} \left(\int_{\mathbb{R}_+^N} dx \frac{e^{\beta x_1}}{2} \left[(K+1)\psi^2 + \eta^2 + K(\nabla\psi)^2 + (\nabla\eta)^2 \right] \right)$$

$$+ \quad \beta \int_{\mathbb{R}_+^N} dx e^{\beta x_1} \left[{}^t\omega \frac{-M_1}{2} \omega - \eta_1 \sigma + \frac{-u_1}{2}\psi^2 + \sum_{i=1}^{N} {}^t\partial_i\omega \frac{-M_1}{2} \partial_i \omega + \sigma_x \ \mathrm{div}\eta - \sum_{i=1}^{N} \sigma_i \ \partial_i \eta_1 \right]$$

$$+ \quad \int_{x=0} dy \left[{}^t\omega \frac{-M_1}{2} \omega + \frac{-u_1}{2}\psi^2 + \sum_{i=1}^{N} {}^t\partial_i\omega \frac{-M_1}{2} \partial_i\omega + \sigma_x \ \mathrm{div}\eta - \ \sigma_x \ \partial_x \eta_1 \right] \le C\delta \| e^{\beta x_1/2}\omega \|_1^2$$

$$\because \quad + \int_{\mathbb{R}_+^N} dx \left[e^{\beta x_1} \eta \cdot \nabla\sigma \right] + \sum_{i=1}^{N} \int_{\mathbb{R}_+^N} dx \left[e^{\beta x_1} \partial_i \eta \cdot \nabla \partial_i \sigma \right]$$

$$= \ - \int_{\mathbb{R}_+^N} dx \left[e^{\beta x_1} \mathrm{div}\eta \, (\sigma - \Delta\sigma) \right] + \cdots$$

$$= \ + \int_{\mathbb{R}_+^N} dx \left[e^{\beta x_1} D_t \psi \psi \right] + \cdots \qquad (\because (\mathrm{P.a})_1, (\mathrm{P.b}))$$

- Derivation of supplementary inequality (c) $\quad \omega := {}^t(\psi, \eta) \in \mathbb{R}^{N+1}$

$$\int dx \left[e^{\beta x_1}\ {}^t\omega \cdot (\mathrm{P.a}) \right] \Rightarrow$$

$$(\mathrm{c}) \quad \frac{d}{dt}\left(\int_{\mathbb{R}^N_+} dx \frac{e^{\beta x_1}}{2} \left[K\psi^2 + \eta^2 + \sigma^2 + (\nabla\sigma)^2 \right] \right) + \int_{x=0} dy \left[{}^t\omega \frac{-M_1}{2}\omega + \frac{-u_1}{2}\sigma_x^2 \right]$$
$$+\beta \int_{\mathbb{R}^N_+} dx e^{\beta x_1} \left[{}^t\omega \frac{-M_1}{2}\omega + \eta_1\sigma + \frac{-u_1}{2}\sigma^2 - \sigma_x \sigma_t - u_1 \left(\frac{(\nabla\sigma)^2}{2} + \sigma_x^2 \right) \right] \leq C\delta \| e^{\beta x_1/2}\omega \|_1^2$$

$$\because \int_{\mathbb{R}^N_+} dx \left[e^{\beta x_1} \eta \cdot \nabla\sigma \right] = -\int_{\mathbb{R}^N_+} dx \left[e^{\beta x_1} \mathrm{div}\eta\sigma \right] + \cdots = \int_{\mathbb{R}^N_+} dx \left[e^{\beta x_1} D_t\psi\sigma \right] + \cdots$$
$$= \int_{\mathbb{R}^N_+} dx \left[e^{\beta x_1} \psi_t\sigma \right] + \int_{\mathbb{R}^N_+} dx \left[e^{\beta x_1} \{ (u\cdot\nabla)\psi \} \sigma \right] + \cdots \equiv (\mathrm{c}_1) + (\mathrm{c}_2) + \cdots$$
$$(\mathrm{c}_1) = -\int_{\mathbb{R}^N_+} dx \left[e^{\beta_1 x} (\Delta\sigma_t - \sigma_t)\sigma \right] + \cdots \left(\because -\Delta\sigma_t = \psi_t e^v - \sigma_t e^{-\phi} \Leftarrow \partial_t (\mathrm{P.b}) \right)$$
$$(\mathrm{c}_2) = -\int_{\mathbb{R}^N_+} dx \left[e^{\beta_1 x} \psi (u\cdot\nabla)\sigma \right] + \cdots = \int_{\mathbb{R}^N_+} dx \left[e^{\beta_1 x} (\Delta\sigma - \sigma)(u\cdot\nabla)\sigma \right] + \cdots$$

- Take constants $\epsilon \ll 1$ and set $\delta = N_{e^{\beta x_1}}(T) + |\phi_b| + \beta^2 \ll 1 \Rightarrow$

$^{\exists} c_0, \cdots, c_7 > 0$ s.t. (a) $+ \epsilon \times$ ((c) $+ \epsilon \times$ (b)) $\Rightarrow$

$$\frac{d}{dt}\left(\int_{\mathbb{R}^N_+} dx e^{\beta x_1}\left[c_1\psi^2 + c_2\eta^2 + c_3(\nabla\psi)^2 + c_4(\nabla\eta)^2 + c_5(\mathrm{div}\eta)^2 + c_6\sigma^2 + c_7(\nabla\sigma)^2\right]\right)$$
$$+ c_0\beta\left(\int_{\mathbb{R}^N_+} dx e^{\beta x_1}\left[\psi^2 + \eta^2 + (\nabla\psi)^2 + (\nabla\eta)^2 + (\mathrm{div}\eta)^2 + \sigma^2 + (\nabla\sigma)^2\right]\right) \le 0$$

$\therefore \quad \int_0^t d\tau\, [e^{\gamma\tau}\ \cdot\ ] \quad (\gamma \ll \beta) \Rightarrow {}^{\exists}C > 0$ s.t.

$$e^{\gamma t}\|e^{\beta x_1/2}(\psi,\eta)(t)\|^2_{H^1} + \int_0^t e^{\gamma\tau}\|e^{\beta x_1/2}(\psi,\eta)(\tau)\|^2_{H^1}\,d\tau \le C\|e^{\beta x_1/2}(\psi_0,\eta_0)\|^2_{H^1} \ (\sharp)$$

For higher order derivatives,

$\partial_{pq}\{(a),(b),(c)\}$ $(p, q = t, x_2, \cdots, x_N)$ + (equivalence of norms) + $(\sharp)$

$\Rightarrow {}^{\exists}C > 0$ s.t.

$$e^{\gamma t}\|e^{\beta x_1/2}(\psi,\eta)(t)\|^2_3 + \int_0^t e^{\gamma\tau}\|e^{\beta x_1/2}(\psi,\eta)(\tau)\|^2_3\,d\tau \le C\|e^{\beta x_1/2}(\psi,\eta)(0)\|^2_3.$$

Main result (Asymptotic stability of sheath : multi-dim case)

$\cdots$ [S.N, M.O., M.S., *SIMA '12*]

Theorem $2'$ (with exponential weight) $(N,m) = (1,2),(2,3),(3,3)$.

Suppose

$$\text{(BSC)}: \quad u_+ < 0, \quad u_+^2 > K+1, \quad K > 0$$

$$(e^{\lambda x_1/2}\psi_0, e^{\lambda x_1/2}\eta_0) \in H^m(\mathbb{R}^N_+)$$

$$\lambda + \left(|\phi_b| + \|(e^{\lambda x_1/2}\psi_0, e^{\lambda x_1/2}\eta_0\|_{H^m}\right)/\lambda \ll 1.$$

Then $^{\exists 1}$ *Time global solution* (ψ,η,σ)

$$e^{\lambda x_1/2}\psi, e^{\lambda x_1/2}\eta \in \bigcap_{i=0}^{m} C^i\left([0,\infty); H^{m-i}(\mathbb{R}^N_+)\right),$$

$$e^{\lambda x_1/2}\sigma \in \bigcap_{i=0}^{m} C^i\left([0,\infty); H^{m+2-i}(\mathbb{R}^N_+)\right).$$

$$^{\exists}C,\gamma > 0 \ s.t. \ \|(e^{\lambda x_1/2}\psi, e^{\lambda x_1/2}\eta)(t)\|_{H^m} + \|e^{\lambda x_1/2}\sigma(t)\|_{H^{m+2}}$$

$$\leq C\|(e^{\lambda x_1/2}\psi_0, e^{\lambda x_1/2}\eta_0)\|_{H^m} e^{-\gamma t}.$$

Main result (Asymptotic stability of sheath : multi-dim case)

$\cdots$ [S.N, M.O., M.S., *SIMA '12*]

$w_{\lambda,\alpha} := (1+\alpha x_1)^{\lambda}$ for $\lambda > 0,\ \alpha > 0$

Theorem 5 (with algebraic weight) $(N,m) = (1,2), (2,3), (3,3)$.

$$u_+ < 0, \quad u_+^2 > K+1, \quad K > 0.$$

If $(w_{\lambda/2,\beta}\psi_0, w_{\lambda/2,\beta}\eta_0) \in H^m(\mathbb{R}^N_+)$ *for* $\lambda \geq 2,\ \beta > 0$, *then* $^\forall \alpha \in (0,\lambda]$ $^\exists\delta = \delta(\alpha) > 0$ *s.t. if* $\beta + (|\phi_b| + \|(w_{\lambda/2,\beta}\psi_0, w_{\lambda/2,\beta}\eta_0)\|_{H^m})/\beta \leq \delta$

$\Rightarrow$ $^\exists 1$ *Time global solution* (ψ,η,σ) *s.t.*

$$w_{\alpha/2,\beta}\psi, w_{\alpha/2,\beta}\eta \in \bigcap_{i=0}^{m} C^i([0,\infty); H^{m-i}(\mathbb{R}^N_+)),$$

$$w_{\alpha/2,\beta}\sigma \in \bigcap_{i=0}^{m} C^i([0,\infty); H^{m+2-i}(\mathbb{R}^N_+))$$

and $^\exists C(\alpha) > 0$ *s.t.* $\|(w_{\alpha/2,\beta}\psi, w_{\alpha/2,\beta}\eta)(t)\|^2_{H^m} + \|w_{\alpha/2,\beta}\sigma(t)\|^2_{H^{m+2}}$

$$\leq C\|(w_{\lambda/2,\beta}\psi_0, w_{\lambda/2,\beta}\eta_0)\|^2_{H^m}(1+\beta t)^{-(\lambda-\alpha)}.$$

Main result (degenerate case) $\cdots$ [S.Nishibata, M.O., M.Suzuki]

Theorem 6 (Asymptotic stability of sheath)
$(N,m) = (1,2)\,,(2,3)\,,(3,3)\,.$

$$u_+ < 0, \quad u_+^2 = K+1, \quad K > 0.$$

Let $\lambda_0 \in \mathbb{R}$ *satisfy* $\lambda_0(\lambda_0-1)(\lambda_0-2)-12(\lambda_0+2)=0,\ \lambda\in[4,\lambda_0)$.
${}^\forall\alpha\in(0,\lambda],\quad {}^\forall\theta\in(0,1],\quad {}^\exists\delta=\delta(\alpha,\theta)>0$
s.t. if $\phi_b\in[-\delta,0),\quad \beta/\Gamma|\phi_b|^{1/2}\in[\theta,1]$ *and*
$(w_{\lambda/2,\beta}\psi_0, w_{\lambda/2,\beta}\eta_0)\in H^m(\mathbb{R}^N_+)$ *with* $\|(w_{\lambda/2,\gamma}\psi_0, w_{\lambda/2,\gamma}\eta_0)\|_m/\beta^3\le\delta$,

$\Rightarrow$ ${}^\exists 1$ *Time global solution* (ψ,η,σ)

$$w_{\alpha/2,\beta}\psi, w_{\alpha/2,\beta}\eta \ \in \ \bigcap_{i=0}^{m} C^i([0,\infty);H^{m-i}(\mathbb{R}^N_+)),$$

$$w_{\alpha/2,\beta}\sigma \ \in \ \bigcap_{i=0}^{m} C^i([0,\infty);H^{m+2-i}(\mathbb{R}^N_+))$$

and ${}^\exists C(\alpha,\theta)>0$ *s.t.*

$$\|(w_{\alpha/2,\beta}\psi, w_{\alpha/2,\beta}\eta)(t)\|_m^2 + \|w_{\alpha/2,\beta}\sigma(t)\|_{m+2}^2 \le C\|(w_{\lambda/2,\beta}\psi_0, w_{\lambda/2,\beta}\eta_0)\|_m^2(1+\beta t)^{-(\lambda-\alpha)/3}.$$

5. Concluding Remarks

We give mathematical definition to the sheath in plasma physics by a monotone stationary solution under (BSC).

- $u_+^2 \geq K + 1$ (BSC), $|\phi_b| \ll 1 \Rightarrow$
  - ♢ Monotone stationary solution exists uniquely.
  - ♢ Monotone stationary solution is asymptotically stable.

- Spectrum analysis supports $\begin{cases} \text{(BSC)} \Rightarrow \text{Linearly stable.} \\ \text{Otherwise} \Rightarrow \text{Linearly unstable.} \end{cases}$
  - ♢ (BSC) may be a necessary condition for stability.

We validate our def. of sheath from the mathematical point of view.